Fully Adaptive AMG

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Support and Collaboration

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- This work has been performed in collaboration with Steve McCormick, Tom Manteuffel, John Ruge, Marian Brezina, and James Brannick at CU-Boulder.

Fully Adaptive Multigrid Framework

- Want to solve new problems in an algebraic multigrid setting
- Design a setup cycle that, given an arbitrary matrix, A, designs an effective multigrid V-cycle
- No parameter tuning (if possible)
- Generalizing classical AMG:

More general techniques for more general problems

- Will be more expensive, but also more robust

FAlosophy

- Ask: If the AMG setup cycle was “free”, how would we design the ideal multigrid algorithm?
- What are the real goals of coarsening?
- What are the ideal properties of coarse grids?
- How do we measure these?
- Once we’ve figured out the ideal case, then ask if we can make it practical

Classical AMG Coarsening

- Strong Connections based on matrix entries:

$$S_i = \left\{ j : -a_{ij} \geq \theta \max_{k \neq i} \{-a_{ik}\} \right\}$$

- Coarse grid chosen by maximal independent set heuristics

H1: For each $i \in F$, every $j \in S_i$ should be either in C_i or should strongly depend on at least one point in C_i

H2: The set, C, should be a maximal subset of the fine grid, such that no C-point strongly depends on another C-point

Weaknesses

- Definition of strong connections based on “nice” M-matrix properties
- Breaks down if near null space of A is far from the constant
  - Diagonal rescaling,
$$A \to DAD$$
  - Finite element anisotropy,
$$-u_{xx} - \epsilon u_{yy} \to \frac{1}{6}\begin{bmatrix} (-1-\epsilon) & (2-4\epsilon) & (-1-\epsilon) \\ (-4+2\epsilon) & (8+8\epsilon) & (-4+2\epsilon) \\ (-1-\epsilon) & (2-4\epsilon) & (-1-\epsilon) \end{bmatrix}$$
- Even for simple problems, size of a_{ij} may not reflect true connection between i and j

What are Strong Connections?

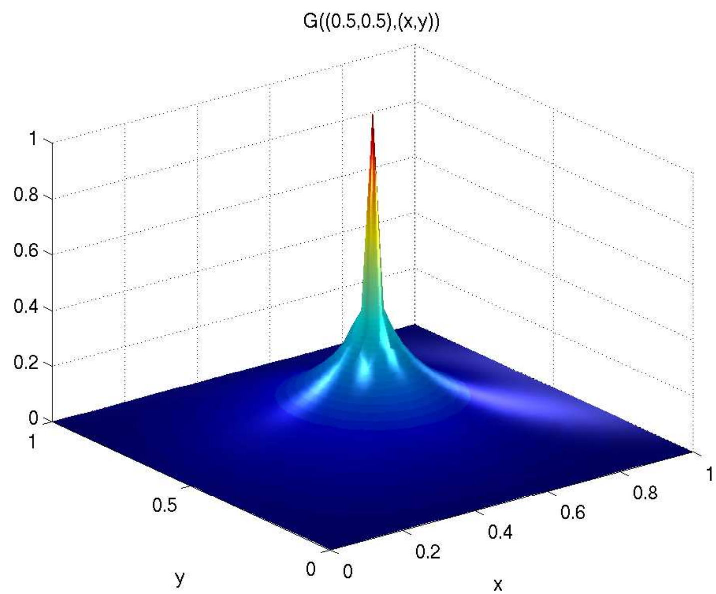
- Point i strongly depends on point j if
  - a change in the right-hand side at point j significantly changes the solution at point i.
  - a change in the residual at point j significantly changes the error at point i
- Good coarse-grid correction depends on identifying strong connections
  - Interpolation to i is most effective from points that it strongly depends on
  - Corrections from weakly connected points have little effect on the error at i

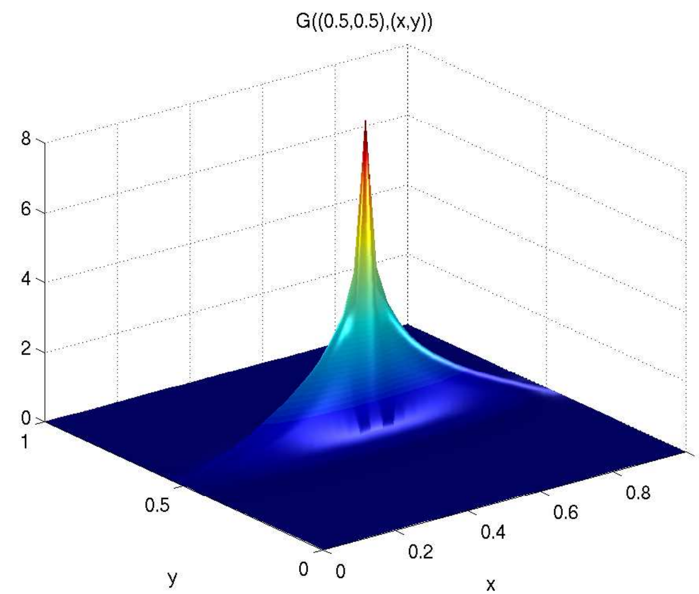
Green's Functions

- Given a PDE, $Lu = f$, the Green's function, G_L relates u and f:

$$u(\mathbf{x}) = \int_\Omega G_L(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) d\mathbf{y}$$

- If a change in $f(\mathbf{x}_j)$ affects a significant change in $u(\mathbf{x}_i)$, then $G_L(\mathbf{x}_i, \mathbf{x}_j)$ must be large
- $\mathbf{x}_i$ strongly depends on $\mathbf{x}_j$ if $G_L(\mathbf{x}_i, \mathbf{x}_j)$ is large compared to other values of $G_L(\mathbf{x}_i, \mathbf{x})$



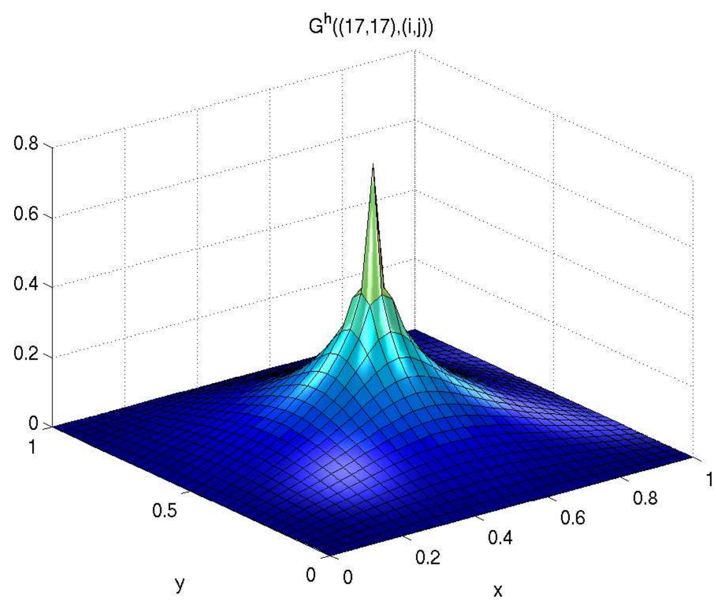


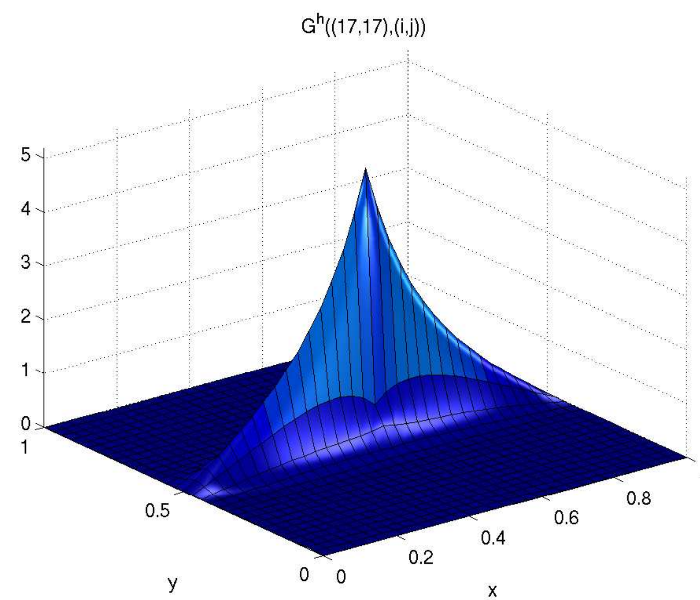
Discrete Green's Function

- For the discrete linear system, $A^h u^h = f^h$, the equivalent of the Green's function is the inverse

$$u^h = \left(A^h\right)^{-1} f^h$$

- If a change in f_j^h causes a significant change in u_i^h, then $(A^h)_{ij}^{-1}$ must be large relative to other values of $(A^h)_{ik}^{-1}$





Measures of Strong Connections

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- How should we measure this size, relative to $(A^h)^{-1}_{ik}$?

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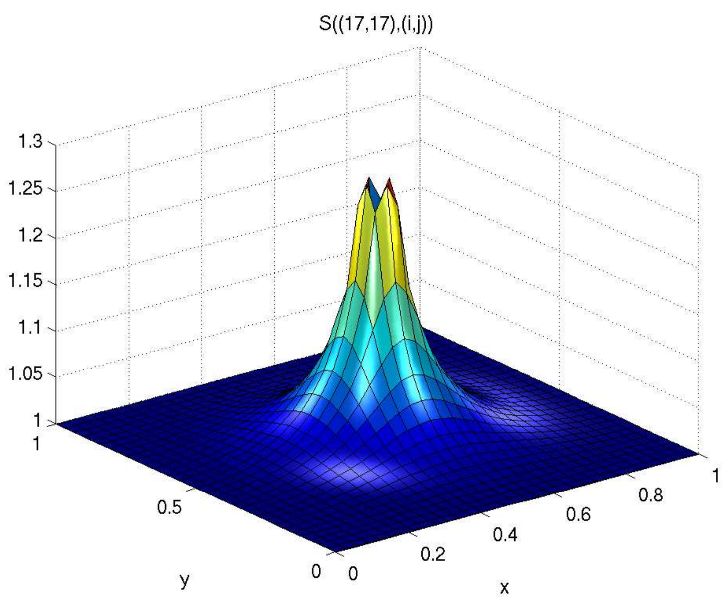
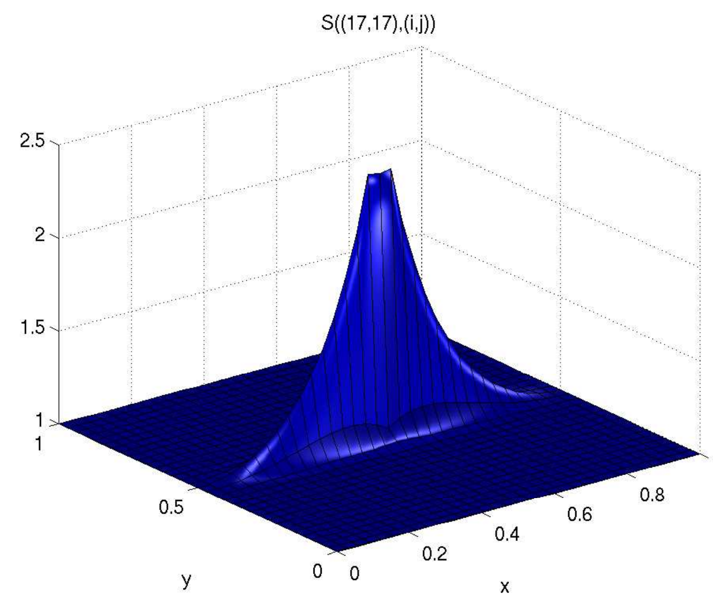
- How should we measure this size, relative to $(A^h)^{-1}_{ik}$?
- L^2 measure: $(A^h)^{-1}_{ij} \geq \theta \max_{k \neq i} \left\{ \left(A^h\right)^{-1}_{ik} \right\}$
- Energy measure: Let $G_j^{(i)} = (A^h)^{-1}_{ij}$, $S_{ij} = \dfrac{\|G^{(i)} - G_j^{(i)} e^{(j)}\|_{A^h}}{\|G^{(i)}\|_{A^h}}$





