One–dimensional long–term equilibria of tidal embayments

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Abstract
A one–dimensional model for a long tidal embayment is developed and analyzed. It is a generalization of earlier work to domains in which vertical tide is spatially nonuniform. The system is driven by a prescribed external $M_2$–tide only, overtides are generated internally. It is demonstrated that the model allows for morphologic equilibria characterized by an equilibrium bed profile that has a prescribed depth at its entrance and a zero water depth at the land–side of the embayment. For short embayments (length much shorter than the tidal wave length), the bed has approximately a constantly sloping profile and the velocity is spatially uniform. If the length is increased the bed becomes concave and the velocity becomes spatially non–uniform. For an embayment lengths larger than 240 km no equilibrium profile can be found anymore.

1 Introduction
The morphology of many tidal inlets and estuaries is characterized by a complex pattern of channels and sandy shoals, both in space and time. This dynamic behaviour is caused by the feedback between the water motion, sediment transport and bottom changes. Observations clearly indicate that morphological behaviour is sensitive to changes in external conditions caused e.g. by sealevel rise and/or human interferences (Van der Spek, 1997). Both for economical and ecological reasons there is a strong need to simulate and predict both the morphological processes and the sensitivity of the morphology on changing conditions.

In this paper the emphasis is on the Western Scheldt, an estuary located in the the south–west of Holland and Belgium (see Fig. 1), which serves as the main navigation lane to the harbour of Antwerp. In the Western Scheldt economical and ecological interests clearly conflict. For shipping purposes it would be advantageous if the channel was kept fixed at its position. On the other hand, the Western Scheldt has important feeding grounds for birds and
other animals for which the system has to be kept as dynamic as possible. To keep the harbour open for large ships, the navigation lane of the Western Scheldt has to be dredged continuously \((8 \cdot 10^6 \text{ m}^3\text{year}^{-1})\) (Verbeek, 1998).

The estuary has a length of approximately 150 kilometers with a marine part of approximately 80 kilometers. The depth near Vlissingen is approximately 25 meters, it reaches a maximum near Terneuzen, 15 kilometers downstream of Vlissingen, of about 50 meters after which the depth decreases again. The water motions are driven at the sea side by a dominant \(M_2\)-tide with amplitude 1.75 m and an \(M_4\)-overtide with amplitude 0.13 m. The phase difference between the \(M_2\) and \(M_4\)-tide is approximately 0 deg at the entrance and increases to 180 deg near Antwerp. Hence the water motions have the characteristics of a travelling wave. The average river discharge is 110 m\(^3\)s\(^{-1}\), which is less than one percent of the tidal water fluxes. Hence the system is well-mixed (Van den Berg et al., 1990).

The sediment consists of fine sand with an average grain size of 0.2 mm and is mainly transported as suspended load. The number of main channels decreases from three at the mouth of the estuary to one at its end. The ebb channels meander on a scale of about 5 km, whereas the flood channels are straight. The secondary channels show a much more complex behaviour. Connecting channels as described in Jeuken (1998) exhibit cyclic behaviour. Furthermore, fractal bottom patterns can be found in parts of the estuary, such as the "Verdronken
In recent years there have been substantial developments regarding the simulation of morphological processes with the use of models based on first principles (Verbeek et al., 1998; De Jong, 1998). However, due to lack of process knowledge, such models are as yet not suitable to simulate the long–term evolution of channel–shoal patterns. This motivates the simultaneous use of idealized models since they capture the essential physics but can still be analyzed using dynamical systems theory. In Schuttelaars (1997) it was shown that with this approach periodic behaviour and the splitting of channels that resembles behaviour as observed in short embayments (i.e. the embayment length is short compared to the tidal wave length), can be found. The important issue is to generalize this method to longer embayments, such as the Western Scheldt, which are considerably more complicated to analyze.

