Mathematical model for the bell SWI: project Old Church Delft

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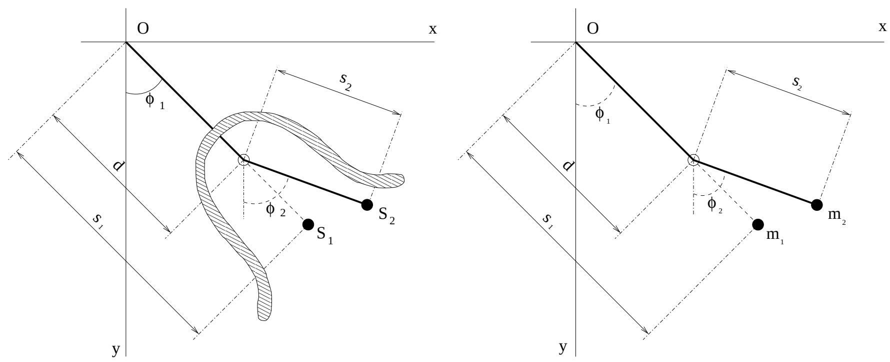
1 Introduction

This article describes a mathematical model for the swing of the Bourdon bell in the tower of the Old Church in Delft. Although a bell with clapper (or bob) is formally a coupled system of 2 pendulums, we will use a single physical pendulum approximation for the computation of the pendulum period because of the relatively small mass of the clapper compared with the bell itself.

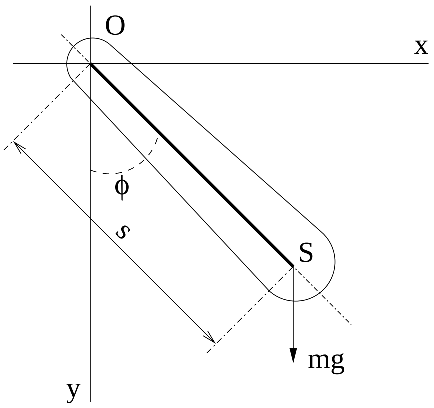
Nevertheless, as the amplitude is relatively high (about $\pm$ 70 degrees) we do not use a linear approximation such as the harmonic oscillator but use elliptic integrals to describe this pendulum in a more accurate way.

Figure (1) shows the double pendulum model and notations while figure (2) shows the single pendulum approximation.



(a) Bourdon bell with bob (b) Generalized double pendulum

Figure 1: Bell and bob parameters



I_0 = moment of inertia around O

m = mass of pendulum

S = center of gravity

Figure 2: Single pendulum approximation (schematic)

2 Analysis for a non-linear singular pendulum

Using Newton's second law for rotational motion on this pendulum yields as usual for the balance of moments :

$$I_0\ddot{\varphi} = -m\ g\ s \sin(\varphi) \tag{1}$$

Here I_0 denotes the moment of inertia, ϕ the amplitude, m the mass, s the distance to the centre of gravity and g the gravity constant (see also fig. 2).

Substituting the commonly used "radius of gyration" i :

$$I_0 = m\ i^2 \tag{2}$$

yields :

$$\ddot{\varphi} + \frac{g\ s}{i^2} \sin(\varphi) = 0 \tag{3}$$

With the angular speed ω defined by :

$$\omega = \frac{\sqrt{g\ s}}{i} \tag{4}$$

we obtain the standard form for the non-linear pendulum :

$$\ddot{\varphi} + \omega^2 \sin(\varphi) = 0 \tag{5}$$

We take as initial conditions (at $t = 0$) :

$$\varphi(0) = \varphi_{max} = \alpha \text{ and } \dot{\varphi}(0) = 0$$

If we multiply (5) with $\dot{\varphi}$ and integrate we find :

$$\frac{1}{2}\int \frac{d}{dt}(\dot{\varphi})^2 = \omega^2 \int \frac{d}{dt}(\cos(\varphi + C))$$