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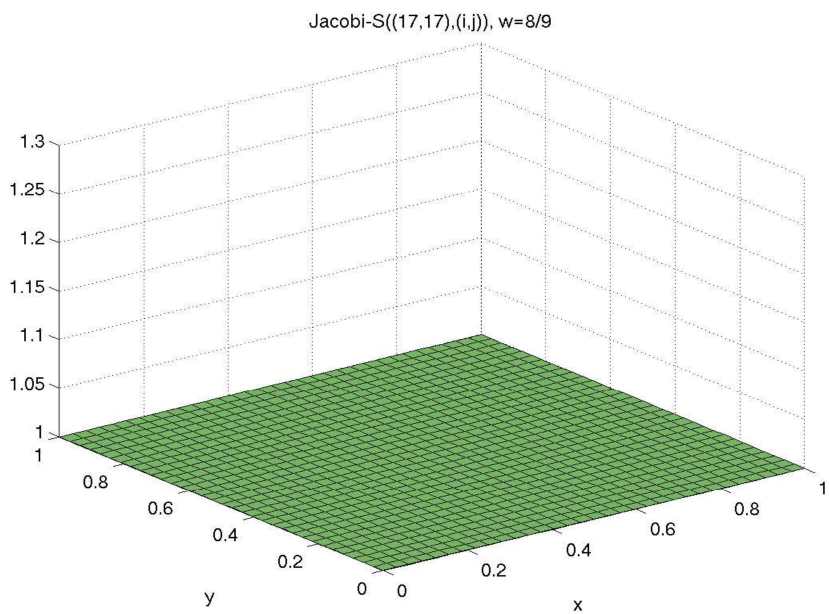

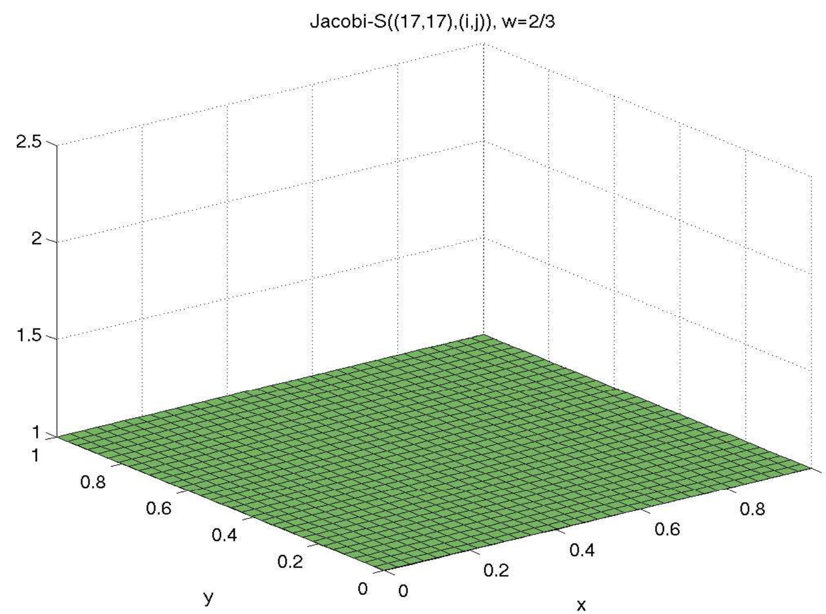

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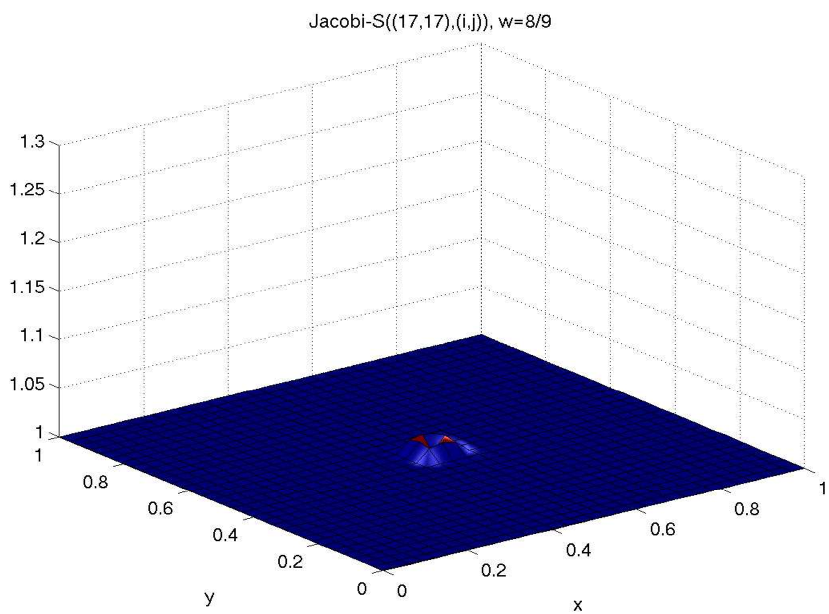

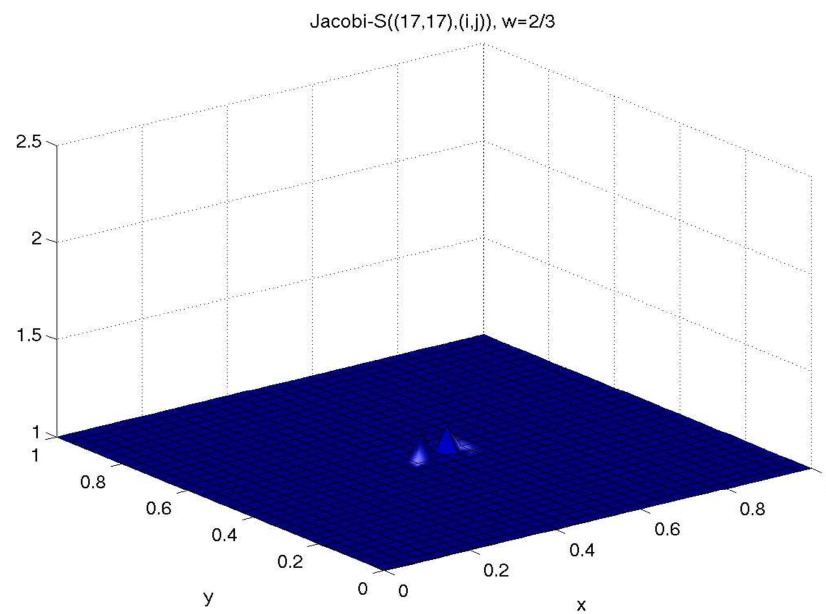

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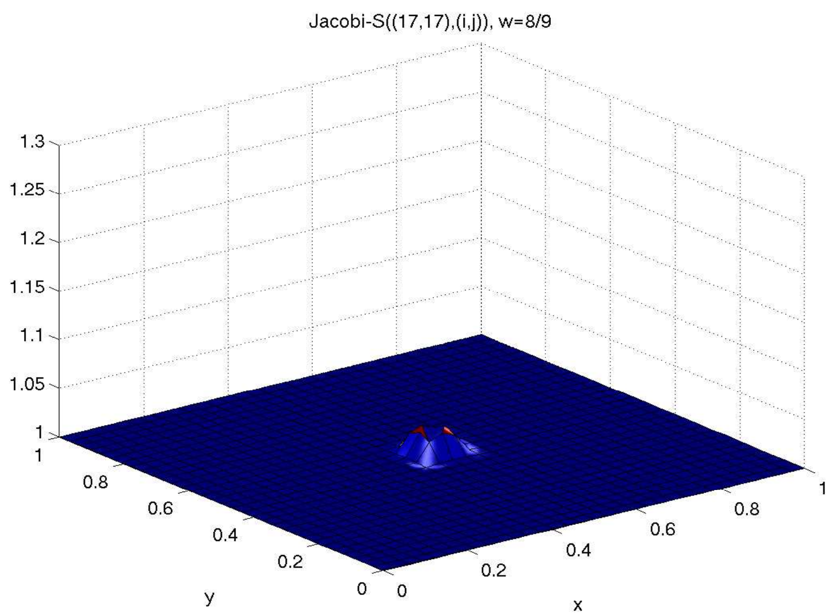

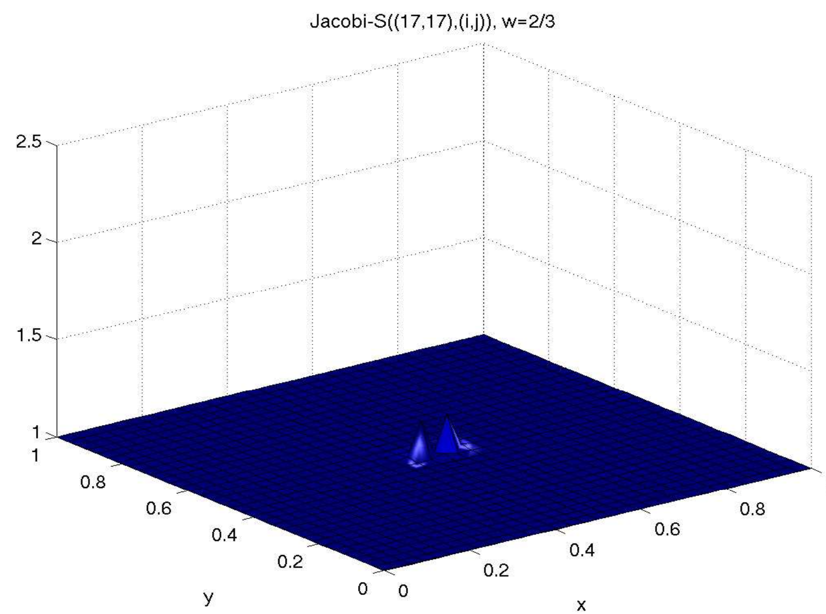

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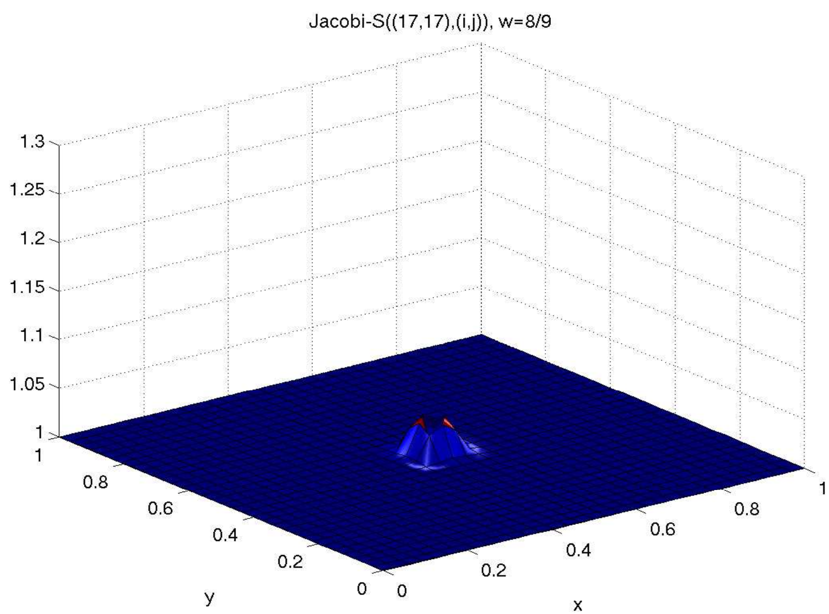

