First order Partial Differential Equations

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1995, 2001, 2002, English version 2010 (KL), v2.1

1 First order wave equation

The equation

$$au_x + u_t = 0 \, , \; u = u(x,t) \, , \; a \, \epsilon \, \mathbb{R} \tag{1.1}$$

describes the motion of a wave in one direction while the shape of the wave remains the same. We'll see that the constant a indicates the speed of the traveling wave. Equation (1.1) is also commonly known as the *transport equation*.
We solve this type of PDE by looking for curves in the xt-plane where (1.1) can be reduced to an ordinary differential equation.
Suppose that C is a parametrized curve $(x(s), t(s))$ in the xt-plane. For the curve C the following applies :

$$u(x,t) = u(x(s), t(s)) \; .$$

Differentiation with respect to s on C yields :

$$\frac{du}{ds} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial t}\frac{dt}{ds} \; . \tag{1.2}$$

Comparing (1.2) with (1.1) we see that when

$$\frac{dx}{ds} = a \quad \text{and} \quad \frac{dt}{ds} = 1 \tag{1.3}$$

the following should hold as well

$$\frac{du}{ds} = 0 \quad \text{on} \quad C \; . \tag{1.4}$$

A curve on which (1.4) is valid is called a *characteristic curve* or simply a *characteristic* of equation (1.1), in this example define by (1.3). From (1.4) it follows that the solution u for (1.1) should be constant on a characteristic.
Using (1.3) we find as solutions :

$$x(s) = as + x_0 \ , \ \ t(s) = s + t_0 \ ,$$

from which after eliminating s we could derive

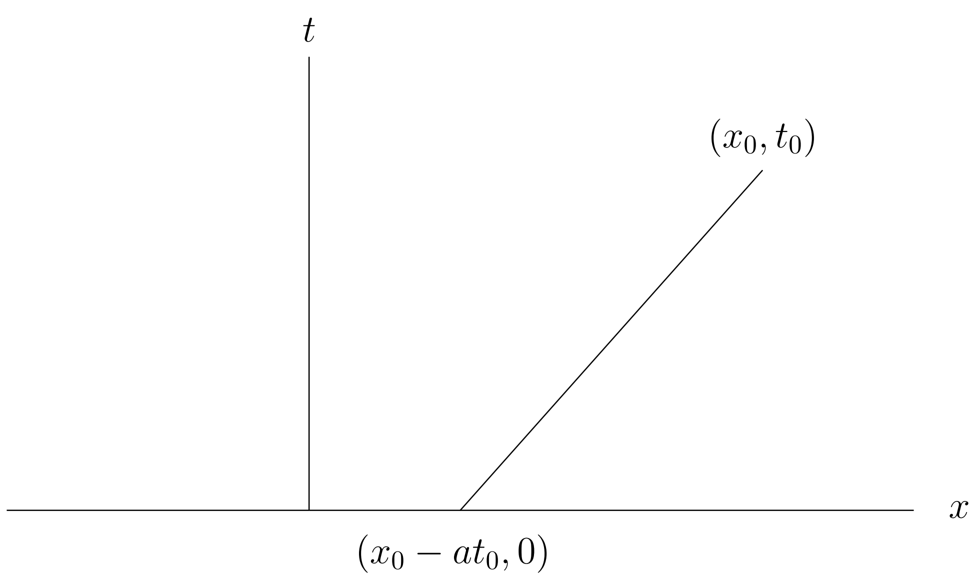
$$x = at + \text{constant} \ .$$

The characteristic curves C are in this case straight lines in the xt-plane.
The solution $u(x,t)$ only changes if we move to a different characteristic and so it only depends on the value for $x - at$. We write this with an arbitrary differentiable function F :

$$u(x,t) = F(x - at) \ . \tag{1.5}$$

This is often referred to as the *general solution* for (1.1), because every solution should be of this form. So, for a unique solution it is necessary to have more information.
A common condition is to prescribe that for a certain value of t (often $t = 0$), $u(x,t)$ must be equal to a continuously differentiable function : an initial condition. When we add such an initial condition to (1.1) we call it a *Cauchy problem* :

$$\begin{cases} au_x + u_t & = 0 \ , \quad x \in \mathbb{R} \ , \quad t > 0 \ , \\ u(x,0) & = f(x) \ , \quad x \in \mathbb{R} \ . \end{cases} \tag{1.6}$$



