Robust Algebraic Multigrid

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Acknowledgements

- This work has been supported by
  - Chancellor's Fellowship, University of Colorado
  - DOE SciDAC TOPS program
  - Center for Applied Scientific Computing at Lawrence Livermore National Lab
- This work has been performed in collaboration with
  - Steve McCormick, Tom Manteuffel, John Ruge, and Marian Brezina at CU-Boulder
  - Rob Falgout at CASC
  - David Moulton at LANL.

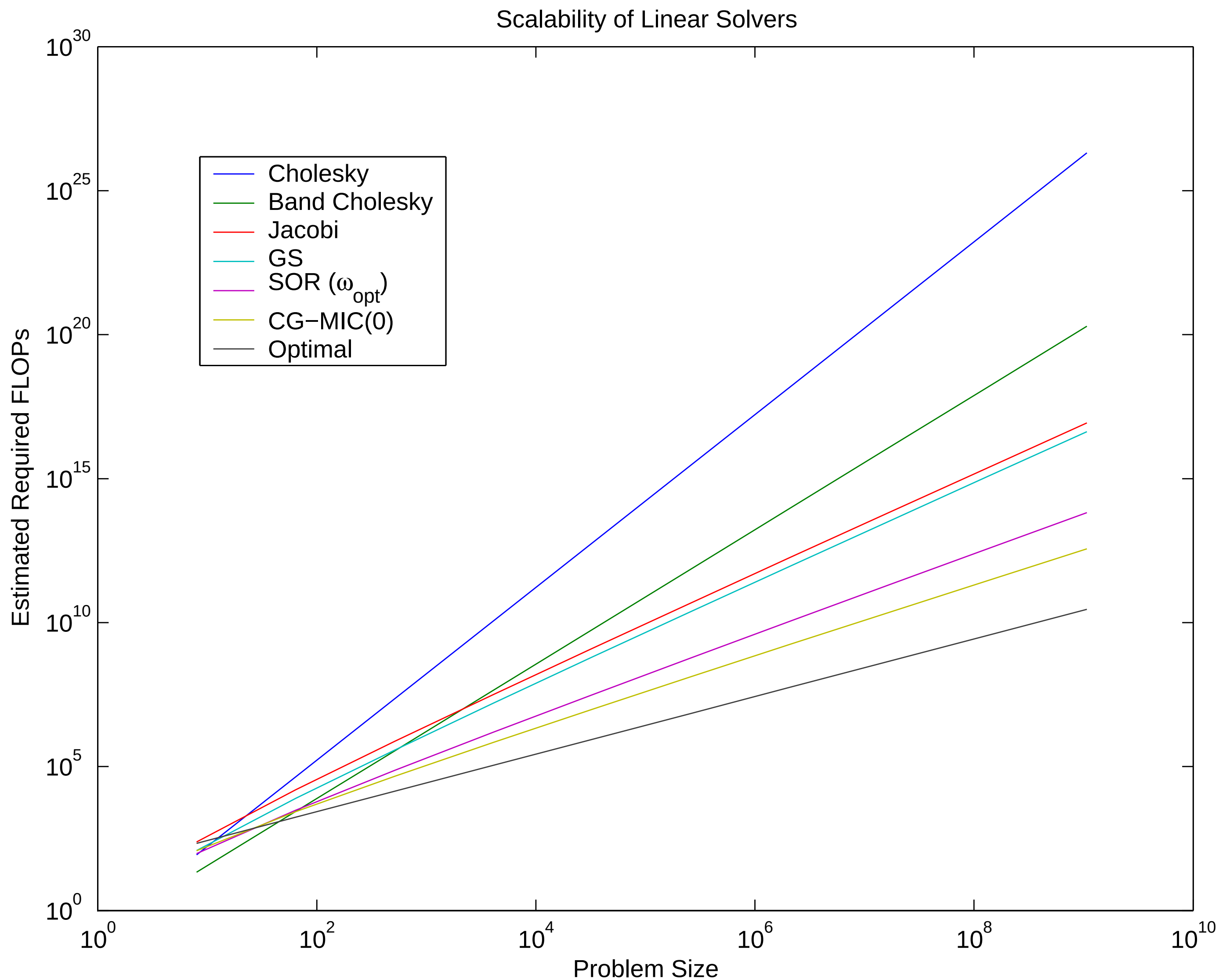
The Need for Optimal Linear Solvers

- Significant interest in simulating complex physical systems with features, and hence solutions, that vary on multiple scales
- Accuracy constraints lead to discretizations with tens of millions, or even billions, of degrees of freedom (DOFs)
  - 3D Tsunami Model: 200 million cells
  - Transport: 500 million to 1 billion degrees of freedom
- Without optimal methods, solving three-dimensional problems can be prohibitively expensive

Properties of Matrices

- We consider (primarily) discretizations of the underlying continuum models (differential equations) via finite elements or finite differences
- The matrices from these discretizations tend to be sparse and ill-conditioned
- The matrices inherit properties of the continuum model (e.g. symmetry, definiteness)

