Pythagoras theorem revised : a + b = c ?

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1 Introduction

In most scientific mathematical publications a yet unsolved problem (or at least one that is thought to be unsolved :) is described in a few lines and then the rest of the article is about ways to solve the problem using a certain new and astonishing approach.

In this article however I'd like to discuss a problem that is thought to be solved already more than 2000 years ago : the Pythagoras theorem about rectangular triangles that gives the length of the distance between the endpoints of 2 perpendicular lines that start in the same point. I'll try to show that the commonly accepted solution for the length of the hypotenuse as given by equation (1) is not in all cases the most obvious solution.

$$a^2 + b^2 = c^2 \tag{1}$$

Although measurement of a rectangular triangle immediately shows you that the formula given by Pythagoras yields indeed a correct answer, I do have reasons to believe that in some cases the answer nevertheless seems to be given by equation (2) :

$$a + b = c \tag{2}$$

Ok, I can hear the laughter now and you may think warily about those silly non-mathematicians that want to interfere with material they have no knowledge about, but before you discard this article as absolute nonsense, please read at least the next section before doing so !

2 Problem description

If we look at figure 1 we see the standard triangle as used in most textbooks to discuss Pythagoras theorem. If we proceed from A over B to C the answer is given by equation

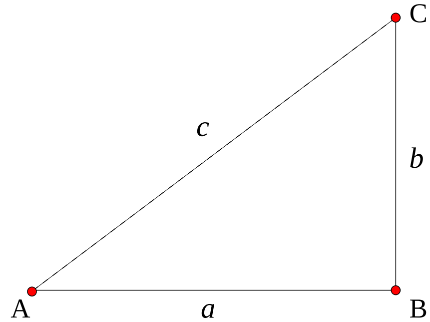

Figure 1: The basic Pythagoras problem

(2), but if we move directly from A to C the answer is given by equation (1). So far no problems whatsoever.

Let us now try to follow the staircase path as shown in figure 2. If we compute the total distance l from A to C over this path it is easy to see that the answer is given by :

$$l = \frac{1}{2}a + \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}b = a + b$$

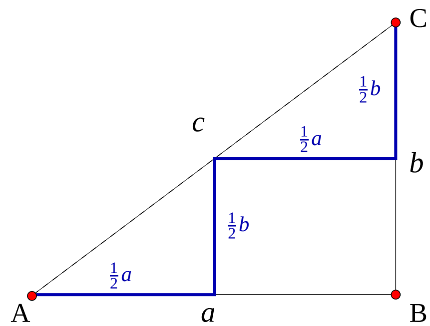

Figure 2: The distance from A to C using 2 steps

In a similar way we can compute the total distance l from A to C for figure 3 :

$$l = 4 * \frac{1}{4}a + 4 * \frac{1}{4}b = a + b$$

And for the general case in figure 4 :

$$l = \sum_n \frac{1}{n}a + \sum_n \frac{1}{n}b = a + b \tag{3}$$

Even in this last case it seems obvious that the distance l always equals $a+b$, no matter how large we choose n, so it looks as if we may write :

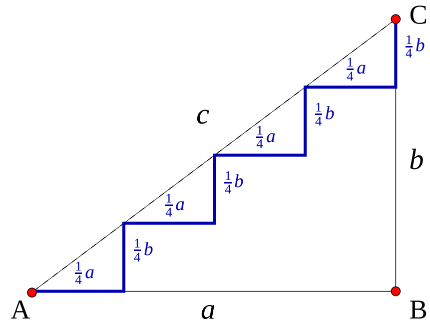

Figure 3: The distance from A to C using 4 steps

$$l = \lim_{n\to\infty}\{\sum_n \frac{1}{n}a + \sum_n \frac{1}{n}b\} = a + b \tag{4}$$

Each step of the staircase is defined by a triangle with sides $\frac{1}{n}a$ and $\frac{1}{n}b$ and in the limiting case for $n \to \infty$ each reactangular side of the triangle must necessarily have a length of zero, so obviously *all points of the triangle including the lowerright corner must be located on the line A-C !* (as is also indicated by the arrow)

However, the distance from A to C is even in that case equal to $a+b$ according to equation (4), which is still quite different from the standard answer as given by the Pythagoras theorem.

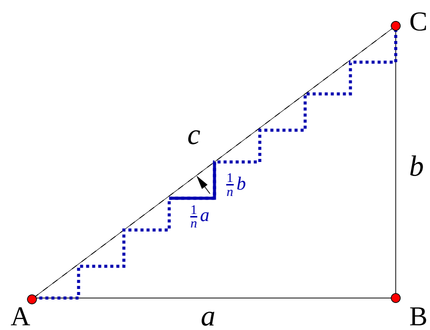

Figure 4: The distance from A to C using n steps

We can also look at the problem from a slightly different point of view : consider a staircase-shaped line going from A to C that consists of an *infinite* number of staircase steps with infinite short sides da and db. In that case the length of the line is given by :

$$l = \int_A^B da + \int_B^C db = a + b$$

which is again the same result as we obtained earlier.

3 Possible Explanations

Although the reasoning above seems plausible enough even a simple example shows that the answer is not what we expect from practice.

Suppose a=12 and b=16 : then according to Pythagoras the hypotenuse c=20, while according to the method from above the distance l=28. Measurement shows that the distance is indeed 20 and not 28, so what's wrong with the theory ?

In my opinion the question should NOT be why the 2 methods give a different answer for the same (appearent) length, but should be whether the 2 distances c and l are or should be indeed the same thing and if not, what is then the difference ?

As my 11 year old son noticed when I confronted him with this problem : in the first case the line has only 2 fixed points which are the starting point and the endpoint. In the second case however the line consists of not only the starting and ending point but has many extra points between.

Of course he was absolutely right (I am little prebiased ;-), but why should that make any difference if all points are exactly on the line A-C ?

A possible explanation may be found from a problem taken from chaos theory : consider a coastline, let's say the distance from Hoek of Holland to Den Helder measured over the Northsea coast. If we have a very simple map this distance won't be any longer than the length of a straight line going from HvH to DH, but the more detailed the map, the more curves we'll observe in the coastline and thus the longer the measured distance will be.

And suppose that we really walk the distance over the beach and measure every tiny curve along the way ? Or that we'll use a microscope to examine each single grain of sand ? Obviously the distance will increase with the amount of detail we take into account and there is no reason to believe that there is any limit to this number. So, the correct answer is any number between the shortest distance over the straight line and infinity !

However, although in our problem from above we do find a longer distance than the shortest one possible, we definitely do not find one that goes to infinity. Furthermore, if we use an infinite number of steps - thereby forcing all points from these steps to be effectively on the same line as A-C and so they shouldn't have any influence at all - we still find a larger distance than simply c, which at least contradicts our intuition.

4 Future research

I haven't been able to solve this problem yet, but of course I am only a physicist, without many mathematical skills :) Anyone with more skills who feels him/herself tempted to give a satisfactory explanation for this problem ? Or maybe the problem was already solved by someone else long ago, but I simply wasn't able to find the result ?

I do have some possible ideas where (part of) the explanation may be found, but haven't investigated them yet. It may however be useful to mention some of them here :

- Principle of the least action ?
- Calculation of variations ?
- Theory of line integrals ?
- Fractal dimensions ?

References

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[2] James Gleick, Chaos, (Penguin, reprint edition December 1998)

[3] Hughes en Brecht, Vicieuze cirkels, (Bert Bakker, tweede druk 1988)