The value of u in (x_0, t_0) is determined by the characteristic on which this point is located. Because the value on the characteristic is a constant, it should be everywhere the same as the value for $t = 0$. The characteristic through (x_0, t_0) intersects the x-axis in $x_0 - at_0$, so

$$u(x_0, t_0) = u(x_0 - at_0, 0) = f(x_0 - at_0) \ . \tag{1.7}$$

Example

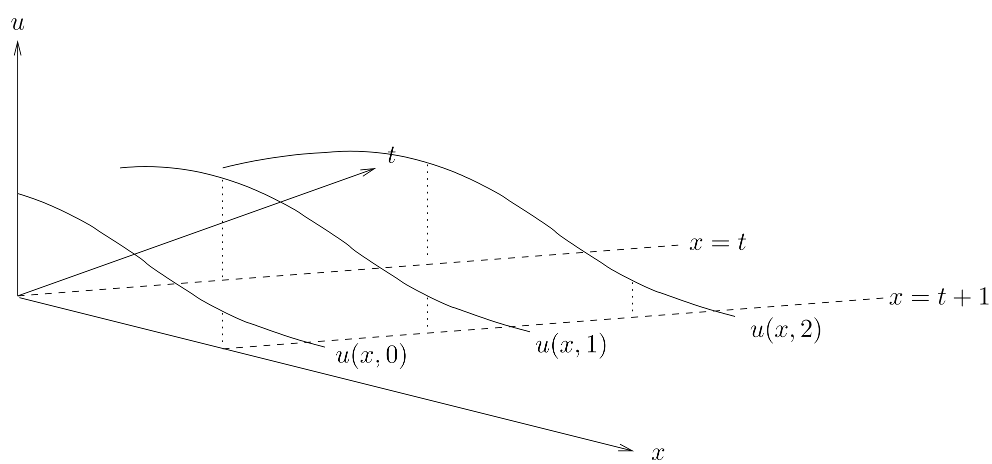
Consider the following initial value problem :

$$\begin{cases} u_x + u_t &= 0 \ , \qquad\qquad x \in \mathbb{R} \ , \quad t > 0 \ , \\ u(x,0) &= \dfrac{1}{1+x^2} \ , \quad x \in \mathbb{R} \ , \end{cases}$$

Here $a = 1$ and $f(x) = \dfrac{1}{1+x^2}$. The solution is given by

$$u(x,t) = \frac{1}{1+(x-t)^2} \ .$$

The solution conserves the shape of the initial curve and simply moves along the characteristics , the lines $x - t =$ constant.



The condition for which u from (1.7) is a solution of (1.6) is that f should be continuously differentiable. Only then u_x and u_t are continuous. This type of solution is often referred to as a *classical* or *strong solution.*

In cases where f is only piecewise continuously differentiable, as for example

$$f(x) = \begin{cases} 1+x \ , & -1 < x < 0 \ , \\ 1-x \ , & 0 \leq x < 1 \ , \\ \quad 0 \ , & \text{all other values for x,} \end{cases}$$

or even where f is only piecewise continuous, it is still possible to describe the solution u in terms of (1.7). We won't explain here in what sense u is still a solution but only mention that in these cases u is referred to as a *weak* or *generalized solution.*