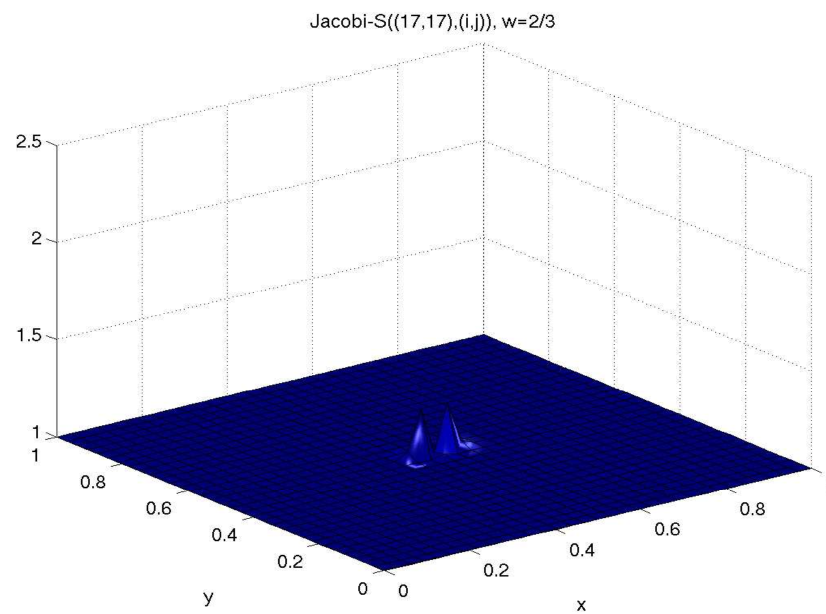

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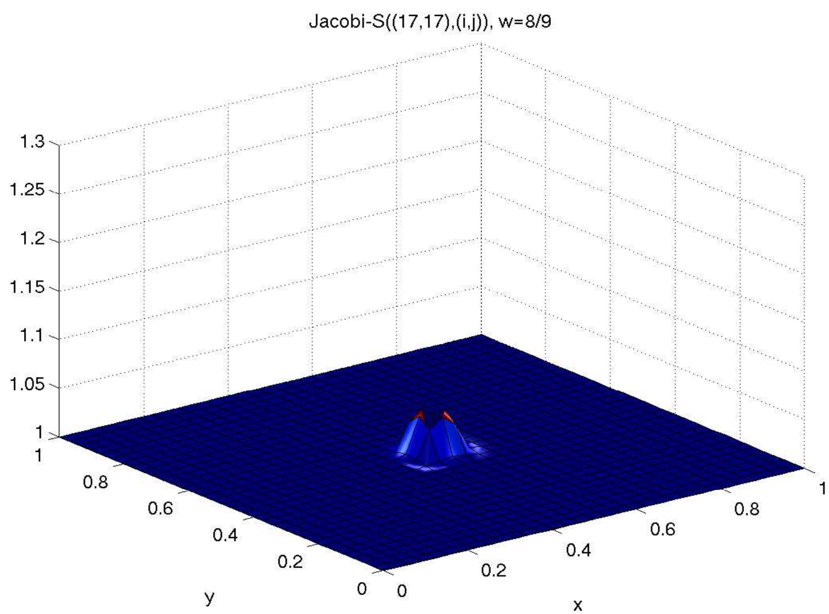

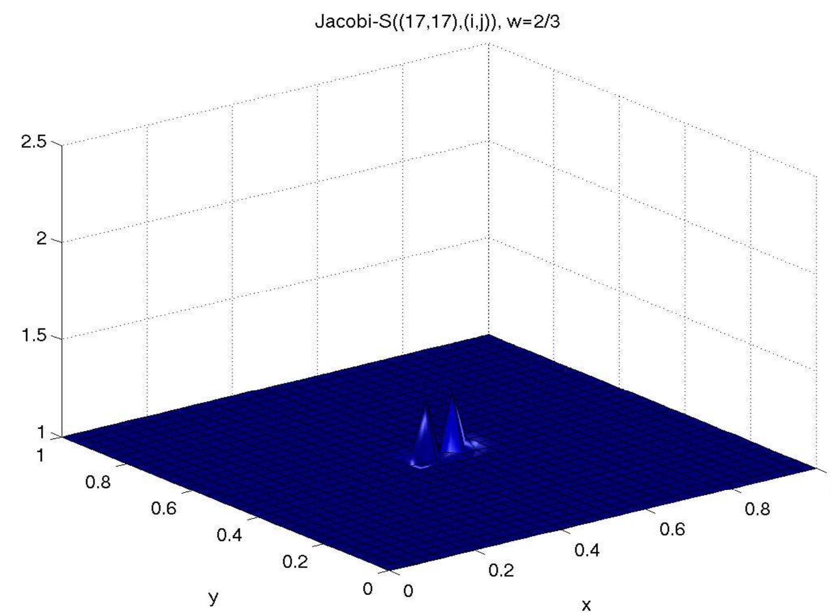

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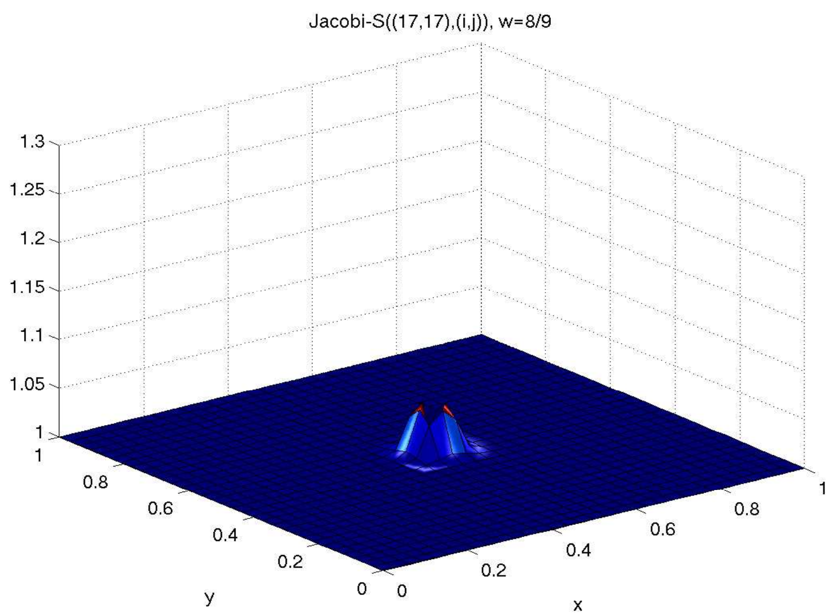

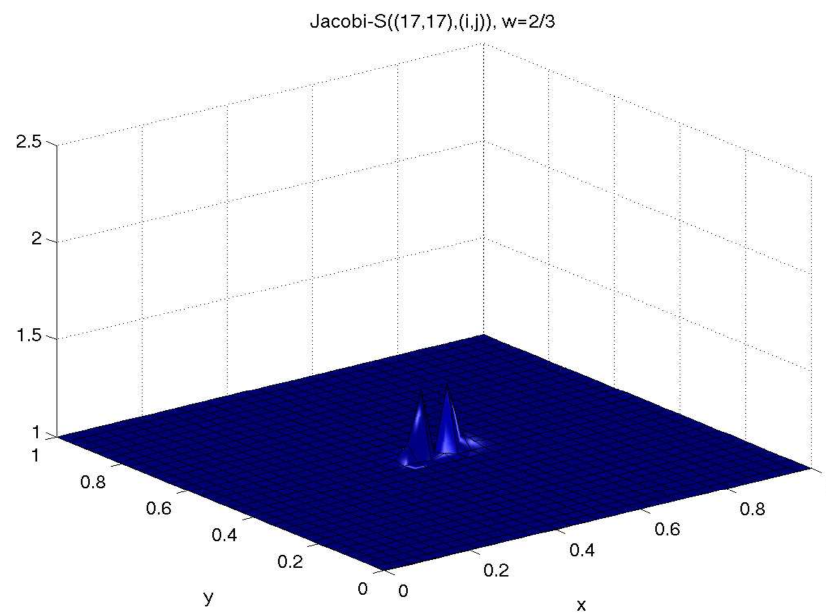

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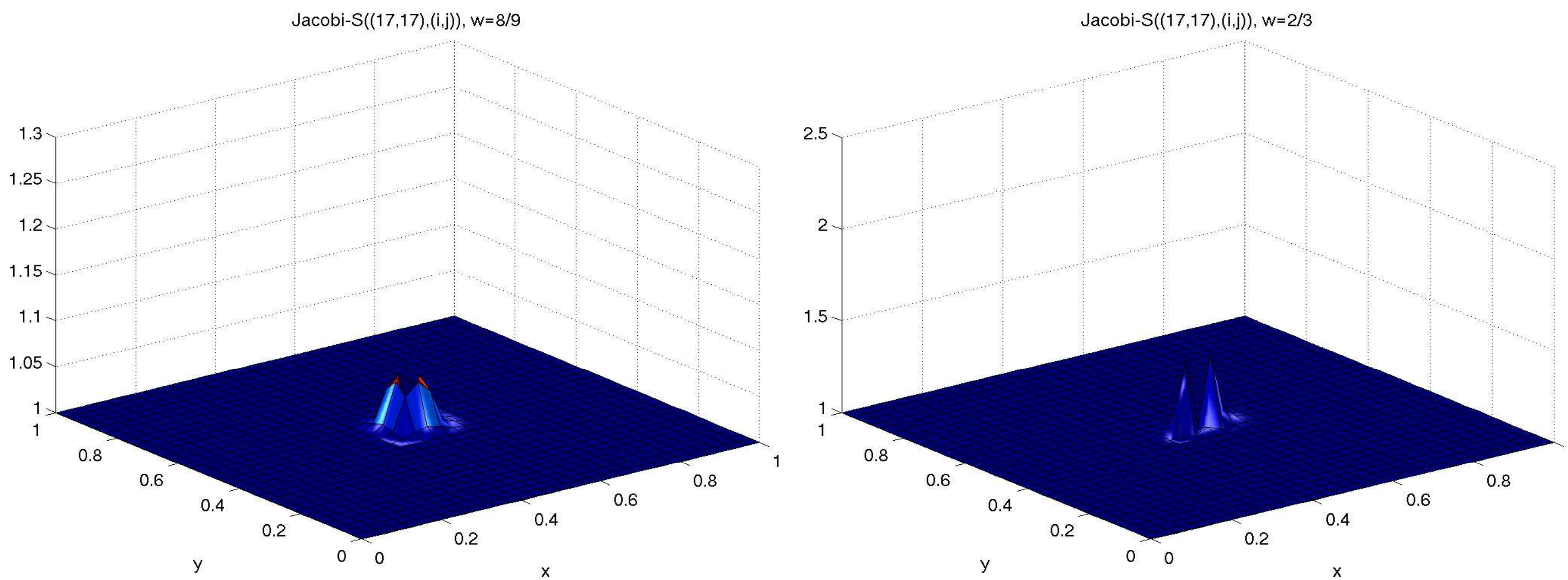

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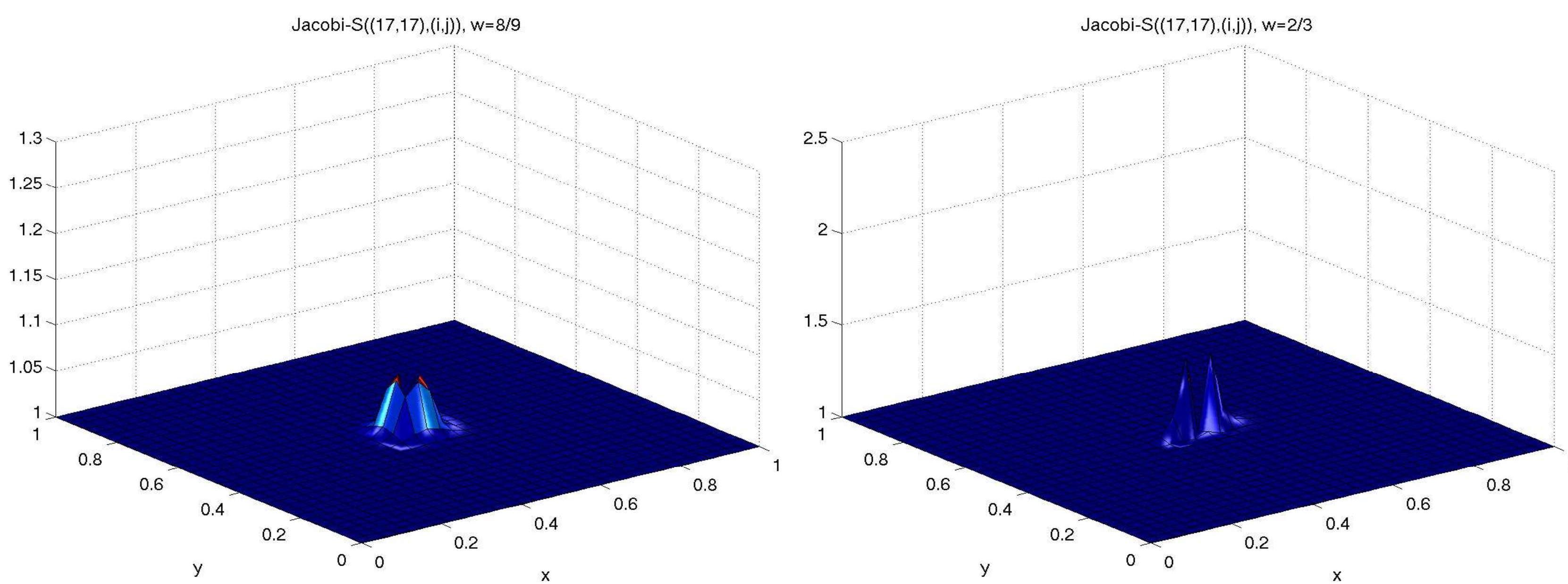