In this paper the analysis will be restricted to a one–dimensional model, since this provides the information necessary to perform a two–dimensional analysis resulting in channel–shoal behaviour. Important generalizations (with respect to Schuttelaars and De Swart (1996)) are the spatial variations of both the vertical and horizontal tide. Furthermore, in a long channel the net transport of suspended sediment will be mainly due to advection, whereas in a short embayment the main transport mechanism is diffusive transport. Main aims are to show that such a system allows for morphodynamic equilibria, characterized by a steady bottom profile. Their character strongly depends on the forcing conditions. In section two the model will be presented, the equations are scaled and an asymptotic expansion in a small parameter is made. In section three the existence of equilibria and an explanation for the observed characteristics of the equilibrium bedforms is given. In the last section some conclusions and outlooks for future work will be made.

2 Model Description

The embayment under study is a rectangular basin with zero water depth at the landward end. Hence it is assumed that there is no river inflow or that the influence of the river inflow can be neglected (as is the case in the Western Scheldt, since the river inflow is only one percent of the tidal water fluxes). The width is much smaller than both the length of the embayment and the Rossby deformation radius, which makes it possible to model the embayment as one–dimensional (no variations in the cross–channel direction).

The water motions are described by the depth–averaged shallow water equations for a homogeneous fluid (Csanady, 1982) with a linearized bottom friction (see Lorentz (1922); Zimmerman (1982, 1992)). At the open boundary the tidal elevation $\zeta$ is prescribed. Note that due to surface elevations the boundary at the end of the embayment is a moving boundary (see Fig. 2, where the length, corresponding to sea level $\zeta_1$ and $\zeta_2$ is denoted by $L_1$ and $L_2$, respectively). At the closed end, the kinematic boundary condition is used. This condition requires the total derivative of the water depth to be zero when the water depth
The sediment in the embayment consists of noncohesive material with only one grain size and is mainly transported as suspended load. This transport is described by a depth-integrated concentration equation (see Van Rijn (1993)) in which only diffusive and advective mechanisms can result in net transport. At the moving boundary the sediment flux (corrected for the inertia of sediment particles) has to be zero. At the entrance, it is imposed that the bed does not change. Since the bed only changes due to tidally averaged erosion and deposition, this condition does not supply us with enough boundary conditions. We therefore require that no diffusive boundary layer develops at the entrance of the embayment. The bottom evolution equation is derived from the continuity of mass and states that the bed level changes due to erosion and settling of sediment.

The equations are made dimensionless using

\[ x^* = Lx, \quad t^* = \tau \sigma^{-1}, \quad u^* = \frac{A\sigma L}{H}u, \]
\[ \zeta^* = A\zeta, \quad h^* = Hh, \quad C^* = \frac{\alpha U^2}{\gamma}C \]  

Using a Taylor expansion around a fixed point in space, the moving boundary conditions are transformed to a fixed space coordinate \( x = 1 \). Here \( L \) is chosen such that \( x = 1 \) when \( \zeta = 0 \), i.e. \( h = 1 \). The radian frequency of the main tidal constituent is denoted by \( \sigma \) and \( A \) is the amplitude of the vertical tide at the entrance. Furthermore, \( \alpha \) is the erosion coefficient, related to sediment properties (see Dyer and Soulsby (1988); Van Rijn (1993)). The constant \( \gamma \) can be related to the settling velocity and a diffusion coefficient \( \kappa_v \) that describes the mixing of the sediment in the vertical. Assuming a constant \( \kappa_v \) it follows from Van Rijn (1993) that \( \gamma = \frac{w_s^2}{\kappa_v} \). Suppressing the asterix, the equations...
\[ \zeta_t + [(1 - h + \epsilon \zeta)u]_x = 0 \quad (2a) \]
\[ u_t + \epsilon uu_x + \lambda_L^2 \zeta_x + r \frac{u}{1 - \eta(h - \epsilon \zeta)} = 0 \quad (2b) \]
\[ a [C_t + (\epsilon uC - \mu C_x)_x] = u^2 - C \quad (2c) \]
\[ h_\tau = - \langle u^2 - C \rangle \quad (2d) \]

Note that in (2d) the bedload contribution is neglected. Here \( \epsilon = U/\sigma L \) is, apart from a factor of 2\( \pi \), the ratio of the tidal excursion (the distance travelled by a fluid particle in a tidal period) and the tidal inlet length. The constant \( \lambda_L \) is the ratio of the tidal wave length \( L_g \) and the embayment length, and \( r = 8C_D A L/(3\pi H^2) \) is a measure for the frictional strength. The parameter \( \eta \) in the bottom friction is used to vary the influence of the depth on the friction; since \( 1 - h + \epsilon \zeta \) becomes zero at the end of the embayment, this term would become infinite, which is not a very realistic parameterization of the bed friction.

