RAIN-WIND INDUCED VIBRATIONS OF A SIMPLE OSCILLATOR

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Abstract

In this paper a relatively simple mechanical oscillator is considered which may be used to study rain-wind induced vibrations of stay cables of cable-stayed bridges. In recent publications mention is made of vibrations of (inclined) stay cables which are excited by a wind-field containing rain drops. The rain drops that hit the cables generate a rivulet on the surface of the cable. The presence of flowing water on the cable changes the cross section of the cable experienced by the wind-field. A symmetric flow pattern around the cable with circular cross section may become asymmetric due to the presence of the rivulet and may consequently induce a lift-force as a mechanism for vibration. During the motion of the cable the position of rivulet(s) may vary as the motion of the cable induces an additional varying aerodynamic force perpendicular to the direction of the wind-field. It seems not to easy to model this phenomenon: several author state that there is no model available yet.

The idea to model this problem is to consider a horizontal cylinder supported by springs in such a way that only one degree of freedom i.e. vertical vibrations are possible. We consider a ridge on the surface of the cylinder parallel to the axis of the cylinder. Let additionally the cylinder with ridge be able to oscillate, with small amplitude, around the axis such that the oscillations are excited by an external force. It may be clear that the small amplitude oscillations of the cylinder and hence of the ridge induce a varying lift and drag force. In this approach it is assumed that the motion of the ridge models the dynamics of the rivulet(s) on the cable. By using a quasi-steady approach to model the aerodynamic forces one arrives at a nonlinear second order equation displaying three different kinds of excitation mechanisms: self-excitation, parametric excitation and ordinary forcing. The first results of the analysis of the equation of motion show that even in a linear approximation for certain values of the parameters involved stable periodic motions are possible. In the relevant cases where in linear approximation unstable periodic motions are found, results of an analysis of the nonlinear equation are presented.

1 INTRODUCTION

Cable-stayed bridges are characterized by inclined stay cables connecting the bridge deck with one or more pylons. Usually the stay cables have a smooth polyurethane mantle and a cross section which is nearly circular. Under normal circumstances for such type of cables one would not expect galloping type of vibrations due to wind-forces. There are however exceptions: in the winter season ice accretion on the cable may induce aerodynamic instability resulting in vibrations with relatively large amplitudes. The instability mechanism for this

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type of vibrations is known and can be understood on the basis of quasi-steady modeling and analysis. In this analysis the so-called Den Hartog’s criterion expressing a condition to have an unstable equilibrium state plays an important part. The other exception concerns vibrations excited by a wind-field containing raindrops. This phenomenon has probably been detected for the first time by Japanese researchers as can be derived from the papers by Matsumoto a.o. ([3], [4]). As has been observed on scale models in wind-tunnels the raindrops that hit the inclined stay cable generate one or more rivulets on the surface of the cable. The presence of flowing water on the cable changes the cross section of the cable as experienced by the wind field. Accordingly the pressure distribution on the cable with respect to the direction of the (uniform) wind flow may became asymmetric, resulting in a lift force perpendicular to direction of the wind velocity.

It is of interest to remark that there is an important difference between the presence of ice accretion and rivulets as far as it concerns the dynamical behaviour. The ice accretion concerns an ice coating fixed to the surface of the cable whereas the rivulet concerns a flow of water on the surface of the cable where the position of the rivulet depends on the resulting wind velocity, the surface tension of the water and the adhesion between the water and polyurethane mantle of the cable.

For the interesting cases the thickness of the ice accretion is not uniform: the evolution process of ice accretion usually results in an ice coating involving a ridge of ice. The case with water rivulets can also be characterized by the presence of the ridge of water be it with the difference that this water ridge is not fixed to the surface of the cable. As long as the water ridge is present, it may be blown or shaken off, one may assume that the position of the ridge varies in time. Subsequently one may assume that this time-dependence has a similar character as the motion of the cable i.e. if the cable oscillates harmonically then one may expect that the water ridge moves accordingly. The observation of this complicated system of an inclined cable, connecting a bridge deck and a pylon, with a moving rivulet leads to the following conclusion.