Example

The initial value problem :

$$\begin{cases} -2u_x + u_t & = 0 \ , \quad x \in \mathbb{R} \ , \ t > 0 \ , \\ u(x,0) & = \begin{cases} 0 \ , & |x| > 1 \ , \\ 1 - |x| \ , & |x| \leq 1 \ , \end{cases} \end{cases}$$

has the generalized solution :

$$u(x,t) = \begin{cases} 0 \ , & x > 1 - t \ , \\ 1 - |x + 2t| \ , & -1 - t < x < 1 - t \ , \\ 0 \ , & x < -1 - t \ . \end{cases}$$

Exercises

1. Compute the solution $u(x,t)$ of the initial value problem

$$\begin{cases} 2u_x + u_t & = 0 \ , \quad x \in \mathbb{R} \ , \quad t > 0 \ , \\ u(x,0) & = f(x) \ , \quad x \in \mathbb{R} \ , \end{cases}$$

for the functions f from below. Also indicate if we have a strong or weak solution here.

(a) $f(x) = e^{-x^2} \ , \ x \in \mathbb{R}$.

(b) $f(x) = e^{|x|} \ , \ x \in \mathbb{R}$.

(c) $f(x) = \begin{cases} 0 , & |x| > 1 \ , \\ 1 , & |x| \leq 1 \ . \end{cases}$

2. Consider the initial value problem :

$$\begin{cases} 3u_x + u_t & = 5 \, , \quad x \in \mathbb{R} \, , \quad t > 0 \ , \\ u(x,0) & = e^x \, , \quad x \in \mathbb{R} \, . \end{cases}$$

Use the transformation $u(x,t) = v(x,t) + 5t$ and compute the solution $v(x,t)$. Also compute $u(x,t)$.

3. Also consider the initial value problem :

$$\begin{cases} 4u_x + u_t & = u \, , \quad x \in \mathbb{R} \, , \quad t > 0 \\ u(x,0) & = \cos x \, , \quad x \in \mathbb{R} \, . \end{cases}$$

Use the transformation $u(x,t) = e^t \, v(x,t)$ and compute the solution $v(x,t)$. Compute again $u(x,t)$.

2 Well-posed problems

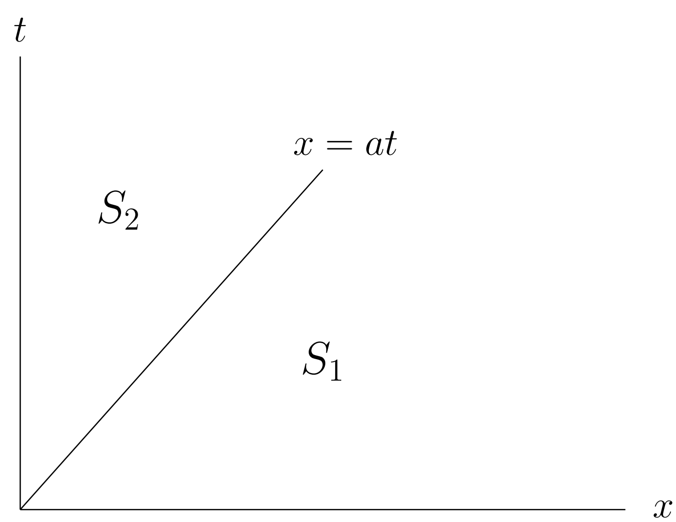
We have seen in §1.1 that the partial differential equation $au_x + u_t = 0$ has infinitely many solution. By applying an extra restriction - in this case an initial condition we were able to find a unique solution.
We call a problem *well-posed* if the following conditions apply :

1. there is a solution for the problem (*existence*),
2. there is exactly one single solution for the problem (*uniqueness*),
3. the solution depends continuously from the initial or boundary conditions (*stability*).

If we make a model of a real-life problem from physics using a partial differential equation we always try to formulate a well-posed problem by using as many additional conditions as required. With too little conditions the problem can be non-unique, and with too many conditions it may be that there is no valid solution at all anymore.
The stability condition for instance implies that a small error in the initial conditions may not induce large perturbations in the solution. We'll discuss this condition more extensively in the next chapters.