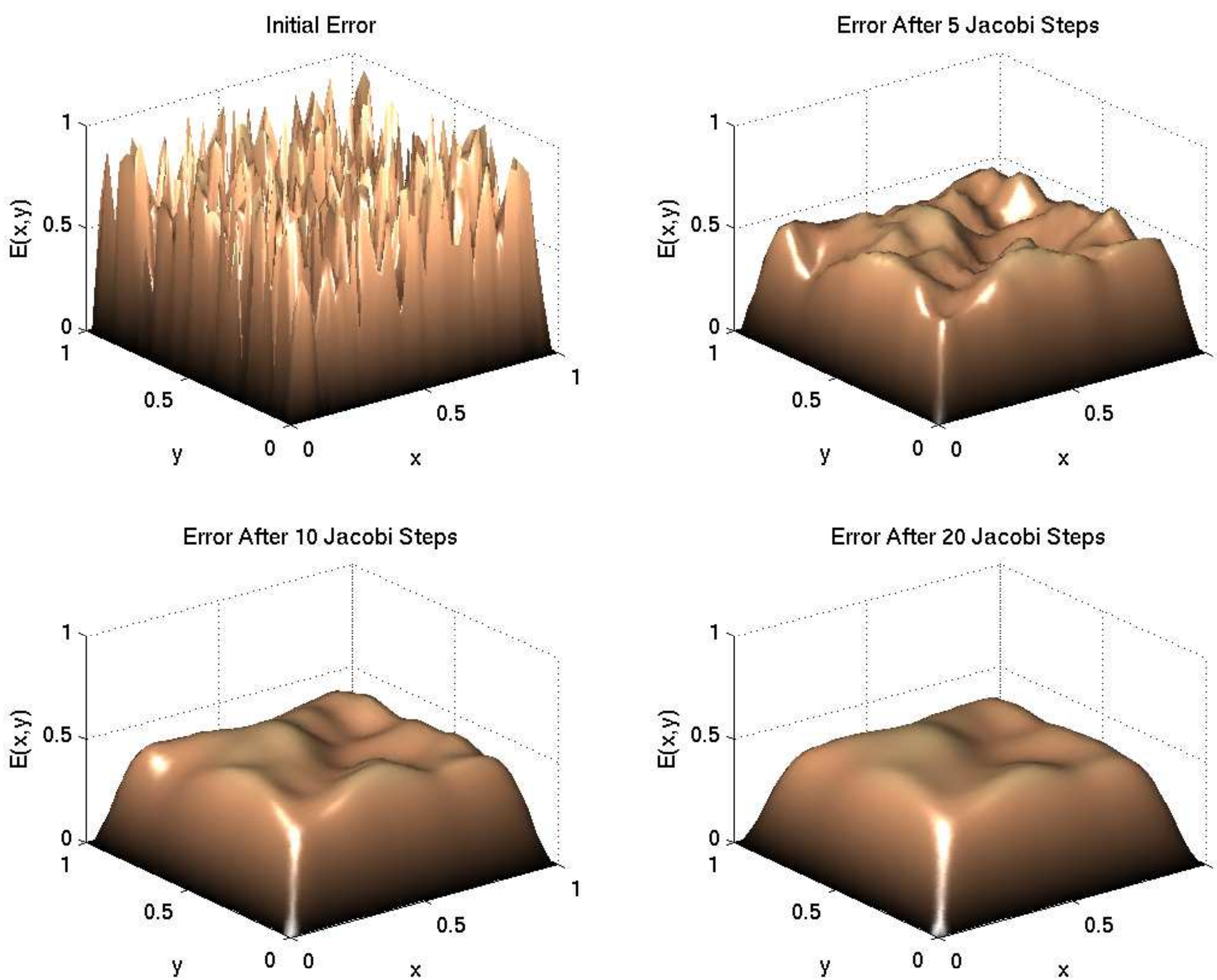
Classical Methods do not Suffice



Stationary Iterative Methods

- The Jacobi and Gauss-Seidel iterations do converge for FE discretizations of elliptic operators, but are not require $O(N^{\frac{5}{3}})$ operations for 3-D problems
- These methods do, however, resolve some components much faster than others
- For example, for the Laplacian, it is the geometrically smoothest components of the solution that are the slowest to be resolved
- For this reason, Jacobi and Gauss-Seidel are often called smoothers - they smooth the error in the approximation

Smoother Performance



Complementing Relaxation

- If the error left after a few Jacobi or Gauss-Seidel sweeps is smooth, it can be accurately represented using fewer degrees of freedom
- Problems with fewer degrees of freedom can be solved with less effort