The inclination of the cable is relevant for having a rivulet. The rivulet however can be viewed as a moving ridge which may be modeled as a solid one. According this way of modeling the inclination of the cable is no longer relevant. Hence we consider as a prototype of an oscillator a one degree of freedom system consisting of a horizontal rigid cable supported by springs with a solid state ridge moving with small amplitude oscillations. From the point of view of the type of equation of motion, we arrive at a second order differential equation with external forcing. A more detailed description of the modeling is presented in the following section.

2 THE MODEL EQUATION FOR RAIN-WIND INDUCED VIBRATIONS OF A PROTOTYPE OSCILLATOR

The modeling principles we use are closely related to the quasi-steady approach as give in [1]. We consider a rigid cylinder with uniform cross-section supported by springs in a uniform rain-wind flow directed perpendicular to the axis of the cylinder. The oscillator is constructed in such a way that only vertical (one degree of freedom) oscillations are possible. The basic cross-section of the cylinder is circular, however on the surface of the cylinder there is a ridge able to carry out small amplitude oscillations. To model the rain-wind forces on the cylinder a quasi-steady approach is used; the type of oscillations which can be studied on the respective assumptions are known as galloping. A more detailed description of the quasi-steady approach can be found in [5]. The basic assumption of the quasi-steady approach is that at each moment in the dynamic situation the rain-wind force can be taken equal to the steady force exerted on the cylinder in static state. In the dynamic situation one should take into account that the flow-induced forces are based on the instantaneous flow velocity which is equal to the vector sum of flow velocity and the time varying vertical
flow velocity induced by the (vertical) motion of the cylinder.
The steady rain-wind forces can be measured in a wind-tunnel and are expressed in the
form of non-dimensional aerodynamic coefficients which depend on the angle of attack $\alpha$.
This angle, an essential variable for the description of the dynamics of the oscillator, is
defined as the angle between the resultant flow velocity and an axis of reference fixed to
the cylinder; measured positive in clockwise direction. The system we will study in more
detail is sketched in fig 1.

The horizontal wind velocity is $U$ and as the cylinder is supposed to move in the positive
$y$ direction, there is a virtual vertical wind velocity $-\dot{y}$. The drag force $D$ is indicated in
the direction of the resultant wind-velocity $U_r$, whereas the lift force $L$ is perpendicular to
$D$ in anti clockwise direction. The ridge on the cylinder bold indicated in fig 1. is able to
carry out small amplitude oscillations. The aerodynamic force $F_y$ in vertical direction can
easily be derived from fig 1.:  
\[ F_y = -D \sin \phi - L \cos \phi \quad (2.1) \]

where $\phi$ is the angle between $U_r$ and $U$, positive in clockwise direction, with $|\phi| \leq \pi/2$.
The drag and lift force are given by the empirical relations:
\[ D = \frac{1}{2} \rho \, d \, l \, U_r^2 \, C_D(\alpha) \quad (2.2) \]
\[ L = \frac{1}{2} \rho \, d \, l \, U_r^2 \, C_L(\alpha) \]
where $\rho$ is the density of air, $d$ the diameter of the cylinder, $l$ the length of the cylinder,
$C_D(\alpha)$ and $C_L(\alpha)$ are the drag and lift coefficient curves respectively, determined by mea-
surements in a wind-tunnel.

From fig 1. it follows that :
\[ \sin \phi = \dot{y}/U_r \quad (2.3) \]
\[ \cos \phi = U/U_r \]
\[ \alpha = \alpha_s + \arctan(\dot{y}/U) \]

The equation of motion of the oscillator readily becomes :
\[ m\ddot{y} + c_y\dot{y} + k_y y = F_y, \quad (2.4) \]
where \( m \) is the mass of the cylinder, \( c_y > 0 \) the structural damping coefficient of the oscillator, \( k_y > 0 \) the spring constant.