From the initial conditions we can replace the value for C :

$$\dot{\varphi}^2 = 2\omega^2\{\cos(\varphi) - \cos(\alpha)\} \tag{6}$$

Using $1-\cos(\varphi) = 2\sin^2(\frac{\varphi}{2})$ and $1-\cos(\alpha) = 2\sin^2(\frac{\alpha}{2})$ and introducing the parameter k according to :

$$k = \sin(\frac{\alpha}{2}) \ , \ 0 \le k \le 1 \tag{7}$$

we can rewrite (6) into :

$$\dot{\varphi}^2 = 4k^2\omega^2(1 - \frac{\sin^2(\varphi/2)}{k^2}) \tag{8}$$

Introduction of a new variable $y = \frac{\sin(\varphi/2)}{k}$ transforms the problem into equation (10) with the help of (9) :

$$\dot{\varphi} = \frac{\delta\varphi}{\delta y}\ \dot{y} = \frac{2k}{\sqrt{(1-k^2y^2)}}\ \dot{y} \quad \text{or}$$

$$\dot{\varphi}^2 = \frac{4k^2}{1-k^2y^2}\ \dot{y}^2 \tag{9}$$

$$\dot{y} = \omega\sqrt{(1-y^2)(1-k^2y^2)} \tag{10}$$

Finally this yields as solution (11) :

$$\omega\ t + C = \int_0^y \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}} \tag{11}$$

3 Elliptic integrals

The integral in the righthand side of (11) is a so-called elliptic integral of the first kind in the Legendre normal form and it's solution is available in a tabulated form ([2]).
It has an inverse $sn(x)$ on $0 \le y \le 1$, which is an elliptic function of Jacobi and where $sn(x)$ is a periodic function with a $4K$ period.
$K(k)$ is defined as follows :

$$K(k) = \int_0^1 \frac{d\eta}{\sqrt{(1-\eta^2)(1-k^2\eta^2)}} \tag{12}$$

So, on the interval $0 \le y \le 1$ $sn(K(k)) = k$ and $sn(K) = 1$. From (11) it follows that the solution can be written as $y = sn(\omega t + C)$. Also $sn(0) = 0$ and with $k \to 0$ we obtain $sn(x) \to \sin(x)$.
Now putting all pieces together we find :

$$y = \frac{\sin(\phi/2)}{k} = sn(\omega t + C) \tag{13}$$

With the initial conditions from above at $t = 0$ ($\phi(0) = \alpha$), we obtain

$$\sin\frac{\alpha}{2} = k\ sn(C),\ \text{or:}\ sn(C) = 1,\ \text{and}\ C = K$$

Finally, we find the solution

$$\begin{aligned} \sin\frac{\phi}{2} &= ksn(\omega\ t + K) \\ \text{and:}\ \phi(t) &= 2\arcsin\{sn(\omega\ t + K)\} \end{aligned} \tag{14}$$

We need the relation between $K(k)$ and k on the interval $< 0, 1 >$ before we are able to see how frequency changes with the bell amplitude. We can search a few points and present the relation between k and $K(k)$ roughly in a figure :
From (12) we find for $k = 0$:

$$K(0) = \int_0^1 \frac{d\eta}{\sqrt{1-\eta^2}} = \arcsin 1 = \pi/2$$

Also, because the integral diverges for $k \uparrow 1$ we have $\lim_{k\to 1}\ K(k) = \infty$.
For $\alpha = 90^o$ or $\pi/2$ (remember α denotes ϕ_{max}) $k = \sin\alpha/2 \approx 0.707$. We can find $K(0.707)$ numerically or by looking it up in a table such as in [2]. Either way we obtain $K(0.707) \approx 1.854$.
These values were used to sketch fig. (3).