Furthermore, \( a = \sigma/\gamma \), the ratio of the timescale of the deposition process and the tidal period and \( \mu = \mu_*/\sigma L^2 \) is the ratio of the tidal period and the diffusive time scale. Here \( \mu_* \) is the dimensional diffusion coefficient. Here \( \tau \) is the slow time coordinate, defined as

\[ \tau = \frac{T_{\text{tidal}}}{T_{\text{morf}}} t \equiv \frac{\alpha U^2}{\rho_b(1-p)H\sigma t} \]

with \( T_{\text{morf}} \) the morphodynamic time–scale, \( T_{\text{tidal}} \) the tidal time–scale, \( p \) the bed porosity and \( \rho_b \) the density of the individual grain particles. So \( \tau \) is a function of \( U \), and hence of the embayment length \( L \), as well.

The boundary conditions at the open end are
\[ \zeta = \cos(t) + \beta \cos(2t - \phi) \quad (3a) \]
\[ h = 0 \quad (3b) \]
\[ C'(x, t, \mu) = C'(x, t, \mu = 0) \quad (3c) \]

where \( C' \) is the oscillatory part of the concentration equation. Condition 3c implies that no boundary layer in the fluctuating concentration can be formed. At the closed end the boundary conditions read
\[ u_x \text{ is finite} \quad (4a) \]
\[ \langle \epsilon uC - \mu C_x \rangle = 0 \quad (4b) \]
\[ C'(x, t, \mu) = C'(x, t, \mu = 0) \quad (4c) \]

Here (4a) is the kinematic boundary condition translated to the fixed point \( x = 1 \). (4b) states that the time–averaged flux is zero at this point. The last

\(^1\)The dimensional equations of motion and the boundary conditions are specified in Appendix 4.
condition (4c) states that no boundary layer in the fluctuating concentration can be formed.

This system of equations will be solved by making an expansion with respect to the small parameter \( \epsilon \), up to first order. Hence for the water elevation \( \zeta \) the following expansion is made

\[
\zeta = \zeta_0 + \epsilon \zeta_1 + \ldots
\]

with

\[
\zeta_0 = \zeta_0^s \sin(t) + \zeta_0^c \cos(t) \quad \zeta_1 = \zeta_0^0 + \zeta_2^s \sin(2t) + \zeta_2^c \cos(2t)
\]

Here \( \zeta_0 \) describes the main tidal constituent, \( \zeta_0^0 \) the mean and \( \zeta_2^c, \zeta_2^s \) the internally generated \( M_4 \)-overtide. Similar expansions for \( u \) and \( C \) can be made, where the superscripts have the same meaning as in the above expansion.

If a bed profile is given, the horizontal velocities, surface elevations and the concentration can be calculated explicitly. This information can be used to calculate a net sediment flux \( F \) and to rewrite the bed evolution equation as

\[
h_{\tau} = -F_x(h, h_x, h_{xx})
\]

, i.e., the bed changes due to divergences or convergences of the sediment flux. An equilibrium bed profile is found if \( h_{\tau} = 0 \). Using the boundary conditions on the sediment flux (4b), this means that in equilibrium the net sediment flux throughout the embayment has to be zero. After expanding the flux in the small parameter \( \epsilon \), this condition reads

\[
F = -a\mu C_0^0 + a\epsilon^2 \left( u_0^0 C_0^0 + \frac{1}{2} [u_0^s C_0^s + u_0^c C_0^c + u_2^s C_2^s + u_2^c C_2^c] \right) = 0
\]

So if an equilibrium bed profile is found, the combination of the velocity and concentration fields as given in (7) is such that the total time–averaged flux is zero. Using the knowledge of the equilibrium solution in case of the short embayment limit (see Schuttraelaars and De Swart (1996)) and standard numerical methods (continuation techniques in slowly varying parameters, see for example Seydel (1994)), equilibrium bed profiles and their corresponding velocity fields, surface elevations and concentration fields for different embayment lengths, tidal forcing etc. can be obtained.