By using (2.2) and (2.3) we obtain for \( F_y \):

\[
F_y = -\frac{1}{2} \rho \ m \ l \ \sqrt{U^2 + \dot{z}^2} \ (C_D(\alpha) \dot{y} + C_L(\alpha) U)
\]  

(2.5)

Setting \( \omega_y^2 = k_y/m, \tau = \omega_y t \) and \( z = \omega_y y/U \) equation (2.4) becomes:

\[
\ddot{z} + 2\beta \dot{z} + z = -K \sqrt{1 + \ddot{z}^2} \ (C_D(\alpha) \dot{z} + C_L(\alpha))
\]

\[
\alpha = \alpha_s + \arctan(\dot{\alpha})
\]

(2.6)

where \( 2\beta = c_y/\omega_y \) and \( K = \rho \ m \ l \ U/2\omega_y \) are non-dimensional parameters, and \( \dot{z} \) now stands for differentiation with respect to \( \tau \).

We study the case where the drag and lift coefficient curve can be approximated by a constant and a cubic polynomial respectively:

\[
C_D(\alpha) = C_{D_o} \]

\[
C_L(\alpha) = C_{L_1}(\alpha - \alpha_o) + C_{L_3}(\alpha - \alpha_o)^3, \]

(2.7)

where \( C_{D_o}, C_{L_1} \) and \( C_{L_3} \) are real parameters with \( C_{D_o} > 0 \) and for the interesting cases \( C_{L_1} < 0 \) and \( C_{L_3} > 0 \). By using \( \alpha = \alpha_s + \arctan \dot{\alpha} \) we obtain for \( C_L(\alpha) \):

\[
C_L(\alpha) = C_{L_1}(\alpha_s - \alpha_o + \arctan \dot{\alpha}) + C_{L_3}(\alpha_s - \alpha_o + \arctan \dot{\alpha})^3
\]

(2.8)

The cases that \( \alpha_s = \alpha_o \) and \( \alpha_s \neq \alpha_o \) where \( \alpha_s \) and \( \alpha_o \) are (time independent) parameters have been studied in [1]. Here we study the case that the position of the (water) ridge varies with time:

\[
\alpha_s - \alpha_o = f(t) = f(\tau/\omega_y)
\]

(2.9)

Substitution of (2.8) and (2.9) in (2.6) and expanding the right hand side with respect to \( \dot{z} \) in the neighbourhood of \( \dot{z} = 0 \) yields:

\[
\ddot{z} + z = -K[C_{L_1} f(t) + C_{L_3} f^3(t) + (C_{D_o} + C_{L_1} + 2\beta/K + 3C_{L_3} f^2(t)) \ \dot{\dot{z}} +
\left(\frac{1}{2} C_{L_1} f(t) + \frac{1}{2} C_{L_3} f^3(t) + 3C_{L_3} f(t)\right) \ \ddot{z}^2 +
\left(\frac{1}{6} C_{L_1} + C_{L_3} + \frac{1}{2} C_{D_o} + \frac{1}{2} C_{L_3} f(t)\right) \ \dot{\dot{z}}^3 + 0(\ddot{z}^4)
\]

(2.10)

Inspection of this equation shows that for \( f(t) \equiv 0 \) one obtains:

\[
\ddot{z} + z = K[-(C_{D_o} + C_{L_1} + 2\beta/K) \ \dot{\dot{z}} - \left(\frac{1}{6} C_{L_1} + C_{L_3} + \frac{1}{2} C_{D_o}\right) \ \dot{\dot{z}}^3].
\]

(2.11)

When the following conditions hold:

\[
C_{D_o} + C_{L_1} + 2\beta/K < 0 \ \text{(Den Hartog’s Criterion)}
\]

\[
\frac{1}{6} C_{L_1} + C_{L_3} + \frac{1}{2} C_{D_o} > 0
\]