As an example we'll show now what well-posedness implies for the transport equation. Let's study the equation on the interval $0 < x < \infty$ for $a > 0$:

$$\begin{cases} au_x + u_t &= 0\,, \qquad 0 < x < \infty\,, \quad t > 0 \\ u(x,0) &= f(x)\,, \quad 0 < x < \infty\,, \end{cases} \tag{2.8}$$



Then the solution $u(x,t) = f(x - at)$ is only defined for $S_1 = \{(x,t) \mid x > at\}$. For $S_2 = \{(x,t) \mid 0 < x < at\}$ the solution is unknown as there is no initial condition on characteristics with $x - at < 0$.

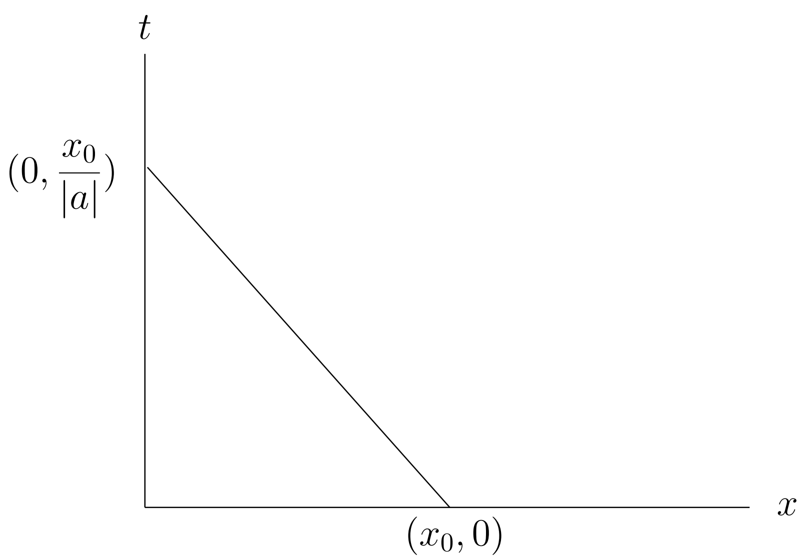
By also specifying the *boundary condition* at $x = 0$ we can arrive at a well-posed *initial-boundary value problem* :

$$\begin{cases} au_x + u_t &= 0 \ , \qquad 0 < x < \infty \ , \quad t > 0 \ , \\ u(x,0) &= f(x) \ , \quad 0 < x < \infty \ , \\ u(0,t) &= g(t) \ , \qquad t > 0 \ . \end{cases} \tag{2.9}$$

In S_1 the solution is determined by the initial condition, in S_2 by the boundary condition :

$$u(x,t) = \begin{cases} f(x-at), & x > at \ , \\ g(t - \dfrac{x}{a}), & 0 < x < at \ . \end{cases} \tag{2.10}$$

For $a < 0$, the problem from (2.9) is ill-posed. For certain values $x_0 > 0$ the points $(x_0, 0)$ and $(0, \frac{x_0}{|a|})$ may be on the same characteristic.



The initial value results in $u(x_0, 0) = f(x_0)$, and the boundary value results in $u(0, \frac{x_0}{|a|}) = g(\frac{x_0}{|a|})$. But because u must be always constant on an characteristic the following should apply :

$$f(x_0) = g(\frac{x_0}{|a|}) \text{ for all } x_0 > 0 \ .$$

For arbitrary functions f en g in the problem (2.9) this won't be generally true.