- Error which appears smooth across many degrees of freedom is more oscillatory when represented on fewer degrees of freedom
- We choose to represent such error using a subset of the fine-grid degrees of freedom

Multigrid Basics

Multigrid Methods achieve optimality through complementarity

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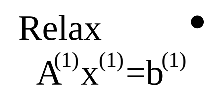
Multigrid Methods achieve optimality through complementarity

Multigrid Components

- Relaxation



- Use a smoothing process (such as Gauss-Seidel) to eliminate oscillatory errors
- Remaining error satisfies $Ae = r$

Multigrid Basics

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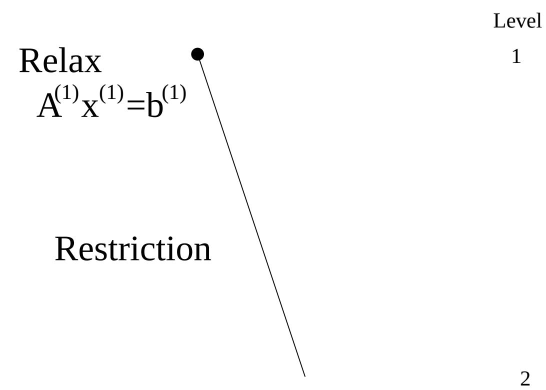
Multigrid Components

- Relaxation
- Restriction



- Need to transfer residual to coarse-grid
  - use Restriction operator

Multigrid Basics

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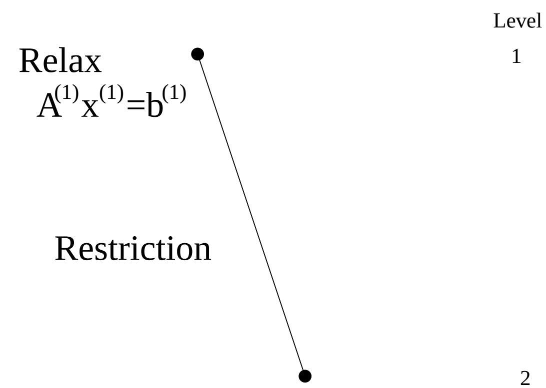
Multigrid Components