(2.12)

the equation can be reduced to the Rayleigh equation, which has, as is well-known, a unique periodic solution (limit-cycle). The linearized version of equation (2.11) has apart from \( z \equiv 0 \) only unbounded solutions if Den Hartog’s criterion applies. Linearization of equation (2.10) however leads to an equation which may have periodic solutions and is hence of interest to study in more detail.
2.1 The linear model

The linearized version of (2.10) can be written as:

$$
\ddot{z} + K(C_{D_o} + C_{L_1} + 2\beta/K + 3C_{L_3}f^2(t)) \dot{z} + z + K(C_{L_1}f(t) + C_{L_3}f^3(t)) = 0.
$$

We consider the case that $f(t) = A \cos \omega t = A \cos(2\eta \tau) = A \cos \Omega \tau$ where $\Omega = \frac{2\eta}{C_{L_3}}$ with

$$
f^2(t) = \frac{1}{2} A^2 (1 + \cos 2\Omega \tau) \quad \text{and}
$$

$$
f^3(t) = \frac{3}{4} A^3 (\cos \Omega \tau + \frac{1}{3} \cos 3\Omega \tau)
$$

(2.13) becomes:

$$
\ddot{z} + (KA_o + KA_1 \cos 2\Omega \tau) \dot{z} + z + K \dot{A}_2 \cos \Omega \tau + K \dot{A}_3 \cos 3\Omega \tau = 0
$$

where

$$
A_o = C_{D_o} + C_{L_1} + 2\beta/K + \frac{3}{2} C_{L_3} A^2,
$$

$$
A_1 = \frac{3}{2} C_{L_3} A^2,
$$

$$
A_2 = C_{L_1} A + \frac{3}{4} C_{L_3} A^3,
$$

$$
A_3 = \frac{1}{4} C_{L_3} A^3.
$$

For the oscillator we study the interesting case $\Omega = 1 + \epsilon \eta$ where $|\epsilon| \ll 1$. By setting $(1 + \epsilon \eta) \tau = \theta$ (2.14) becomes:

$$
(1 + \epsilon \eta)^2 \ddot{z} + (1 + \epsilon \eta)(KA_o + KA_1 \cos 2\theta) \dot{z} + z + K \dot{A}_2 \cos \theta + K \dot{A}_3 \cos 3\theta = 0
$$

where a dot now stands for differentiation with respect to $\theta$.

Let the coefficients $KA_i$, $i = 0, 1, 2, 3$ be of $O(\epsilon)$. Then (2.15) can be written as:

$$
\ddot{z} + (KA_o + KA_1 \cos \theta) \dot{z} + (1 - 2\epsilon \eta)z + K \dot{A}_2 \cos \theta + K \dot{A}_3 \cos 3\theta + O(\epsilon^2) = 0.
$$

This equation has been studied in [2] for case that $A_3 = 0$. The analysis in this paper can be applied in an analogous way for equation (2.16). By setting

$$
z = y_1 \cos \theta + y_2 \sin \theta
$$

$$
\dot{z} = -y_1 \sin \theta + y_2 \cos \theta
$$

and substitution of (2.17) in (2.16) we obtain a first order system for $y_1$ and $y_2$. After averaging this system becomes:

$$
\dot{\bar{y}}_1 = (-\frac{1}{2}KA_o + \frac{1}{4}KA_1)\bar{y}_1 - \epsilon \eta \bar{y}_2 \quad (2.18)
$$

$$
\dot{\bar{y}}_2 = \epsilon \eta \bar{y}_1 + (-\frac{1}{2}KA_o - \frac{1}{4}KA_1)\bar{y}_2 - \frac{1}{2}KA_2 \quad (2.19)
$$

where $\bar{y}_1$ and $\bar{y}_2$ are $O(\epsilon)$ approximations on a long $1/\epsilon$ time-scale.