Exercises

1. Compute the solution of the following initial-boundary value problem. Make a rough sketch of the solution at times $t = 0, 1$ and 2.

$$\begin{cases} u_x + u_t &= 0\ , \quad 0 < x < \infty\ , \quad t > 0\ , \\ u(x,0) &= e^{-x^2}\ , \quad 0 < x < \infty\ , \\ u(0,t) &= 1\ , \quad t > 0\ . \end{cases}$$

2. Compute the solution of the following initial-boundary value problem. Make a rough sketch of the solution at times $t = 0, 1$ and 2.

$$\begin{cases} 3u_x + u_t &= 0\ ,\ 0 < x < \infty\ ,\ t > 0\ , \\ u(x,0) &= 0\ ,\ 0 < x < \infty\ , \\ u(0,t) &= \begin{cases} t\ , & 0 < t < 1\ , \\ 1\ , & t \geq 1\ . \end{cases} \end{cases}$$

3. What restrictions should be imposed to f and g for the problem (2.9) to ensure the solution (2.10) is continuously differentiable ?

3 Linear first order equations

If we can write a first order partial differential equation as

$$a(u,x,t)u_x + b(u,x,t)u_t = c(u,x,t) \tag{3.11}$$

we call it *quasi-linear*.
A special subclass are the *linear* equations that can be written as :

$$a(x,t)u_x + b(x,t)u_t = c(x,t)u + d(x,t)\,. \tag{3.12}$$

And if $d \equiv 0$ we call the equation *homogeneous*.

We will now for the general linear initial value problem below :

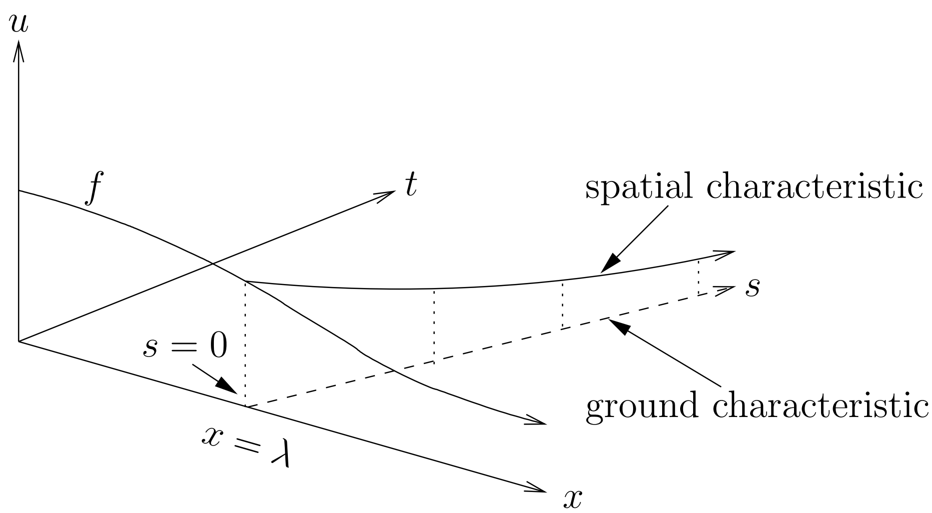
$$\begin{cases} a(x,t)u_x + b(x,t)u_t = c(x,t)u + d(x,t)\,, & x \in \mathbb{R}\,, \quad t > 0\ , \\ u(x,0) = f(x)\,, & x \in \mathbb{R} \end{cases} \tag{3.13}$$

construct a solution using the *method of the characteristics* in the same way as for the problem in (1.6).

We have already seen that the solutions for equation (1.1) can be given using characteristics, the lines for which $x - at =$ constant . A natural parametrization of the characteristics is then

$$x = as + \lambda\ ,\ t = s\ .$$

The parametrization is chosen in such a way that s indicates the position on the characteristic, while $s = 0$ corresponds with the intersection $(\lambda, 0)$ of the characteristic with the x-axis. It is the starting point of the characteristic at the initial value , where $t = s = 0$.