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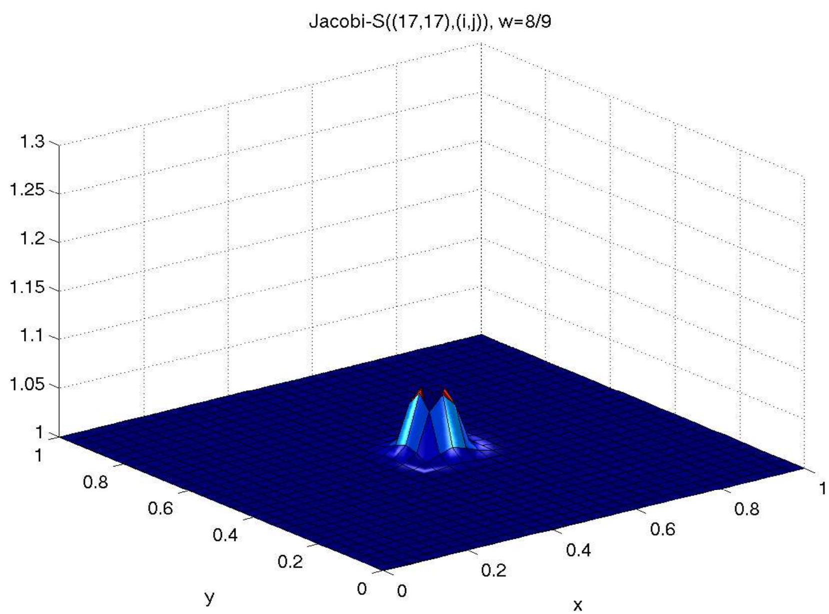
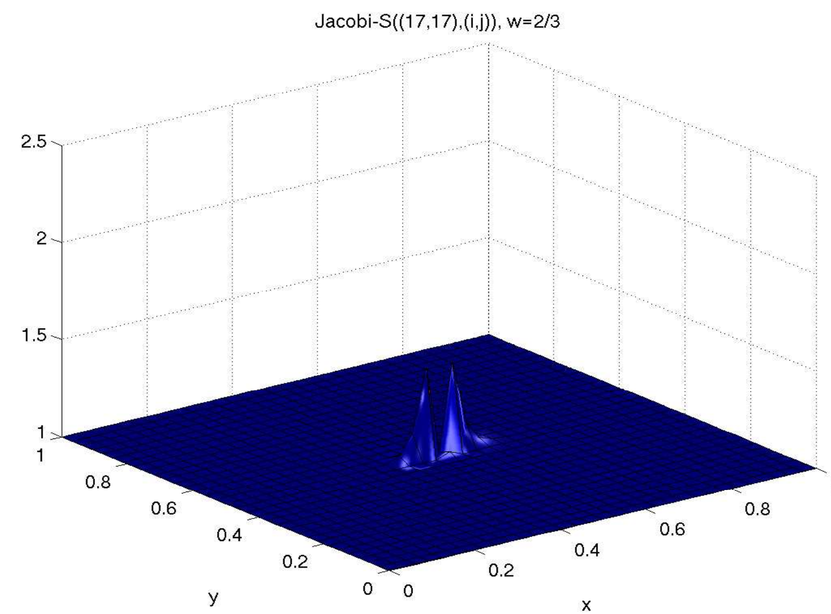
- Weighted Jacobi, 9 steps:





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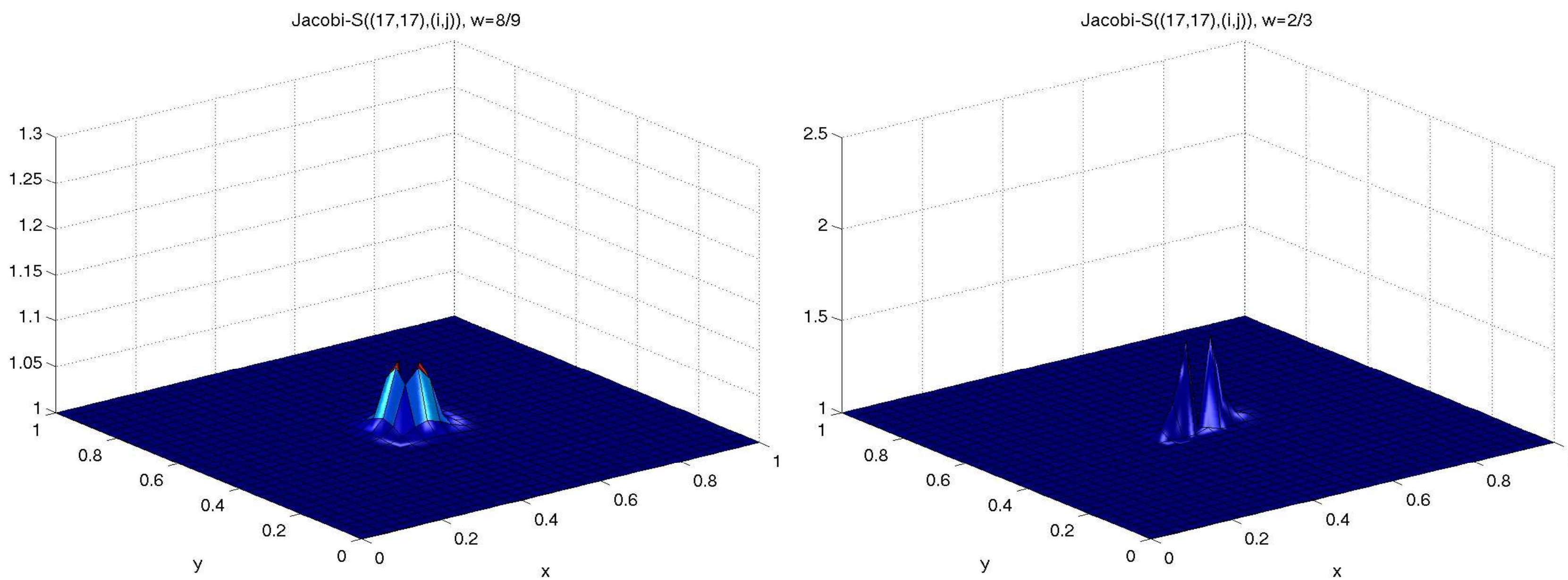
- Weighted Jacobi, 10 steps:



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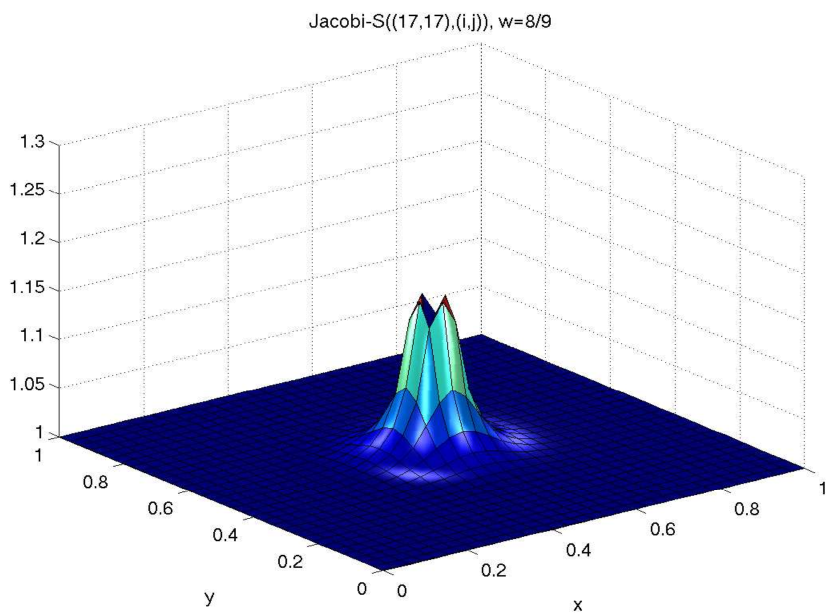
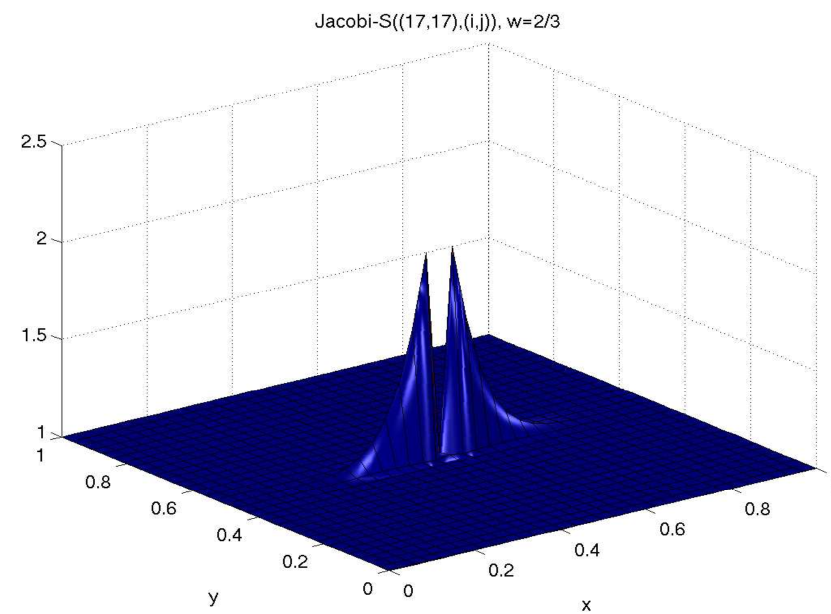
- Weighted Jacobi, 50 steps:





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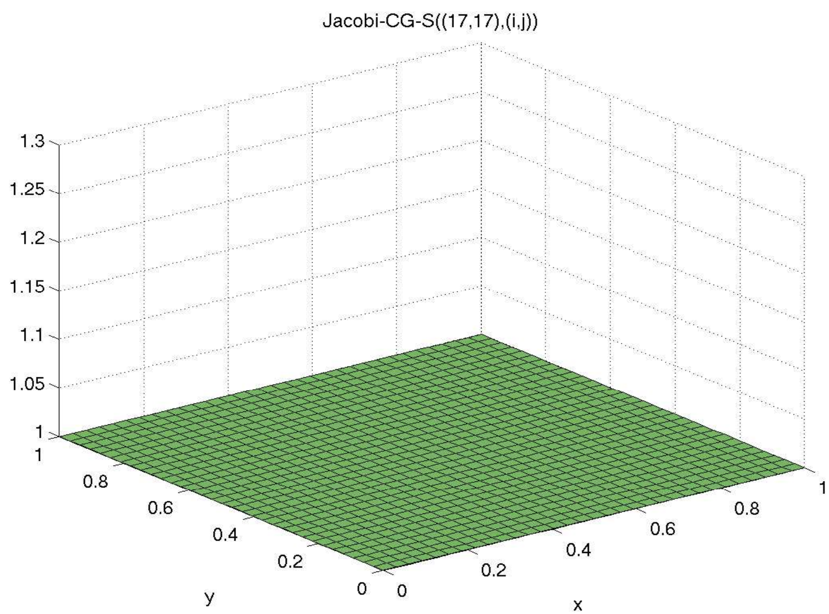