The critical point of (2.18) reads:

$$
\bar{y}_{10} = \frac{2KA_2 \epsilon \eta}{K^2 A_o^2 + 4\epsilon^2 \eta^2 - \frac{1}{4}K^2 A_1^2},
$$

$$
\bar{y}_{20} = \frac{K \dot{A}_2 (\frac{1}{2}KA_1 - KA_o)}{K^2 A_o^2 + 4\epsilon^2 \eta^2 - \frac{1}{4}K^2 A_1^2}.
$$
The eigenvalues of the coefficient matrix are given by
\[ \lambda_{1,2} = -\frac{1}{2}KA_o \pm \sqrt{\frac{1}{16}K^2A_1^2 - \epsilon^2\eta^2}. \]

The stability of the critical point follows from these eigenvalues. If for instance \( \eta = 0 \) then to have two negative eigenvalues one should have \( 0 < A_1 < 2A_o \). From this inequality it follows that for having stable periodic solutions the damping coefficient \( K(A_o + A_1 \cos 2\Omega \tau) \) in equation (2.14) may even become negative during a part of the cycle with period \( \pi/\Omega \). When the damping coefficient is negative one usually refers to self-excitation, corresponding with an unstable equilibrium point. Apparently with self-excitation during a part of the cycle of the damping coefficient stable periodic solutions are possible. The coordinates of the critical point (2.19) become for \( \eta = 0 \):
\[ \bar{y}_{10} = 0, \quad \bar{y}_{20} = -\frac{A_2}{A_o + \frac{1}{2}A_1}. \]

This point corresponds with the periodic solution:
\[ \bar{z} = -\frac{A_2}{A_o + \frac{1}{2}A_1} \sin \theta. \] (2.20)

In the original variables (2.20) reads:
\[ \bar{z} = -\frac{AC_{L_1} + \frac{3}{4}A_3C_{L_3}}{C_{D_o} + C_{L_1} + \frac{2\beta}{K} + \frac{2}{7}C_{L_3}A^2} \sin \theta. \]

It is clear that \( z \to 0 \) if \( A \to 0 \), moreover to have \( O(1) \) amplitude oscillations the nominator and denominator should be of the same order. For \( |A| \ll 1 \) this implies that \( -AC_{L_1} \) should be of the same order as \( C_{D_o} + C_{L_1} + \frac{2\beta}{K} \). We recall that \( C_{D_o} + C_{L_1} + \frac{2\beta}{K} > 0 \) implying that the Den Hartog’s criterion does not apply. The excitation mechanism is clearly not self excitation but ordinary forcing due to the motion of the ridge. When however \( A \) is not small with respect to \( 1 \) then it follows from (2.20) that the amplitude depends both on \( A_2 \) as on \( A_1 \). Accordingly the excitation is a combination of parametrical and ordinary forcing. It is clear that the amplitude of this solution is of \( O(1) \). Increasing \( A_1 \) leads to reduction of the amplitude which can be understood as \( A_1 \) represent the amplitude of varying damping coefficient. One should also note that in case \( A_o < 0 \) there are no stable periodic solutions. This case in however important in applications. In order to find stable periodic solutions in this case one should additionally consider nonlinear terms i.e. the full equation (2.10).

### 2.2 The non-linear model

In this section we present some results on the analysis of the non-linear model equation (2.10) which can be written as:
\[ \ddot{z} + z = -K[A_2 \cos \Omega \tau + A_3 \cos 3\Omega \tau + (A_o + A_1 \cos 2\Omega \tau) \dot{z} + (2\epsilon\eta/K)z + (A_4 \cos \Omega \tau + \frac{1}{2}A_3 \cos 3\Omega \tau) \dot{z}^2 + (A_5 + \frac{1}{6}A_1 \cos 2\Omega \tau) \dot{z}^3] \] (2.21)