This immediately yields us a parametrization of the initial curve, as for $s = 0$ the following should be valid :

$$\begin{cases} x(0) = & \lambda \ , \\ t(0) = & 0 \ , \\ u(0) = & f(\lambda) \ . \end{cases} \quad , \ \lambda \in \mathbb{R} \ , \tag{3.14}$$

If we write a generic parametrization of the characteristic C_λ

$$x = x(s) \ , \ t = t(s) \text{ op } C_\lambda \ ,$$

then C_λ is in accordance with (3.13) if we make sure that x,t and u are solutions of the following *characteristic system*

$$\frac{dx}{ds} = \quad a(x,t) \ , \ x(0) = x_0(\lambda) \ , \tag{3.15}$$

$$\frac{dt}{ds} = \quad b(x,t) \ , \ t(0) = t_0(\lambda) \ , \tag{3.16}$$

$$\frac{du}{ds} = \quad c(x,t)u + d(x,t) \ , \ u(0) = u_0(\lambda) \ . \tag{3.17}$$

Because the functions a en b do not depend on u we can solve the equations (3.15) and (3.16) independent from (3.17). Solutions of these 2 equations define characteristics in the xt-plane and are often referred to as *ground characteristics*.
Solutions for all 3 equations (3.15)-(3.17) define so-called *spatial characteristics*.
For every value of λ this system has a solution that represents a single spatial characteristic. All these characteristics together describe the solution surface $u(x,t)$ in the xtu-space, albeit only through the parameters s en λ:

$$x = x(s,\lambda) \ , \ t = t(s,\lambda) \ , \ u = u(x(s,\lambda), t(s,\lambda)) \ . \tag{3.18}$$

Example

The initial value problem :

$$\begin{cases} xu_x + u_t &= -u\,, \qquad x \in \mathbb{R}\,, \quad t > 0 \\ u(x,0) &= \sin(x)\ , \quad x \in \mathbb{R}\,, \end{cases}$$

has the following characteristic system :

$$\begin{aligned} \frac{dx}{ds} &= \quad x\ , \quad x(0) = \lambda\ , \\ \frac{dt}{ds} &= \quad 1\ , \quad t(0) = 0\ , \\ \frac{du}{ds} &= \ -u\ , \quad u(0) = \sin\lambda\ . \end{aligned}$$

And the parametrization of the solution is :

$$x = \lambda e^s\,,\ t = s\,,\ u = e^{-s}\sin\lambda\ .$$

It should be obvious that although $u(x,t)$ and $u(s,\lambda)$ describe the same solutions they are not the same in terms of their parameters : $u(x,t) \neq u(s,\lambda)$!
In general :

$$u(x,t) = u(x(s,\lambda), t(s,\lambda)) = U(s,\lambda)$$

According to the implicit function theorem the inverse functions

$$s = s(x,t)\,,\ \lambda = \lambda(x,t)$$

only exist in the surrounding of $s = 0$ if the *Jacobian* is not equal to zero :

$$J(x,t) = \left|\frac{\partial(x,t)}{\partial(s,\lambda)}\right| = \begin{vmatrix} x_s & x_\lambda \\ t_s & t_\lambda \end{vmatrix} = x_s t_\lambda - x_\lambda t_s \neq 0 \tag{3.19}$$

This condition prescribes that the initial curve may **never** have the same direction as the direction of any of the characteristics. In other words : if the initial value is defined on a arbitrary differentiable curve in the xt-plane , equation (3.19) implies that this initial curve hasn't got the characteristic direction in *any* point.

Example

For the parametrization from the previous example the Jacobian looks as follows
:

$$J(x,t) = \begin{vmatrix} \lambda e^s & e^s \\ 1 & 0 \end{vmatrix} = e^s \neq 0 \text{ for all } s \ .$$

Inverting $x = \lambda e^s \ , \ t = s$ yields :

$$s = t \ , \ \lambda = xe^{-t}$$

,

which results in the solution u being given by
$u = u(x,t) = \sin(xe^{-t}) \ e^{-t}$.