- Relaxation
- Restriction
- Coarse Grid Correction



- Use coarse grid correction to eliminate smooth errors

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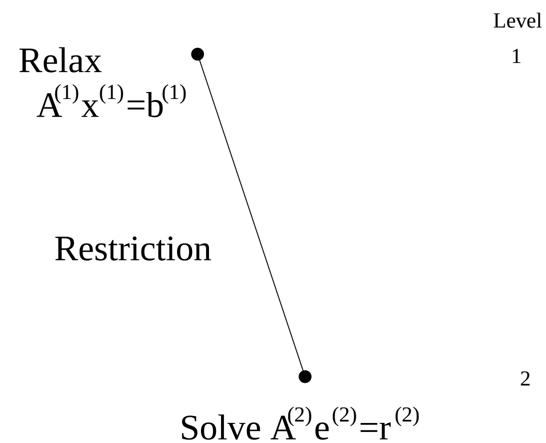

- To solve for error on coarse-grid, use residual equation

$$A^{(2)}e^{(2)} = r^{(2)}$$

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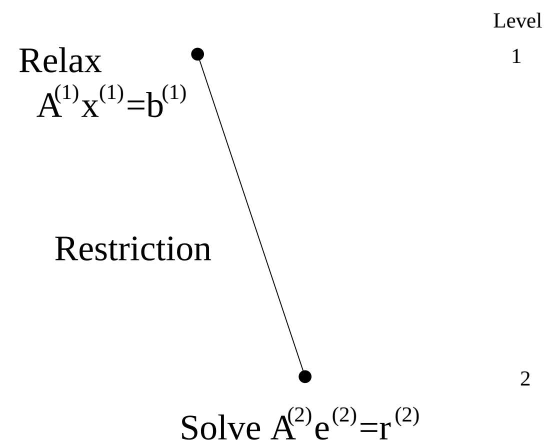


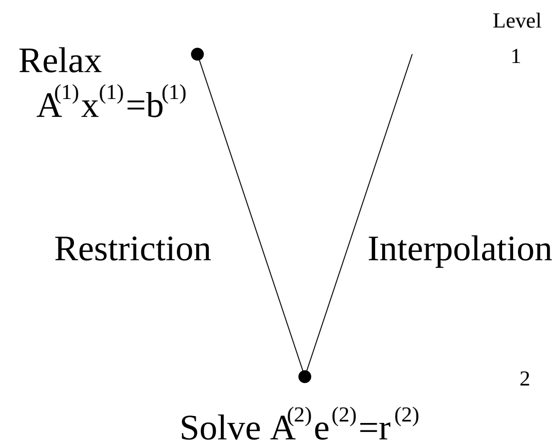
- Solving on coarse-grid requires an operator on this grid which well-approximate the fine-grid operator
- The coarse-grid operator can be formed by rediscretization or using a variational principal

Multigrid Basics

Multigrid Methods achieve optimality through complementarity

Multigrid Components

- Relaxation
- Restriction
- Coarse Grid Correction
- Interpolation



- Need to transfer correction to fine-grid
  - use Interpolation (Prolongation) operator
- Often pick a form of interpolation (P) and take restriction $R = P^T$ (theoretical benefits)

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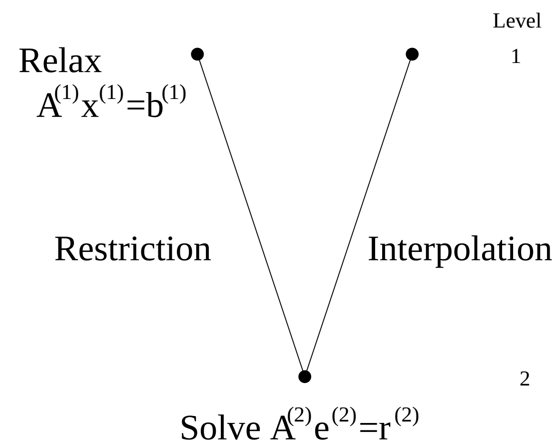
Multigrid Components