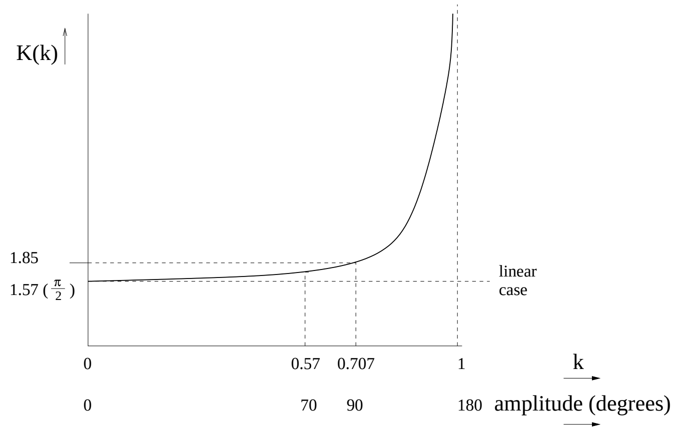


Figure 3: Relation between k and $K(k)$

For a more accurate approximation we have to do some extra work :

From (14) it follows, since the period of sn is $4K$, that the period of the pendulum equals :

$$T = \frac{4K}{\omega}$$

The influence of the nonlinearity can be estimated from :

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}}$$

obtained by substituting $\eta = \sin\theta$ in the integral from (12).

An expansion of $(1 - k^2 \sin^2\theta)^{-1/2}$ in the binomial series gives :

$$(1 - k^2 \sin^2\theta)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} (-k^2)^n \sin^{2n}\theta$$

Integrating term by term and using :

$$\int_0^{\pi/2} \sin^{2n}(\theta) d\theta = 2^{-2n} \binom{2n}{n} \frac{\pi}{2}$$

it follows that :

$$K(k) = \frac{\pi}{2} \sum_{n=0}^{\infty} \binom{-1/2}{n} \binom{2n}{n} (-k/2)^{2n} = \frac{\pi}{2}\{1 + \frac{k^2}{4} + \frac{9k^4}{64} + \ldots\} \qquad (15)$$

So for the period T we find :

$$T = \frac{4K}{\omega} = \frac{2\pi}{\omega}\{1 + \frac{k^2}{4} + \frac{9k^4}{64} + \ldots\} \qquad (16)$$

where the first term $2\pi/\omega$ represents the period of the linearized system (the harmonic oscillator !).

For $k = 0.707$ (or the amplitude $\alpha = \frac{\pi}{2}$), it follows by using the first 2 terms :

$$k^2/4 + 9k^4/64 \approx 0.16$$

So for an amplitude of $\frac{\pi}{2}$ the nonlinear terms raise T with 16% as compared to the linearized case.

$K(0.707) \approx \frac{\pi}{2}$ $(1 + 0.16) \approx 1.83$ whereas 1.854 is the value from ([2] we already used above.

For k close to 1 (amplitude $\alpha = \pi$) the series converges only slowly and there we need more and more terms to find a reasonable accurate approximation, but for the Bourdon bell this is not necessary.

Bell :		
Height	1820 mm	Royal Eijsbouts
Mass (without crown)	7700 kg	Royal Eijsbouts
Centre of Gravity (from bottom bell)	728 mm	Royal Eijsbouts
Moment of inertia in CG (S)	5110 $kg.m^2$	Royal Eijsbouts
Rotation axis (from bottom bell)	1460 mm	TNO Delft
Crown :		
Mass	550 kg	Royal Eijsbouts
Distance from rotation axis	300 mm	estimated
Counterweight :		
Mass	1000 kg	estimated
Distance from rotation axis	1100 mm	TNO Delft / estimated

Table 1: Some properties of the Bourdon bell

4 Computing the bell period

To be able to use the formulas above we need detailed values for many properties of the Bourdon bell. Unfortunately some of the values below are at this moment only rough estimations.
The Royal Eijsbouts Company and TNO Delft (both in the Netherlands) provided us with some numbers. Other were estimated by ourselves using photographs. The most important ones are listed in table (1).