Exercises

1. Determine to what class (linear, quasi-linear, ...) the following equations belong and also tell whether the equation is homogeneous or inhomogeneous :

   (a) $u_x + tu_t = 0$

   (b) $u_x + uu_t = x$

   (c) $u_x + tu_t = u^2$

   (d) $u_x + \sin x \, u_t = x$

   (e) $u_x + u_t^2 = 0$

2. Is it possible to solve the following problem using characteristics ? Give an explanation as well.

$$\begin{cases} u_x + u_y &= 0 \ , \quad x, y \in \mathbb{R} \\ u(x,y) &= 2 \ , \quad \text{for} \quad y = x \end{cases}$$

3. Determine using the method of the characteristics the solution of the following initial value problems. Also indicate on what (sub) area the solution is valid.

   (a) $\begin{cases} u_x + xu_t &= u^2 \ , \quad x > 0 \ , \quad t > 0 \\ u(x,0) &= 1 \ , \quad x > 0 \ . \end{cases}$

   (b) $\begin{cases} tu_x + u_t &= x \ , \quad x \in \mathbb{R} \ , \quad t > 0 \\ u(x,0) &= x^2 \ , \quad x \in \mathbb{R} \ . \end{cases}$

   (c) $\begin{cases} xu_x - tu_t &= 0 \ , \quad x \in \mathbb{R} \ , \quad t > 0 \\ u &= x^2 \ , \quad \text{op } x = t \ . \end{cases}$

4 Quasi-linear equations

The method from the previous section can be extended to quasi-linear partial differential equations. We can also consider completely arbitrary initial curves.
Let's study the following generalized initial value problem :

$$\begin{cases} a(x,t,u)u_x + b(x,t,u)u_t = c(x,t,u)\ , & (x,t)\,\epsilon\, D\ \subset\ \mathbb{R}^2\ , \\ x = x_0(\lambda)\ ,\ t = t_0(\lambda)\ ,\ u = u_0(\lambda)\ , & \lambda\,\epsilon\,\Lambda\ \subset\ \mathbb{R}\ . \end{cases} \tag{4.20}$$

The initial curve B for this quasi-linear equation is a spatial curve (x_0, t_0, u_0), parametrized by λ .
The solution $u = u(x,t)$ is a certain *surface* in the xtu-space. The initial curve B must - as before - be located completely on this surface. The method we'll derive hereafter to determine this surface has a close resemblance to the methods from the sections before.

First we consider the vectorfield $(a(x,t,u), b(x,t,u), c(x,t,u))$. In every point (x,t,u) of the xtu-space this vectorfield defines a direction. Now we search for curves that in every point (x,t,u) take the direction of this vectorfield. Along this curve C_λ we take the parameter s such that we find for C_λ the following parametrization :

$$(x(s), t(s), u(s))\ .$$

We also demanded that this curve has in every point the direction of the vectorfield. Then the following must hold :

$$\begin{aligned} \frac{dx}{ds} &= a(x(s), t(s), u(s)) \\ \frac{dt}{ds} &= b(x(s), t(s), u(s)) \\ \frac{du}{ds} &= c(x(s), t(s), u(s)) \end{aligned} \tag{4.21}$$

All the curves that fit this system of ordinary differential equations are called *characteristics* of equation (4.20).
Now we look at the collection of curves that intersect the initial curve $B(\lambda)$. In order to do so we'll add the following initial conditions to the system of (4.21) :

$$\begin{aligned} x(0) &= x_0(\lambda) \\ t(0) &= t_0(\lambda) \\ u(0) &= u_0(\lambda) \end{aligned} \tag{4.22}$$