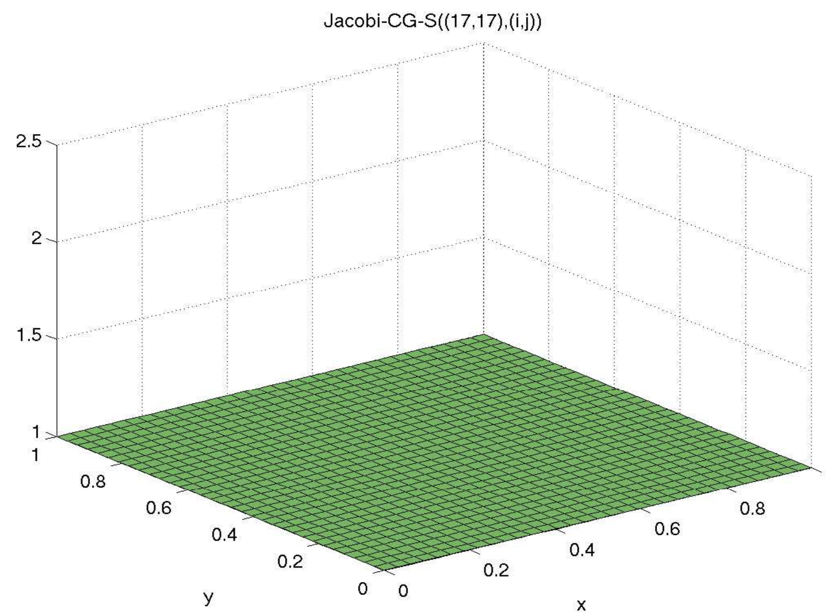

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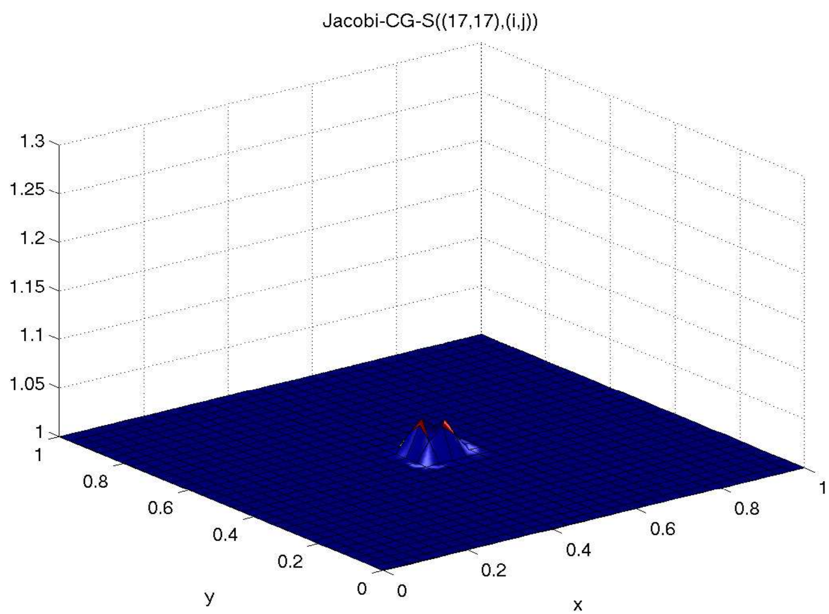

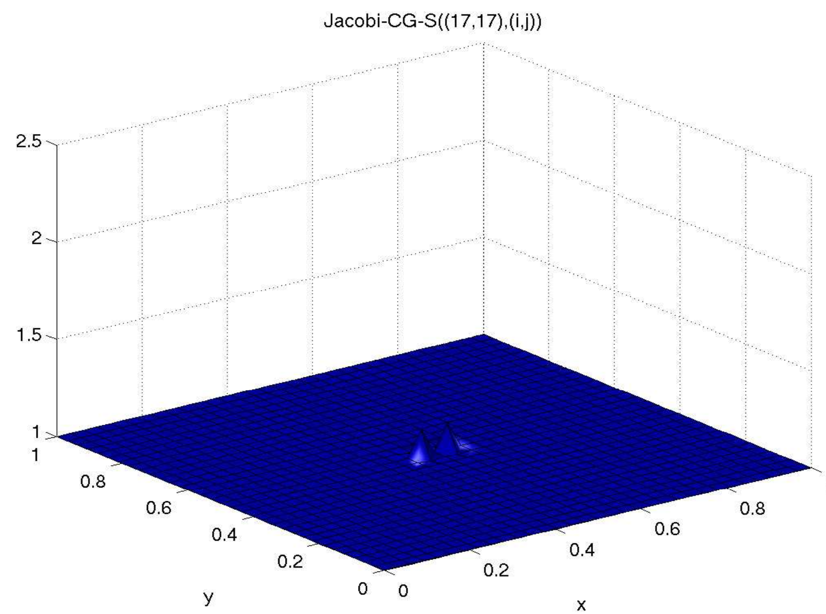

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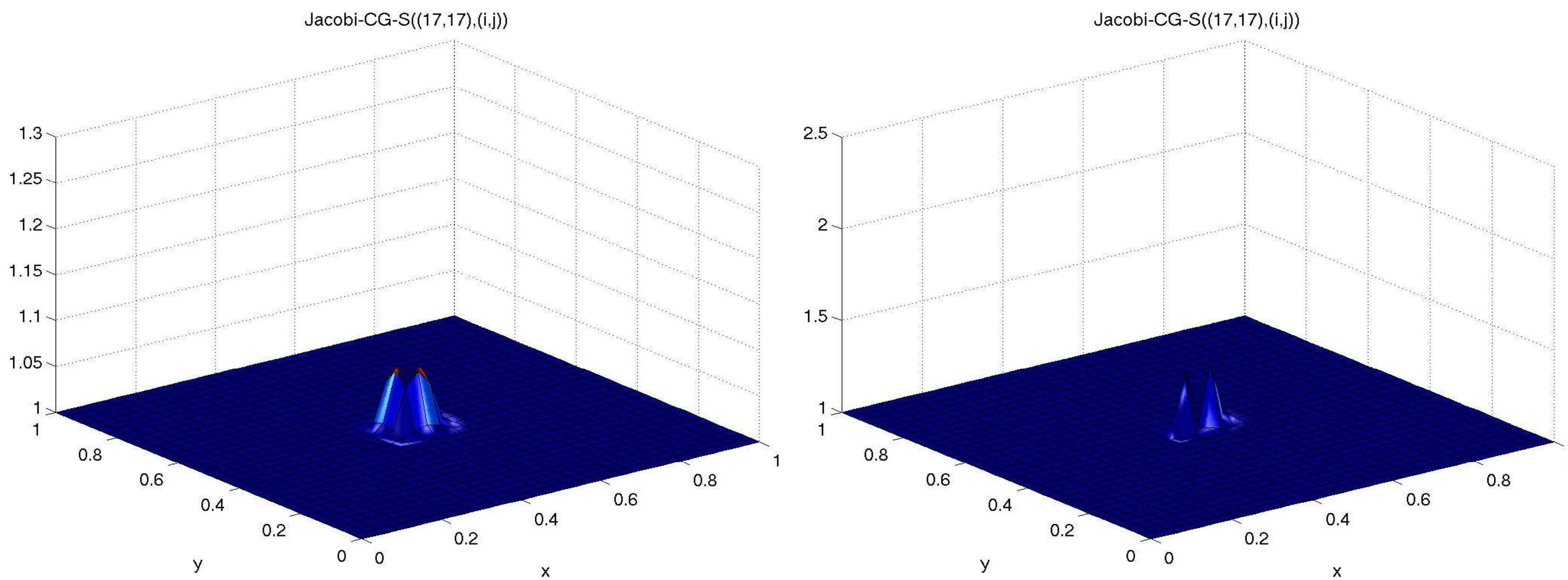

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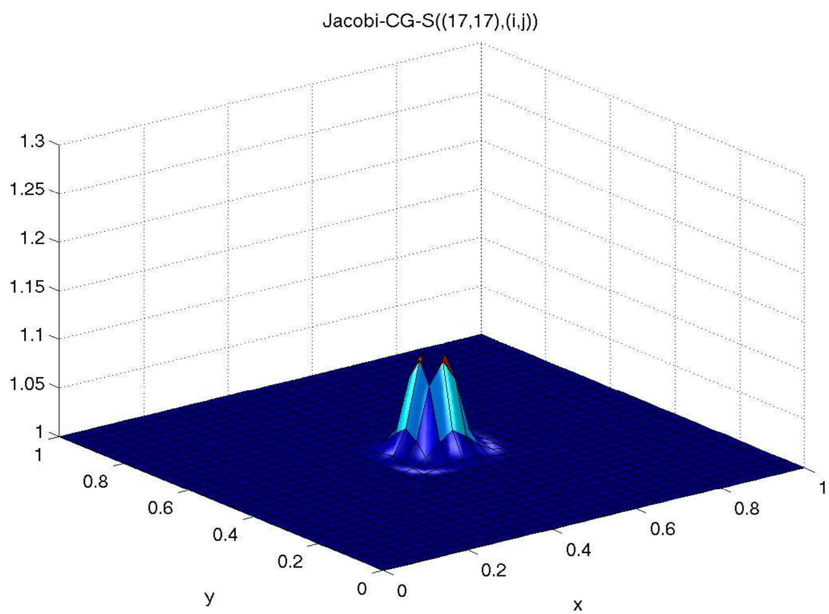

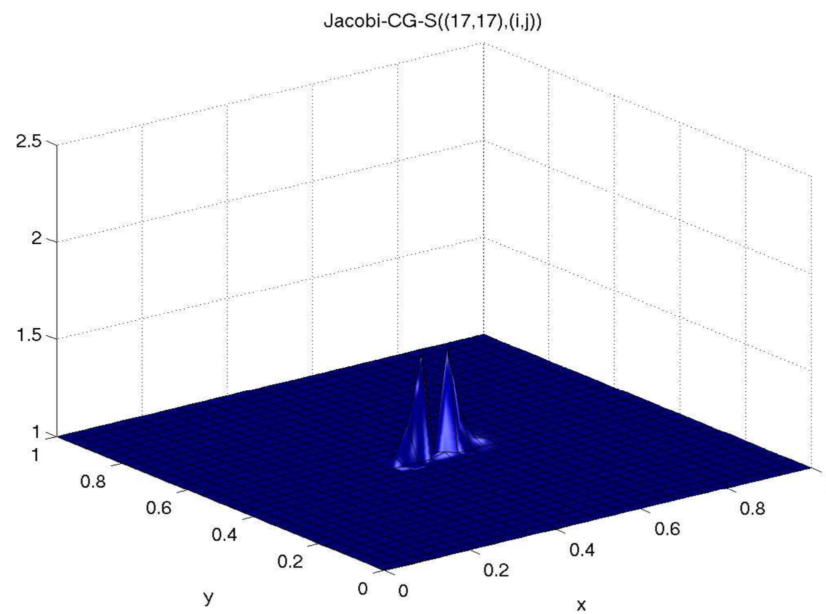

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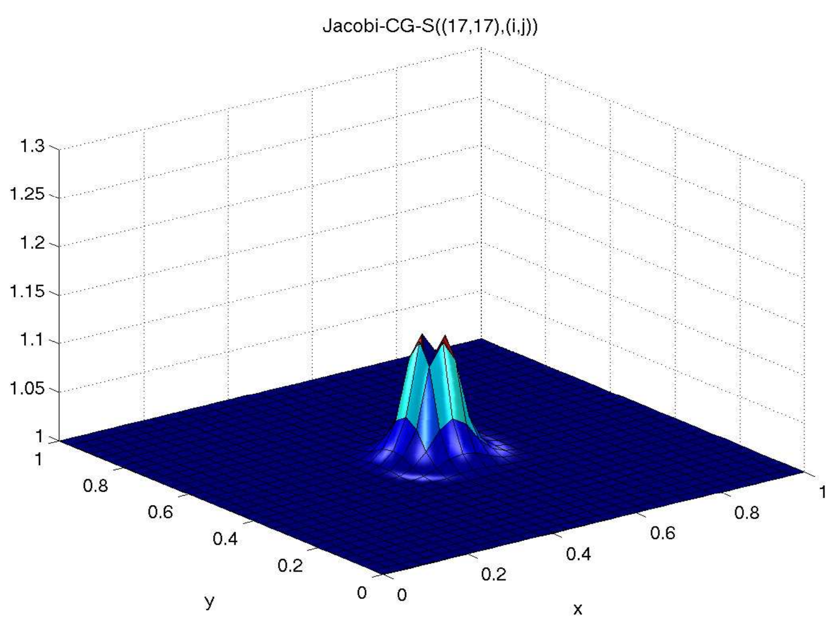