If we assume that the maximum amplitude of the Bourdon bell $\alpha = 70^o (= 7/18\pi)$ we have $k = \sin(\frac{\alpha}{2}) = 0.57$. Using the formula from (15) or the graph from figure (3) we find :

$$K(0.57) \approx \frac{\pi}{2}\{1 + \frac{0.57^2}{2} + 9\frac{0.57^4}{64}\} = 1.73$$

The Bourdon bell has a counterweight on the opposite side of the rotation axis O (see also figure (4)). This implies that the new center of gravity S_{tot} for the total combined system must be shifted towards O.
This new center can be easily computed :

$$\begin{aligned} M_{counter}.(S_{counter} + s_{tot}) &= M_{bell}.(S_{bell} - s_{tot}) \\ 1000.(1100 + s_{tot}) &= 7700.(732 - s_{tot}) \\ s_{tot} &= 4536400/8700 = 521\ mm\ (0.52\ m) \end{aligned}$$

M_{tot} bell + crown + counterbalance : 9250 kg
I_{bell} around O (using Steiners rule) : $5110 + 7700.(732)^2 = 9235\ kg.m^2$

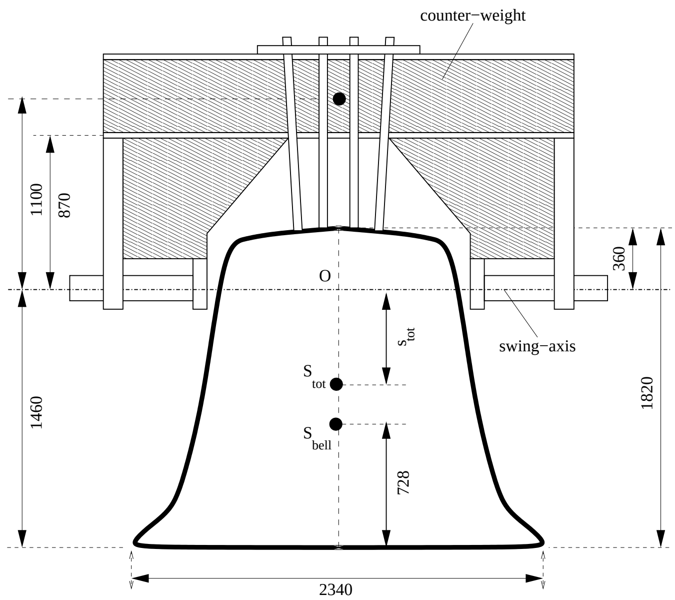


Figure 4: Bourdon bell dimensions

Total moment of inertia :

$$\begin{aligned} I_0 &= I_{bell} + I_{counter} + I_{crown} \\ &= 9235 + 1000.(1.1)^2 + 550.(0.03)^2 \\ &\approx 10500\ kg.m^2 \end{aligned}$$

Now we can gather all other numbers and compute the period :

$$\begin{aligned} i &= \sqrt{\frac{I_0}{M}} = \sqrt{\frac{10500}{9250}} \approx 1.07\ m \\ \omega &= \frac{\sqrt{g\ s_{tot}}}{i} = \frac{\sqrt{9.81.0.52}}{1.07} \approx 2.11\ rad/sec. \\ T &= \frac{4\ K}{\omega} = \frac{4.1.73}{2.11} \approx \mathbf{3.28\ sec} \end{aligned}$$

As there are about 2 chimes per period (the clapper hits the bell twice per period) this implies $n = \frac{2}{T}.60 = \frac{2\ .\ 60}{3.28} \approx 37$ chimes/min.

Linear approximation : If we would have used the linear approach (simple harmonic oscillator) the result would be $T_{lin} = \frac{2\Pi}{\omega} = 2.98$ and 40 chimes/min.
So the difference is $\frac{K(0)}{K(\alpha)} = \frac{1.57}{1.73} = 0.91$ or about 9 %.