where \( A_o, A_1, A_2, A_3 \) are defined in (2.14) and
\[ A_4 = \frac{1}{2}C_{L_1} + 3C_{L_3}A_1 + \frac{3}{8}C_{L_3}A_3, \]
\[ A_5 = \frac{1}{6}C_{L_1} + C_{L_3} + \frac{1}{2}C_{D_o} + \frac{1}{4}C_{L_3}A_2. \]
We consider the case that $\Omega = 1$. By application of transformation (2.17):

$$
\begin{align*}
    z &= y_1 \cos \tau + y_2 \sin \tau \\
    \dot{z} &= -y_1 \sin \tau + y_2 \cos \tau
\end{align*}
$$

we obtain after averaging:

$$
\begin{align*}
    \ddot{y}_1 &= K\left[-\frac{1}{2}A_o + \frac{1}{4}A_1\right] \ddot{y}_1 + \left(\frac{3}{8}A_5 \dddot{y}_1 + \frac{1}{24}A_1 - \frac{3}{8}A_5\right) \ddot{y}_2 - \\
    \ddot{y}_2 &= -K\left[\frac{1}{2}A_2 + \frac{1}{2}A_o + \frac{1}{4}A_1\right] \ddot{y}_2 + \left(\frac{3}{8}A_4 - \frac{1}{16}A_3\right) \dddot{y}_1 + \\
    &\quad + \left(\frac{3}{8}A_4 + \frac{1}{16}A_3\right) \dddot{y}_2 + \left(\frac{3}{8}A_5 + \frac{1}{24}A_1\right) \dddot{y}_2^2
\end{align*}
$$

(2.23)

The linearized system (2.23) has the critical point $(0, -\frac{A_2}{A_o + \frac{1}{2}A_1})$ which is unstable if $A_o - \frac{1}{2}A_1 < 0$. With the original variables this inequality reads:

$$
C_{D_o} + C_{L_i} + 2\beta/K + \frac{3}{4}C_{L_3}A^2 < 0
$$

which can be considered as a modified Den Hartog criterion expressing that the instability is due to self-excitation combined with parametric excitation. Critical points of (2.23) may correspond to periodic solutions of equation (2.21). One important critical point can be found by setting $\ddot{y}_1 \equiv 0$. The second coordinate can be found by solving the cubical equation obtained from the right hand side of the second equation in (2.23):

$$
\left(\frac{1}{2}A_o + \frac{3}{8}A_5 \dddot{y}_1^2\right) \ddot{y}_2 = -\frac{1}{2}A_2 + \frac{1}{4}A_1 \ddot{y}_2 + \\
&\quad + \left(\frac{3}{8}A_4 + \frac{1}{16}A_3\right) \dddot{y}_1 + \frac{1}{24}A_1 \dddot{y}_2^2
$$

(2.24)

The right hand side of this equation can be made arbitrary small by letting $A \to 0$. Accordingly a solution can be found by setting:

$$
\ddot{y}_2 = y_{20} + A\dddot{y}_{21}
$$

(2.25)

where

$$
y_{20}^2 = -\frac{1}{2}A_o/\frac{3}{8}A_5, \quad A_o < 0 \quad \text{and} \quad A_5 > 0.
$$

(2.26)

$y_{21}$ can be computed by substitution of (2.25) in (2.24) and applying a straightforward perturbation procedure. It can be shown that the critical point $(0, \ddot{y}_{20})$ corresponds with a stable periodic solution (limit cycle) of equation (2.23) in the sense of an approximation accurate up to $O(A)$.

If $A = 0$ the critical points $(\ddot{y}_{10}, \ddot{y}_{20})$ of (2.23) are on the circle $\ddot{y}_1^2 + \ddot{y}_2^2 = -\frac{1}{2}A_o/\frac{3}{8}A_5$, implying that the original equation has periodic solutions which can be approximated by time harmonic functions of constant amplitude $\sqrt{-\frac{1}{2}A_o/\frac{3}{8}A_5}$ and arbitrary phase.

