Strategies for Self-Correcting Algebraic Multigrid Methods

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Basic Multigrid V-Cycle

- Relax on $Ax = b$
- Restrict residual equation to coarser grid
- Solve coarse grid equation recursively
- Interpolate coarse grid correction and add to current approximation
- Relax on $Ax = b$

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- Develop an algebraic multigrid solver with increased robustness properties while not sacrificing optimality
- Develop a solver which defaults to simplicity if given a simple problem

Basic Multigrid Properties

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- For efficient multigrid performance, relaxation and coarse grid correction must be complementary

Current Assumptions

- Coarse grids are predetermined and sufficient for full multigrid efficiency
  - Currently choosing coarse grids based on geometric or “classical” AMG criteria
  - Eventually hope to determine coarse grids adaptively as well (Compatible Relaxation)

Main Ideas

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- Complementarity of relaxation and coarse grid correction means that if relaxation is inefficient on a component then that component must be treated by coarse grid correction
- Components that are slow to converge for $Ax = b$ will also be slow for $Ax = 0$

Details of the Method

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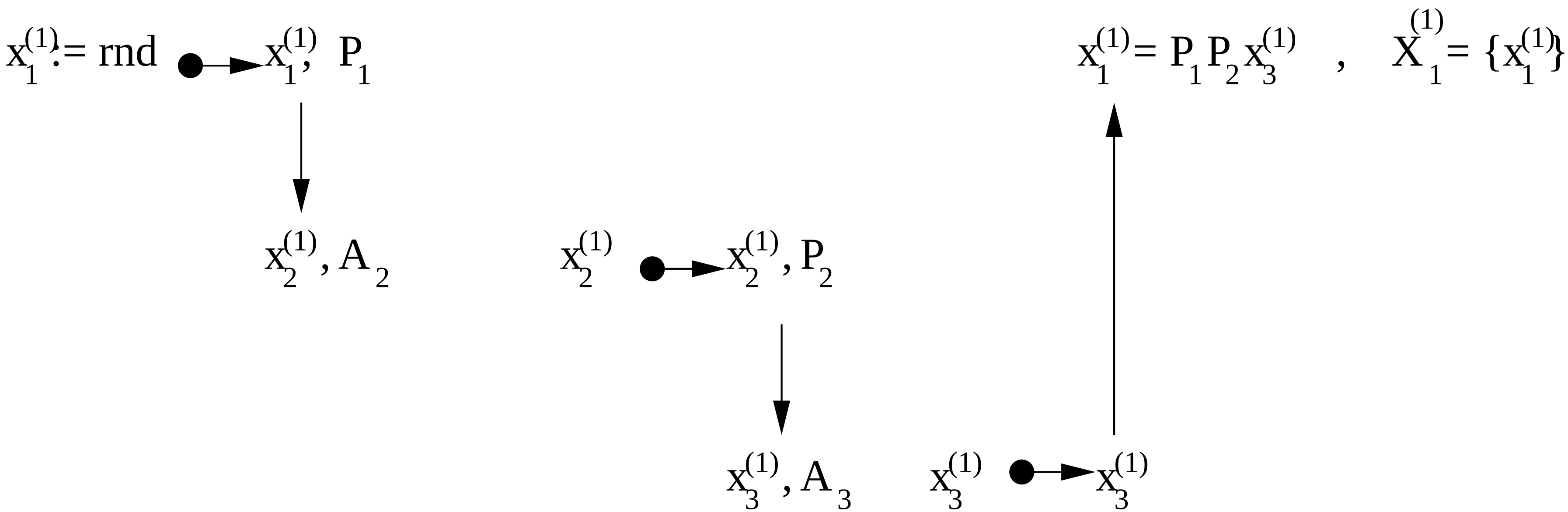
- Go to a multigrid method by recursion

Details . . .


$x_1^{(1)} := \text{rnd}$
$x_1^{(1)}, \ P_1$
$x_1^{(1)} = P_1 P_2 x_3^{(1)} \quad , \quad X_1^{(1)} = \{x_1^{(1)}\}$
$x_2^{(1)}, A_2$
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- Iterate . . .