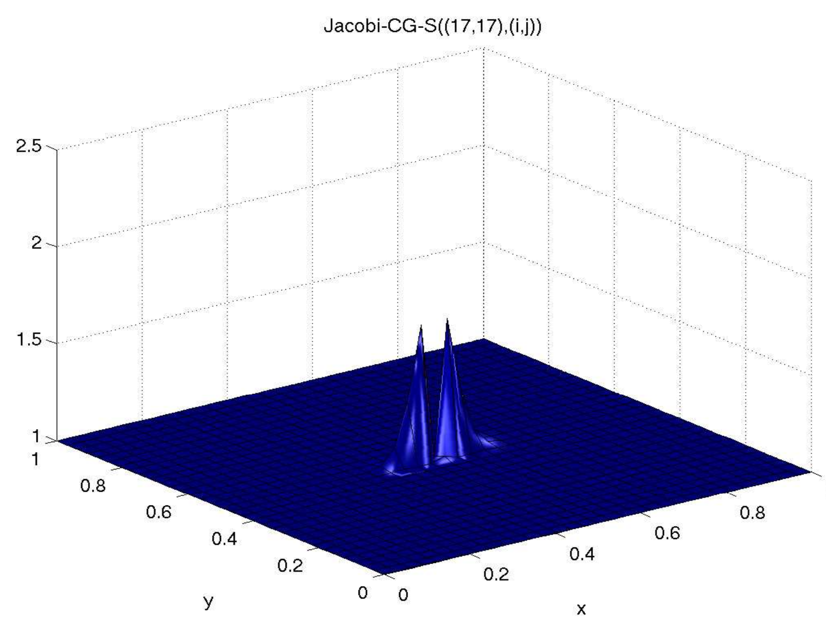

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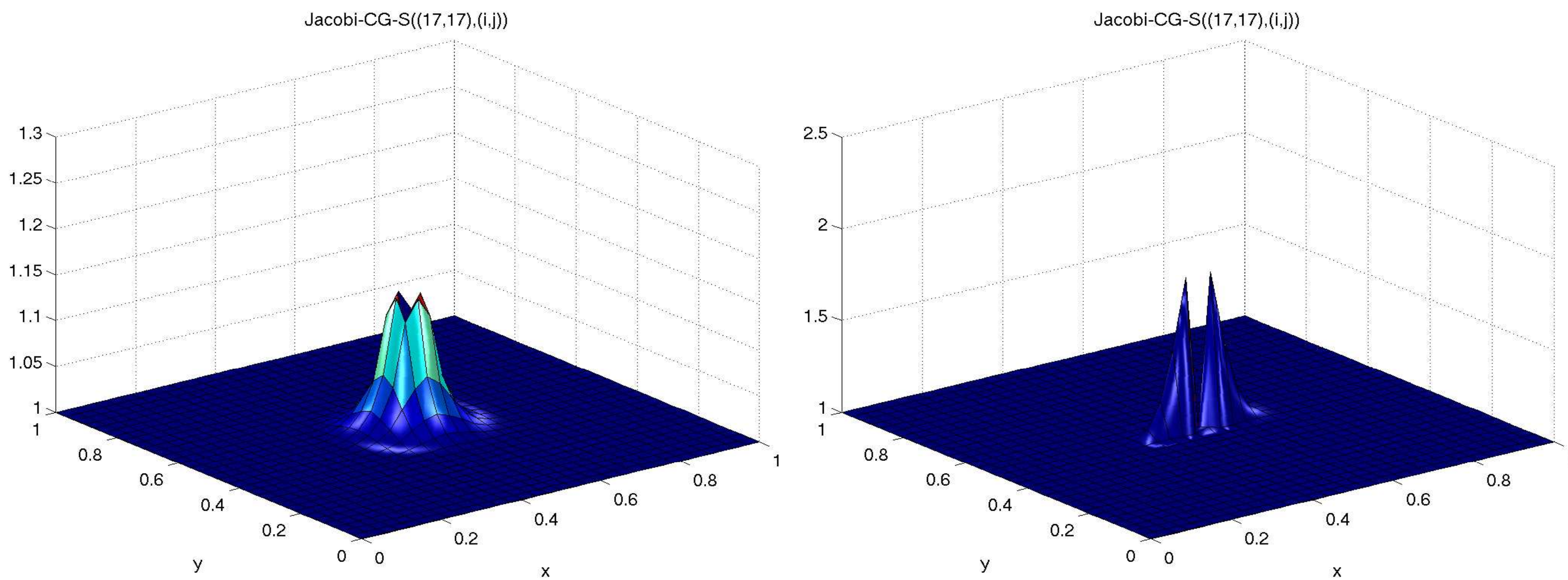

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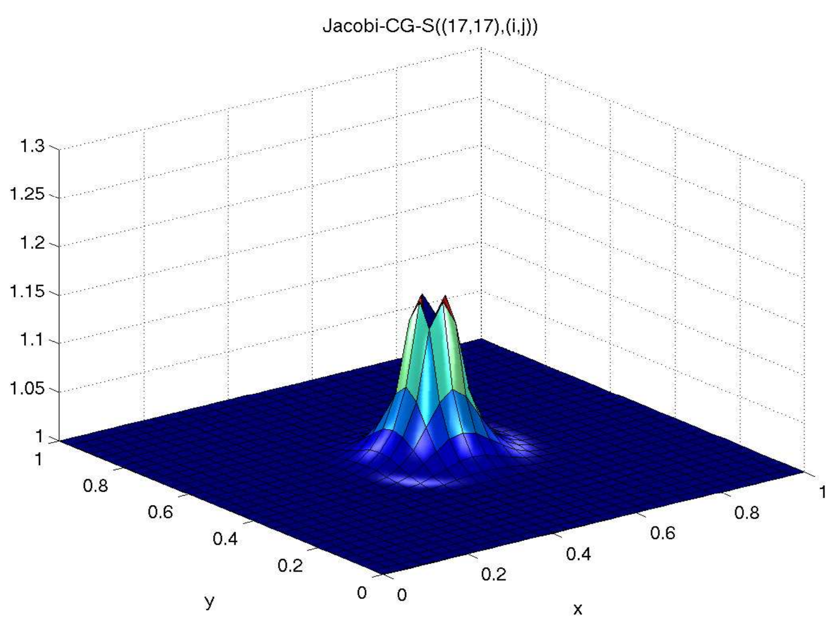

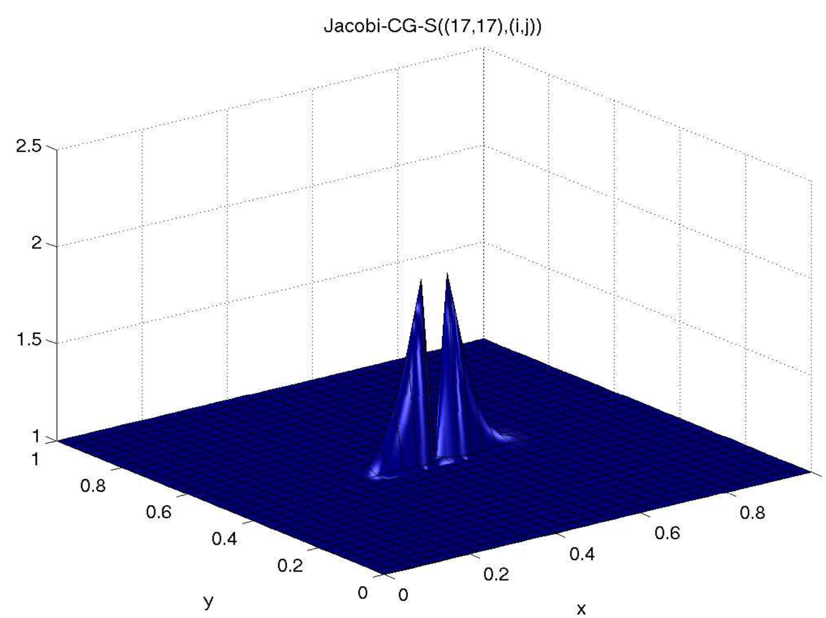

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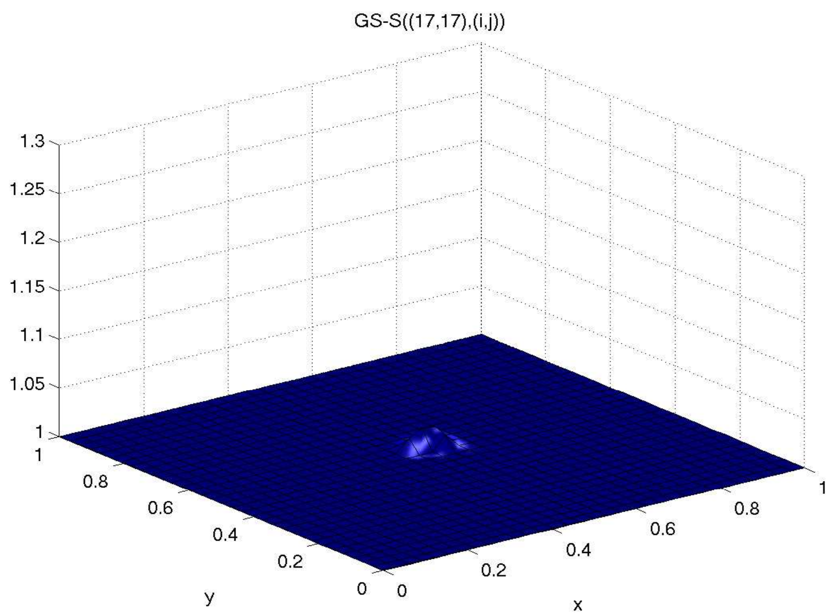

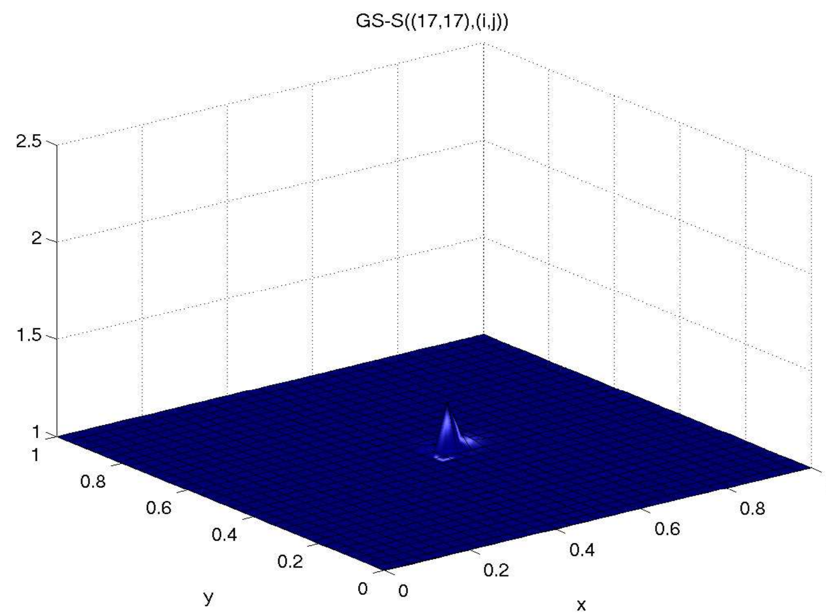

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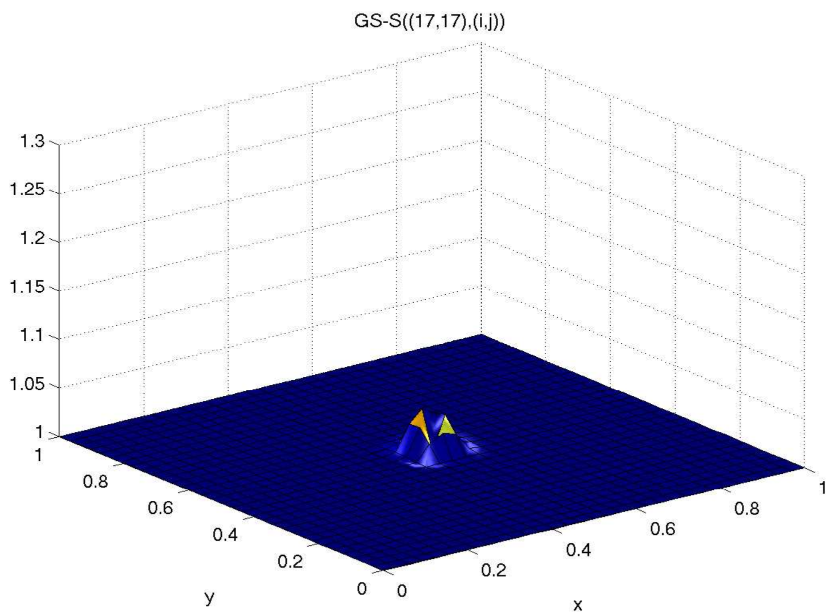