The solution of the system (4.21) with the initial conditions (4.22) can be formally written as :

$$x = x(s,\lambda)\ ,\ t = t(s,\lambda)\ ,\ u = u(s,\lambda)\ . \tag{4.23}$$

Example

The initial value problem :

$$\begin{cases} yu_x - xu_y &= u \ , \quad x > 0 \ , \ y > 0 \ , \\ u &= 1 \ , \quad \text{on } xy = 1 \ , \end{cases}$$

has a characteristic system :

$$\begin{aligned} \frac{dx}{ds} &= \quad y \ , \quad x(s=0) = \lambda \ , \\ \frac{dy}{ds} &= -x \ , \quad y(s=0) = \frac{1}{\lambda} \ , \\ \frac{du}{ds} &= \quad u \ , \quad u(s=0) = 1 \ . \end{aligned}$$

The solution for the equation for u is easy to compute : $u = e^s$. The equations for $x(s)$ en $y(s)$ however form a system of coupled ordinary differential equations, with the somewhat more complicated solutions :

$$x(s,\lambda) = \frac{\sin(s) + \lambda^2 \cos(s)}{\lambda} \ , \ y(s,\lambda) = \frac{-\lambda^2 \sin(s) + \cos(s)}{\lambda} \ .$$

An explicit relation $u = u(x,y)$ can be found if we can determine the inverse transformation $s = s(x,y)$, $\lambda = \lambda(x,y)$.

$$u(x,y) = U(s(x,y), \lambda(x,y)) \tag{4.24}$$

Example

The Jacobian for the functions $x(s,\lambda)$ en $y(s,\lambda)$ from the previous example is :

$$J = \begin{vmatrix} \dfrac{\cos(s) - \lambda^2 \sin(s)}{\lambda} & \dfrac{\lambda^2 \cos(s) - \sin(s)}{\lambda^2} \\ \dfrac{-\sin(s) - \lambda^2 \cos(s)}{\lambda} & \dfrac{-\lambda^2 \sin(s) - \cos(s)}{\lambda^2} \end{vmatrix} = \frac{\lambda^4 - 1}{\lambda^3} \ .$$

As $x > 0$ and because of this also $\lambda > 0$, we can only expect problems for $\lambda = 1$.

NOTE: in this example it is not possible to rewrite s and λ as functions of x en y and so we'll be left here with only a parametrized solution.

In this section we (sort of) derived a method for the quasi-linear Cauchy problem from (4.20). The method consists of solving the characteristic system from (4.21) with initial conditions (4.22) and then expressing s and λ as functions of x and t (if possible).
We didn't bother that much here about what the practical use of this method would be, but that doesn't mean that there are no practical applications at all !

Exercises

1. Solve the following quasi-linear initial value problem using the *characteristic method* and indicate where the solution is valid.

$$\begin{cases} x^2 u_x + t^2 u_t = u^2 \ , & x \in \mathbb{R} \ , \ t > 0 \ , \\ u(x, 4x) = 1 \ , & x \in \mathbb{R} \ . \end{cases}$$

2. Solve the following quasi-linear initial value problems using the *characteristic method.*

   (a) $\begin{cases} u_x + u_t = u \ , & x \in \mathbb{R} \ , \ t > 1 \ , \\ u(x, 1) = e^{-x^2} \ , & x \in \mathbb{R} \ . \end{cases}$

   (b) $\begin{cases} x u_x + t u_t = 1 \ , & x \in \mathbb{R} \ , \ t > 1 \\ u(x, 1) = \sin x \ , & x \in \mathbb{R} \ . \end{cases}$

3. Consider the following coordinate-transformation :

$$\begin{cases} \xi = 3x + y \\ \eta = e^x \end{cases}$$

   (a) Determine ξ_x, ξ_y, η_x en η_y. Why is the transformation invertible ?

   (b) Determine x_ξ, x_η, y_ξ en y_η by implicit differentiation of the formulas from above.

   (c) Express x en y in terms of ξ en η (in other words : give the inverse transformation). Check the answer for (b) using this inverse transformation.