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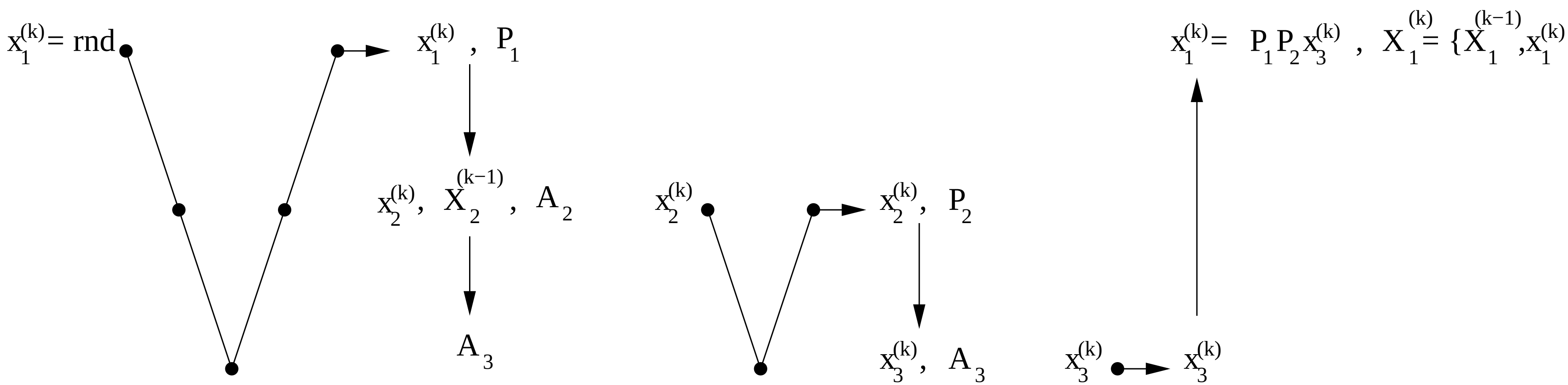

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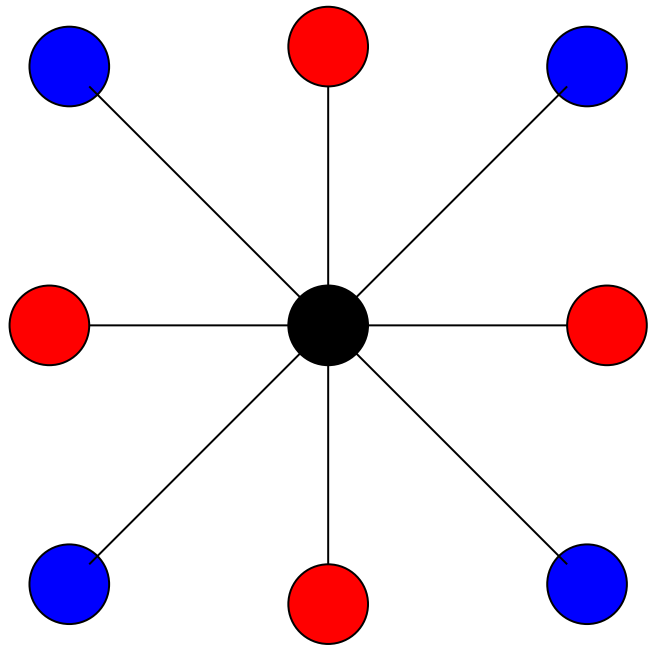
- To define interpolation, need to collapse connections from F_i to C_i

N_i, the neighbourhood of i



Fine Grid Points

Coarse Grid Points

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- When we have accumulated enough vectors (more vectors to fit than points in C_i), we must solve this in a least-norm sense

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- Then, using the definition of algebraic smoothness, we have

$$a_{ii} x_i \approx - \sum_{j \in C_i} a_{ij} x_j - \sum_{k \in F_i} a_{ik} x_k$$

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$$x_i \approx - \sum_{j \in C_i} \frac{a_{ij} + \sum_{k \in F_i} a_{ik} w_{kj}}{a_{ii}} x_j$$

Choosing Interpolation . . .

So, we define interpolation to a fine grid point i as

$$x_i = -\sum_{j \in C_i} \frac{a_{ij} + \sum_{k \in F_i} a_{ik} w_{kj}}{a_{ii}} x_j$$

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- Relieve method from these presumptions
- Coarse grid correction must address errors which look (locally) like representatives
- Get new representatives through method, so they must be distinct
- Get new interpolation by maintaining performance on previous representatives while accounting for new representative

Preliminary Numerics

- $-\nabla \cdot D(x,y)\nabla u(x,y) = 0$ on $[0,1]^2$
- Dirichlet boundary conditions
- Geometric choice of coarse grids
- Interpolation chosen as above, for 1 smooth vector

Preliminary Numerics . . .

- $D(x, y) = 1$ (Laplace)

size	convergence factor
32×32	0.096
64×64	0.14

- $D(x, y) = r_1 * 10^{2r_2}$; r_1, r_2 random, uniform on $[0, 1)$

size	convergence factor
32×32	0.30
64×64	0.50

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- Adaptive choice of coarse grids
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- Ensure optimal multigrid efficiency

Conclusions

- Have framework for self correcting multigrid solvers
- Preliminary numerics suggest this approach is feasible
- Method defaults to simple relaxation for problems where this is sufficient
- Much work still to be done