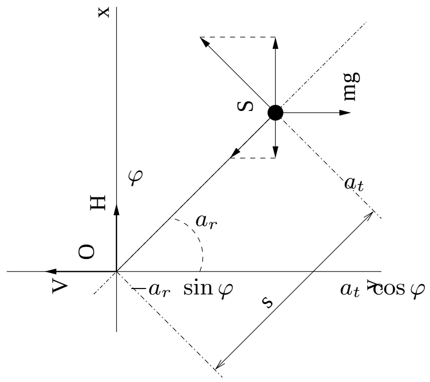


Figure 5: Contribution of accelerations to force (here only shown for H)

5 The forces on the axis of rotation

The acceleration of the centre of gravity S consists of 2 components (see also figure 5) :

$$\begin{aligned}\text{in radial direction} &: \quad a_r = s\dot{\varphi}^2 \\ \text{in tangential direction} &: \quad a_t = s\ddot{\varphi}\end{aligned}$$

So using the contributions of both in the x-direction we find :

$$\begin{aligned}\ddot{x}_s &= -a_r \sin\varphi + a_t \cos\varphi \\ &= -s\ \dot{\varphi}^2 \sin\varphi + s\ddot{\varphi}\cos\varphi \\ &= -s\ \omega^2 \sin\varphi\{4(k^2 - \sin^2(\frac{\varphi}{2})) + \cos\varphi\}\end{aligned}$$

The last line was obtained by using equations (5) and (8)).
With $\sum \vec{F_x} = m\vec{\ddot{x}_s}$ the horizontal force H on the pendulum in the reference point O (to the rightside) is given by :

$$H = -m\ s\ \omega^2\{4(k^2 - \sin^2(\frac{\varphi}{2}))\sin\varphi + \sin\varphi cos\varphi\} \tag{17}$$

and with $\sum \vec{F_y} = m\vec{\ddot{y}_s}$ the vertical force V in the reference point 0 (in upwards direction) is given in a similar way by :

$$\ddot{y}_s = a_r\cos\varphi + a_t \sin\varphi = s\ \omega^2\{4(k^2 - \sin^2(\frac{\varphi}{2}))\cos\varphi - \sin^2\varphi\}$$

This yields by using the same substitutions as for the horizontal force :

$$V = m\ s\ \omega^2\{4(k^2 - \sin^2(\frac{\varphi}{2}))\cos\varphi - \sin^2\varphi\} + m\ g \tag{18}$$

Here of course φ is defined by equation (5).

By taking the derivative for H and V and see where these are equal to 0 we obtain the maximum amplitudes for H and V. Doing this in Maple yields :

$$\begin{aligned} V_{max} \text{ occurs for } \quad \phi = & \quad 0 \, rad. \\ H_{max} \text{ occurs for } \quad \phi \approx & \quad 0.7 \, rad. \end{aligned}$$

Using the numerical values we found in the previous section this yields for the forces :

$$\begin{aligned} H &\approx 23500 \, N \\ V &\approx 115000 \, N \text{ (or 30300 N without the } m \, g \text{ component)} \end{aligned}$$

Note that the signs of the forces are relative to the directions as shown in figure (5).

6 Conclusion

Given the fact that for several parameters we only have a very rough estimation (especially for the counterweight), the error caused by a linear approximation of the system is after all not such a big issue and may be even less than the error caused by incorrect parameters ...

The bell is only tolled at very special occasions of which the most important one is a funeral of a member of the Royal Dutch family, which is of course not a very common occasion.
However, just one day after the workshop ended the former Queen of the Netherlands "Juliana" died unexpectedly and so the bell was tolled at her funeral only a couple of days later.
We were able to obtain a short video recording from the swinging bell at the funeral, so we could measure the period straight from this video. This turned out to be **3.2 sec** or about 37 chimes/min, so notwithstanding the estimated and therefore possibly incorrect parameters the results from above resemble reality quite well !

References

[1] Ferdinand Verhulst, *Nonlinear Differential Equations and Dynamical Systems*, (2th ed.), (Berlin: Springer Verlag, 1996).

[2] Jahnke-Ende, *Tafeln Höhern Funktionen*, (5th ed.), (Leipzig: B.G. Teubner, 1960).

[3] Maria L. Beconcini, Stefano Bennati, Walter Salvatore, *Structural characterisation of a medieval bell tower: First historical, experimental and numerical investigations*, (University of Pisa, Dept. of structural engineering, Pisa, Italy, 2001).