If $A = O(1)$ equation (2.24) may have one, two or three real zeros. As an example the case is considered where $A$ varies and where the other parameters are given the following numerical values: $C_{D_o} = 0.52, C_{L_1} = -6.0, \beta/K = 2.0, \text{ and } C_{L_3} = 2.0$. Then by using Cardano’s formula one finds that for $A = 0.1420$ there are two real zeros. Whereas for $0 < A < 0.1420$ three real zeros exist and for $A > 0.1420$ only one. The phase portraits for these three cases are given in fig 2. including the case $A = 0$. 


It is of interest to consider the case $\Omega \neq 1$, implying that the frequency ratio of the natural frequency of the oscillator and the frequency of the motion of ridge differs (slightly) from 1. By setting $\Omega = 1 + \epsilon \eta$ one obtains after averaging:

$$
\dot{y}_1 = K\left[-\frac{1}{2}A_o + \frac{1}{4}A_1\right]\tilde{y}_1 + \left(\frac{1}{8}A_3 - \frac{1}{4}A_4\right)\tilde{y}_1 \tilde{y}_2 - \frac{3}{8}A_5\tilde{y}_1 \tilde{y}_2^2 + \left(\frac{1}{24}A_1 - \frac{3}{8}A_5\right)\tilde{y}_1^2 + \epsilon \eta \tilde{y}_2
$$

$$
\dot{y}_2 = -K\left[\frac{1}{2}A_2 + \frac{1}{4}A_1\right]\tilde{y}_2 + \left(\frac{1}{8}A_4 - \frac{1}{16}A_3\right)\tilde{y}_2^2 + \frac{3}{8}A_5\tilde{y}_1^2 \tilde{y}_2 + \left(\frac{3}{8}A_4 + \frac{1}{16}A_3\right)\tilde{y}_1^3 + \frac{3}{8}A_5\tilde{y}_1^2 \tilde{y}_2 + \frac{3}{8}A_5 + \frac{1}{24}A_1\right] - \epsilon \eta \tilde{y}_1
$$

For $A \equiv 0$ and by using the transformation

$$
\tilde{y}_1 = r \cos \theta \quad (2.28)
$$

$$
\tilde{y}_2 = r \sin \theta
$$

system (2.27) becomes:

$$
\dot{r} = -K\left[\frac{1}{2}A_o r + \frac{3}{8}A_5 r^3\right] \quad (2.29)
$$

$$
\dot{\theta} = \epsilon \eta
$$

It is clear that by using (2.22) and (2.27) a periodic solution from (2.28) is found:

$$
\bar{z} = r \cos(\tau - \theta)
$$
Let now additionally $A = 0.1$ and consider $\eta$ as a parameter. By using a Groebner basis algorithm one can show that for $\eta = 0.194$ there are two real critical points of (2.27), and for $0 < \eta < 0.194$ three real critical points exists and for $\eta > 0.194$ only one. In fig 3. a sketch is given of this result that is how the number of critical points depend on $\eta$. In fig 4. the phase portraits are given for $\eta = 0.1$ and $\eta = 0.2$. For $\eta = 0.1$ there is a stable critical point corresponding with a limit cycle. Surprisingly for $\eta = 0.2$ there exists a periodic orbit correspond with slowly varying amplitudes i.e. modulated oscillations.

### 3 CONCLUDING REMARKS

In this paper a model equation for the study of rain-wind induced vibrations of a simple oscillator has been introduced. Both for the linear model equation as for the non-linear one periodic solution are presented. For the linear equation the excitation mechanism may be ordinary forcing as well as combined parametrical and ordinary forcing. In both cases intermittent periodic self-excitation may additionally be present. For the non-linear
model equation all three excitation mechanisms play a part in the excitation of a periodic motion as can be derived from equation (2.24) where all coefficients $A_i, i = 0, 1, 2, 3, 4, 5$ are present. In case that $\Omega \neq 1$ the averaged system may have a periodic orbit corresponding to modulated oscillations of the original equation.

References