3 Varying the embayment length

One of the most interesting questions is whether an equilibrium exists for all embayment lengths for a given set of parameters. Therefore, the length will be varied while other parameter values are kept fixed. Remember that due to variation of the embayment length the scaled diffusion constant \( \mu = \mu_s / \sigma L^2 \), the friction coefficient \( r \) and the tidal time–scale \( \delta_s \) change. The default values used in the experiments are given in Table 1 and are based on parameters
Quantities in the dimensional model

\[
\begin{align*}
H &= 15 \text{ m} & \xi &= 1.0 \text{ m} & \sigma &\sim 1.4 \cdot 10^{-4} \text{ s}^{-1} \\
\mu_* &= 10^2 \text{ m}^2 \text{s}^{-1} & \alpha &= 0.01 & C_D &= 1 \cdot 10^{-3} \\
\alpha &= 0.9 & a &= 0.01 & &
\end{align*}
\]

Table 1: Quantities and parameter values used in the numerical experiments.

obtained from observations in the Western Scheldt estuary. Note that no external overtides are taken into account, higher constituents are only generated internally. Since \( \mu \) becomes smaller with increasing \( L \) (since \( \mu = \mu_* / (\sigma L^2) \)) and the strength of the independent advective contributions are independent of the embayment length, advective contributions will become more important with increasing embayment length.

\[\frac{L/L_g}{0.04 \text{ to } 0.41}\]

Figure 3: The equilibrium bed profile for different embayment lengths as a function of \( x/L \). Here \( L_g \sim 550 \text{ km.} \)

From Fig. 3 it can be seen that the equilibrium bed profile is a straight line for small \( L \), if \( L \) is increased the bed becomes deeper near the entrance of the embayment. It can be shown that a maximum depth is reached near \( L = 210 \text{ km.} \)

The tendency of the bed profile to become deeper can be understood as
follows: assume that one starts with a constantly sloping bed profile in a rather short embayment ($L = 20$ km). It can be shown, using (2b), that $|u|$ increases towards the end of the embayment. Since the concentration is in first order proportional to $|u|^2$, the concentration increases as well. For a short embayment advective processes behave as diffusive ones (Schuttelbaars and De Swart, 1996). Due to diffusive processes, sediment has to be moved out of the estuary and the initial bed profile has to become concave.

For $L \gtrsim L_{\text{MAX}}$, with $L_{\text{MAX}} \sim 240$ km no equilibrium bed profile can be found anymore: if one starts with an embayment length $\tilde{L}$ larger than 240 km, the undisturbed water becomes zero near $x = L_{\text{MAX}}/\tilde{L}$. So effectively the embayment length is again $L_{\text{MAX}}$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig4}
\caption{Plots of the $M_2$ and internally generated $M_4$ horizontal and vertical tide for different embayment lengths as a function of $x/L$. Here $L_g = 2\pi\sqrt{gT/\sigma} \sim 550$ km.}
\end{figure}

In Figs. 4(a)–4(d), the equilibrium amplitudes of the horizontal and vertical tides are plotted for different embayment lengths. The $M_2$ tide becomes resonant for an embayment length of approximately 110 km. Since there is no external $M_4$ tide, no independent resonance of this first overtide is present. From Figs. 5(a)–
5(b), it is clear that the character of the tidal wave changes with increasing length: for short embayments, a standing wave is observed in the embayment (no phase difference throughout the embayment), if \( L \) increases the characteristics of a progressive wave are observed.