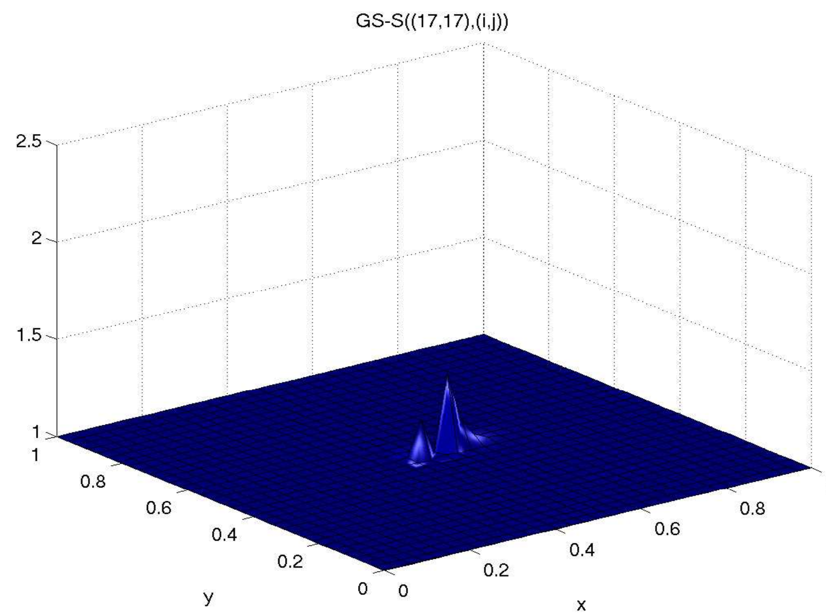

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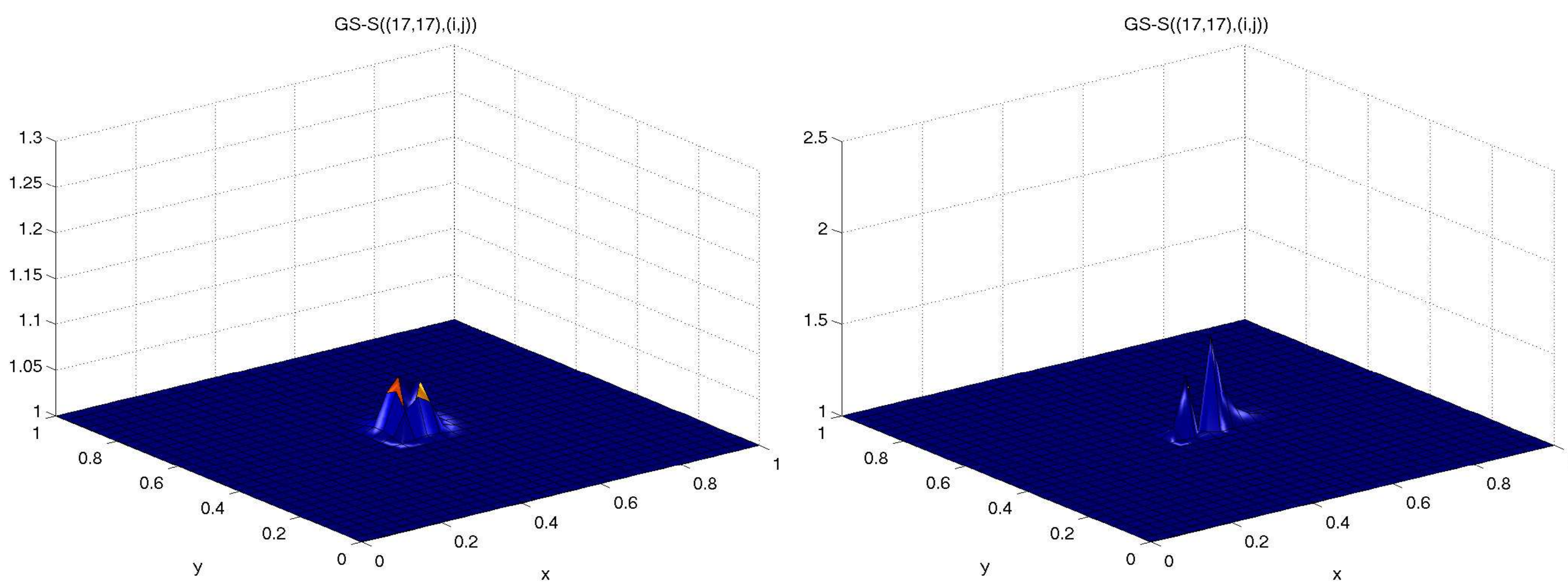

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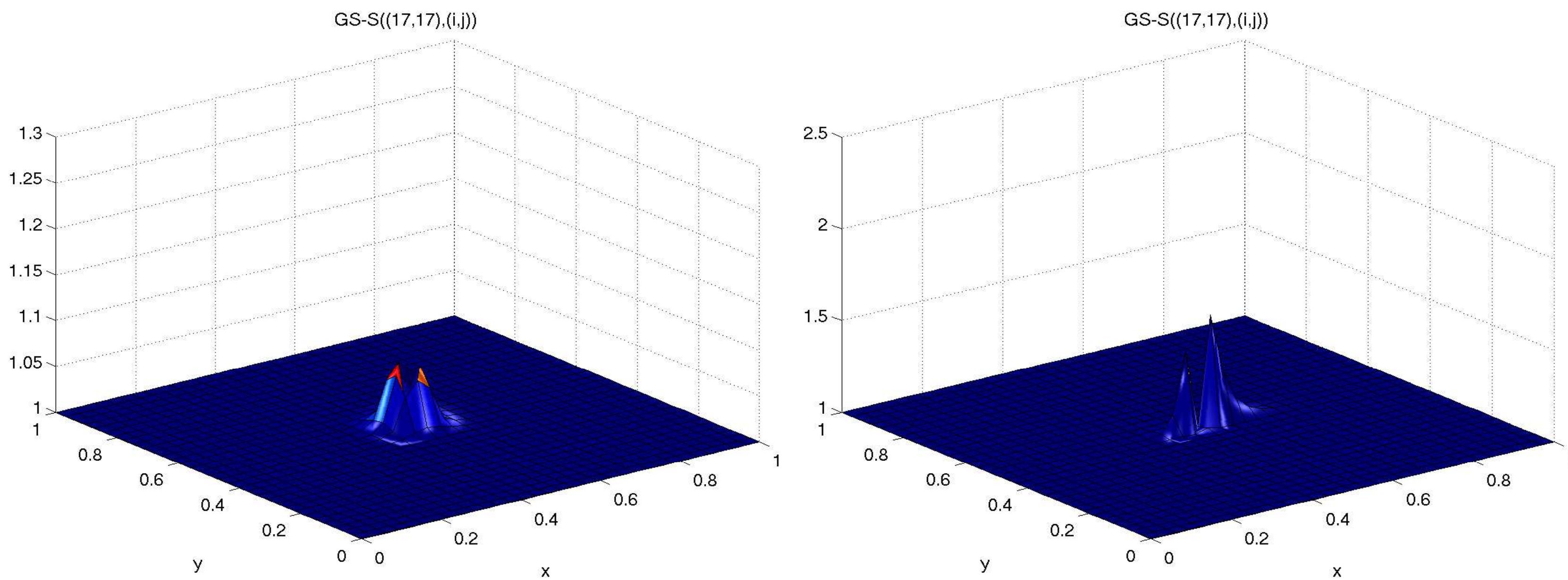

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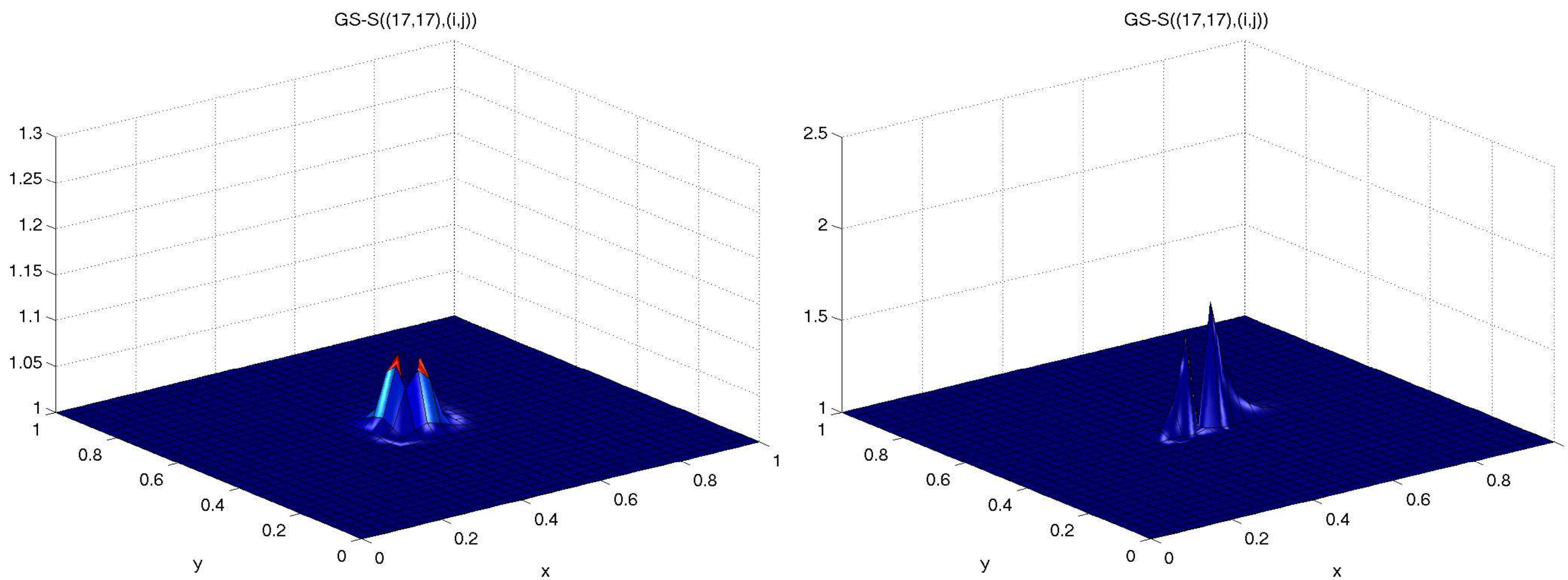