- Relaxation
- Restriction
- Coarse Grid Correction
- Interpolation
- Relaxation



Multigrid Basics

Multigrid Methods achieve optimality through complementarity

Multigrid Components

- Relaxation
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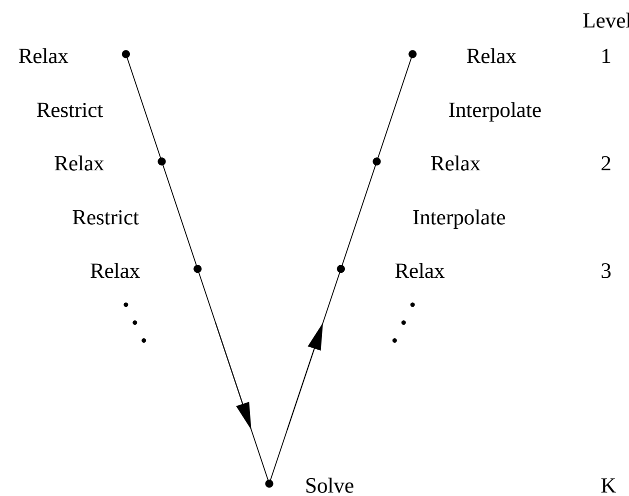
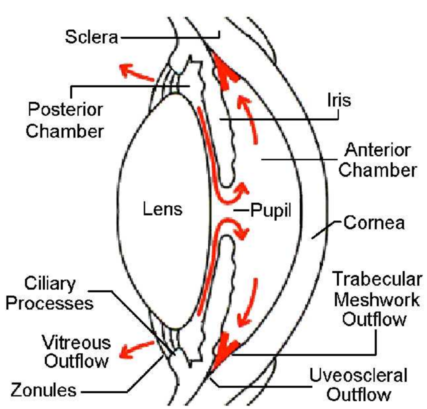
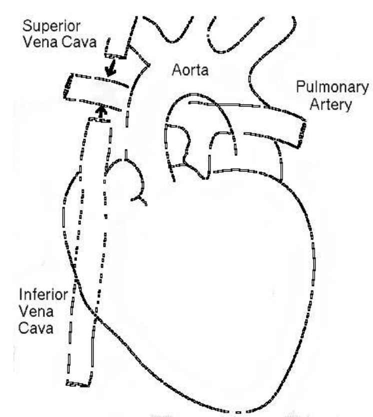


Obtain optimal efficiency through recursion

Challenges

- It is complicated to design multigrid schemes for complex geometries, nonsmooth PDE coefficients, and systems of PDEs
- Algebraic multigrid methods extend the geometric multigrid ideas, but use only the matrix coefficients
- Classical (Ruge-Stueben) AMG assumes that error after smoothing varies slowly along strong matrix connections (i.e., it is essentially locally constant)





Improving Robustness

- Complementarity is key in multigrid: error components that are not quickly reduced by relaxation must be reduced by coarse-grid correction
- A component can only be corrected from the coarse-grid if it is properly interpolated from that grid
- Error components that are not reduced by relaxation are exposed by relaxation on $Ax = 0$
- Such components are then treated by our definition of interpolation and coarse-grid operators