### 4 Conclusions

In this paper an idealized one–dimensional model has been analyzed to gain more understanding of the morphodynamics in long embayments. If the embayment length is much smaller than the tidal wavelength, the momentum equation describes in leading order a pumping mode, if the length is increased, both resonance effects and bottom friction become more important.

Sediment is transported as suspended load, both due to diffusive and advective processes. For a short embayment, the diffusive transport mechanism is most important. However, if the channel length is increased, diffusion becomes less effective and advection is the main transport mechanism.

The main aim in this paper was to investigate the possible existence of equilibria. A system is said to be in morphodynamic equilibrium when the averaged sediment flux vanishes throughout the embayment. The water motions in the embayment were only forced by an externally prescribed \( M_2 \)–tide, overtides were generated internally. Using the knowledge of the short embayment limit and numerical continuation techniques, it was shown that equilibrium profiles exist for embayment lengths smaller than \( L = L_{\text{MAX}} \sim 240 \text{ km} \). Embayments longer than \( L_{\text{MAX}} \) tend to fill up until the maximum embayment length is reached. It can be shown that these one–dimensional equilibria are linearly stable. Of course, the maximum embayment length depends on external parameters, such as the depth at the entrance and the drag coefficient.

The equilibrium bed profiles were concave. For short embayments, this could
be explained by noting that the velocity and hence the concentration increased towards the end of the embayment for a constantly sloping bed profile. Since diffusion was the main transport mechanism, sediment was transported out of the embayment and the equilibrium bed became concave.

It was observed that the embayment became resonant for a length of approximately 0.2 times the gravitational wave length \(( = 2\pi\sqrt{gH/\sigma})\). It is well known that the resonance length \(L_{res}\) of an embayment with a flat bottom is 0.25 times the gravitational wave length. Due to frictional effects and the non-flat bed profile, the resonance length shifts towards shorter embayments. Furthermore, it was shown that character of the tidal wave changed from a standing wave character towards a running wave.

In a forthcoming paper, the model will be extended to handle both a prescribed \(M_2\) and \(M_4\)-tide at the entrance. Preliminary results show that in this case more than one one-dimensional equilibrium can exist. The maximum embayment lengths of these equilibria depend sensitively on the ratio of the amplitudes of the externally prescribed \(M_2\) and \(M_4\)-surface elevations and their phase difference. The sensitivity on these and other external parameters and the global stability of these equilibria will be investigated in more detail. Furthermore, a comparison with field data should be made.

\section*{A \ Dimensional Model}

The dimensional equations are

\[
\begin{align*}
\frac{\partial \zeta}{\partial t} + [(H - h + \zeta)u]_x &= 0 \quad (8a) \\
\frac{\partial u}{\partial t} + uu_x + g\zeta_x + \frac{u}{H - h + \zeta} &= 0 \quad (8b) \\
\left[ \frac{C_x + (uC - \mu^*C_x)}{\mu^*} \right] &= \alpha u^2 - \gamma C \quad (8c) \\
\rho_s(1 - p)h_t &= -\langle \alpha u^2 - \gamma C \rangle \quad (8d)
\end{align*}
\]

The boundary conditions at the entrance \(x = 0\) are

\[
\begin{align*}
\zeta &= A [\cos(\sigma t) + \beta \cos(2\sigma t - \phi)] \quad (9a) \\
h &= 0 \quad (9b) \\
C'(x, t, \mu) &= C'(x, t, \mu = 0) \quad (9c)
\end{align*}
\]

and at \(x = L\)

\[
\begin{align*}
u_x & \text{ is finite} \quad (10a) \\
\langle uC - \mu^*C_x \rangle &= 0 \quad (10b) \\
C'(x, t, \mu) &= C'(x, t, \mu = 0) \quad (10c)
\end{align*}
\]

Additional parameters are \(\hat{r} = (8/3)\pi C_d U \sim 10^{-3} ms^{-1}\), the density of grains \(\rho_s \sim 2650 kg m^{-3}\), the bed porosity \(p \sim 0.4\) and the erosion parameter \(\alpha \sim 10^{-4} kgs^2 m^{-4}\). The other parameters are explained in the text.
References


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