Adaptive Approach

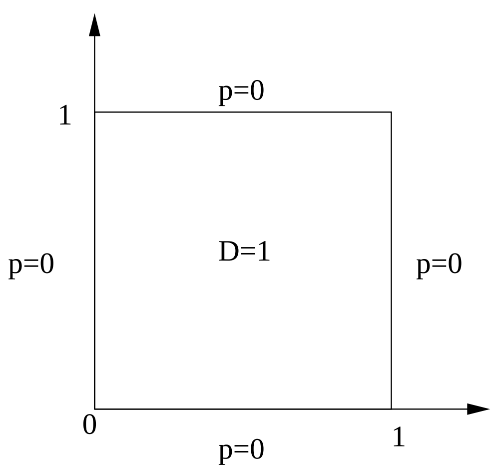
- Our method is also adaptive: as a better representation of the error not reduced by relaxation is found, it is integrated into the algorithm
- Our method reduces to the classical, Ruge-Stueben method when relaxation is least-efficient for a constant vector
- A priori knowledge of the errors left after relaxation yields "textbook multigrid efficiency"
- We use a bootstrap approach to allow the algorithm to optimally adapt itself

Adaptive Approach

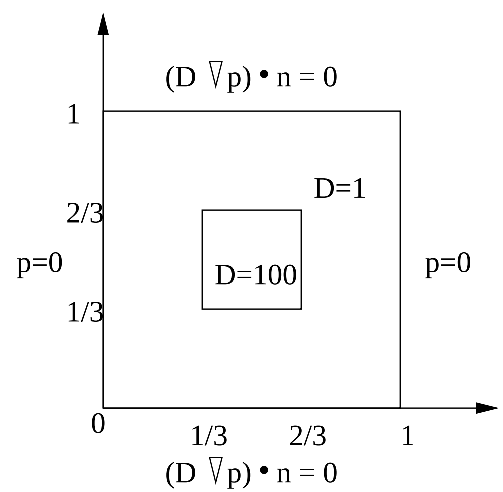
- Start with a random initial guess and relax on $Ax = 0$ to expose error not reduced by relaxation
- Relaxation alone requires too much effort to do this
- Instead, we use an adaptive approach to creating the multigrid V-cycle
- This provides us with a mechanism for the multilevel development of the error to be captured by coarse-grid correction
- We are developing a two-level theory which shows that each bootstrap cycle improves the overall performance of the algorithm

Numerical Results

- Coarse-grid selection is currently done geometrically and coarse-grid operators are determined using a variational principle
- Problems 1 and 2 are standard, bilinear FE discretizations of $-\nabla \cdot D(x, y)\nabla p(x, y) = 0$



Problem 1



Problem 2

Bilinear FE matrices

Work units required to reduce error by 10^{-6}

	Standard AMG		Adapted AMG	
h	Problem 1	Problem 2	Problem 1	Problem 2
$1/32$	12.9	14.5	12.9	14.9
$1/64$	13.4	15.6	13.4	15.6
$1/128$	13.6	14.9	13.6	15.2
$1/256$	13.8	16.4	13.8	16.4
$1/512$	13.9	15.2	13.9	15.2
$1/1024$	13.9	16.7	13.9	16.7

Scaled Problems

- The second pair of problems come from diagonally scaling Problems 1 and 2
- To scale, we use the node-wise scaling function

$$1 + \sin(547\pi x_i)\sin(496\pi y_j) + 10^{-7}$$

- This function gives variable scaling on each node, but does not change its character with h

Scaled Matrices

Work units required to reduce error by 10^{-6}

	Standard AMG		Adapted AMG	
h	Problem 3	Problem 4	Problem 3	Problem 4
$1/32$	1297	59.4	12.9	14.9
$1/64$	4075	112.1	13.5	15.3
$1/128$	6122	218.7	13.7	15.3
$1/256$	6122	430.6	13.8	16.4
$1/512$	7350	858.6	13.9	15.2
$1/1024$	7350	1656	13.9	16.8

Summary

- Optimal ($O(N)$) solution methods are required for modern computational science and engineering applications
- Classical methods (direct and iterative) are not optimal
- Multigrid methods (geometric and algebraic) offer optimal performance for many problems
- Implicitly incorporating information from relaxation into interpolation yields improved solver performance at the cost of a more complex setup procedure
- Optimality and efficiency of these methods are supported by a theory under development