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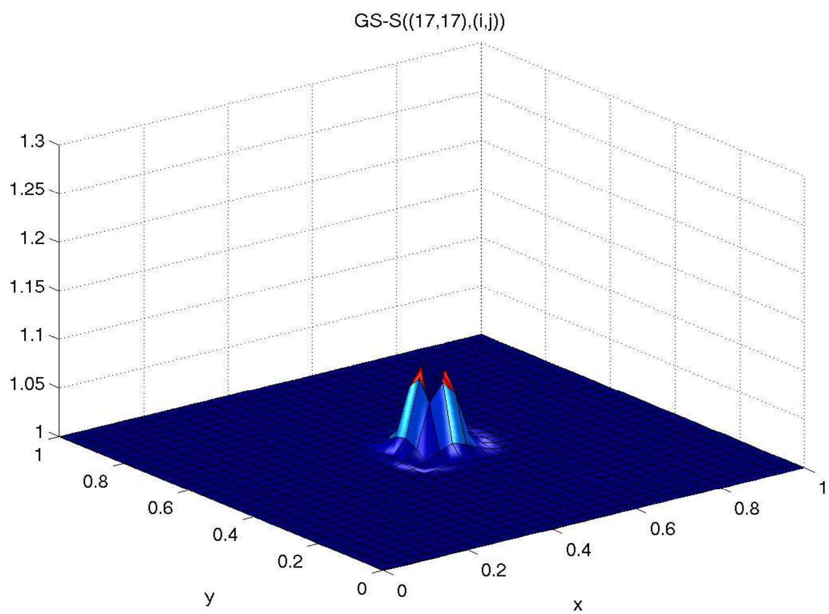

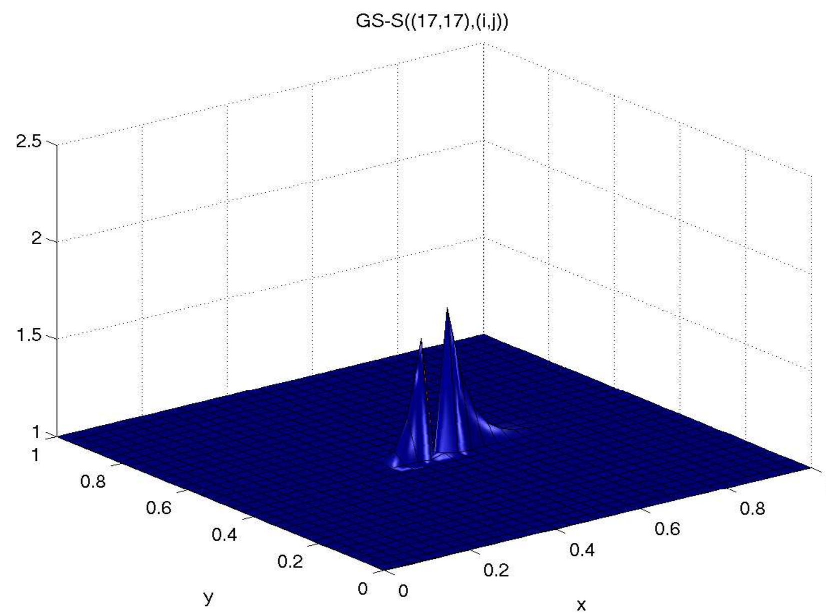

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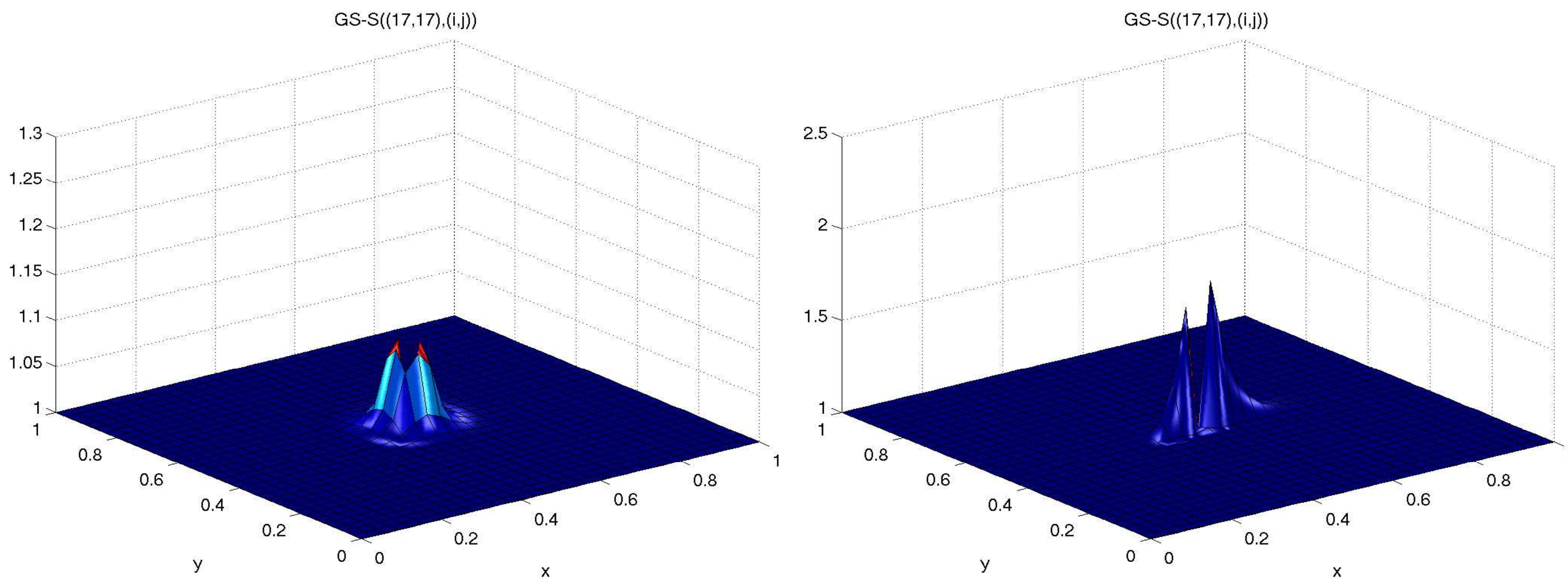

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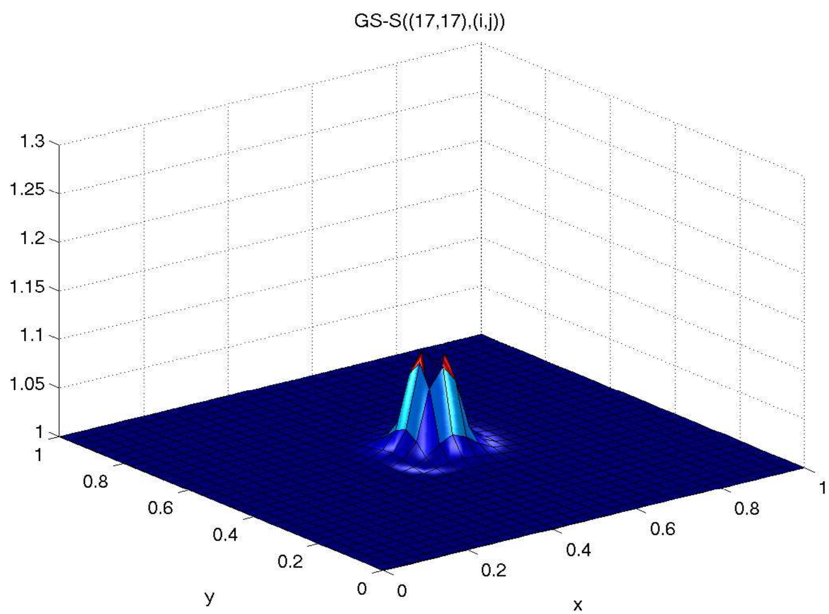
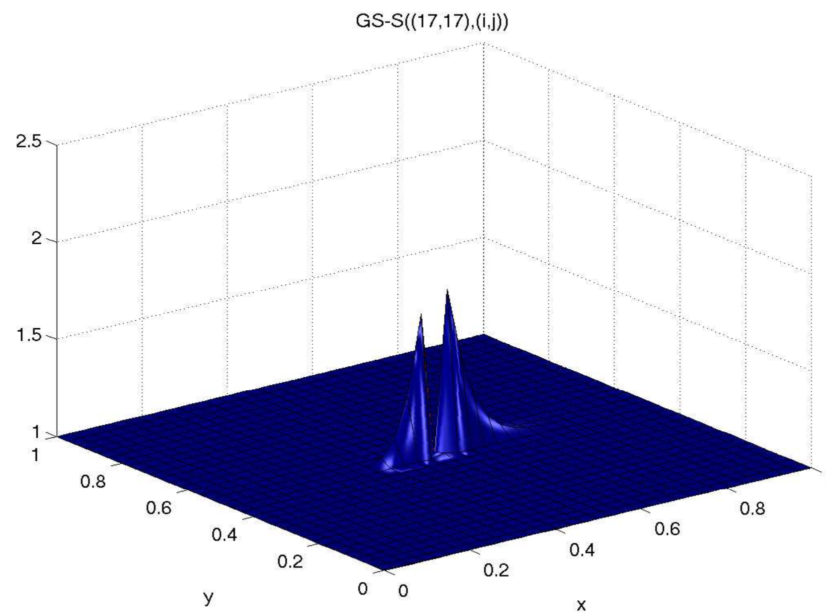
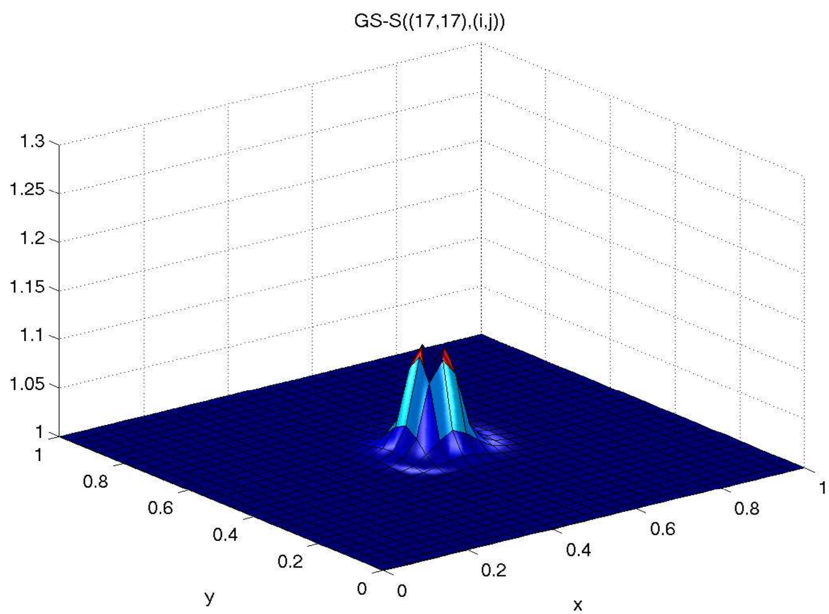
- Gauss-Seidel, 8 steps:

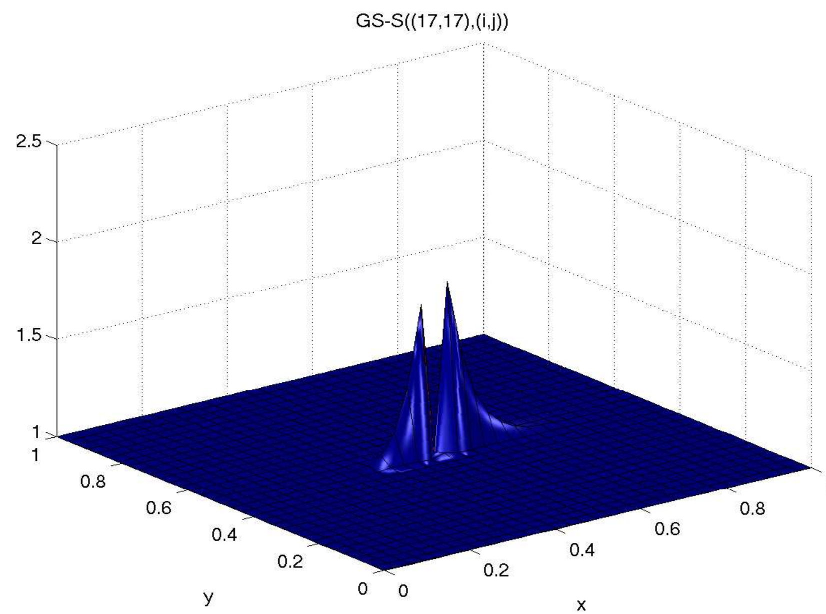




Approximating S_{ij}

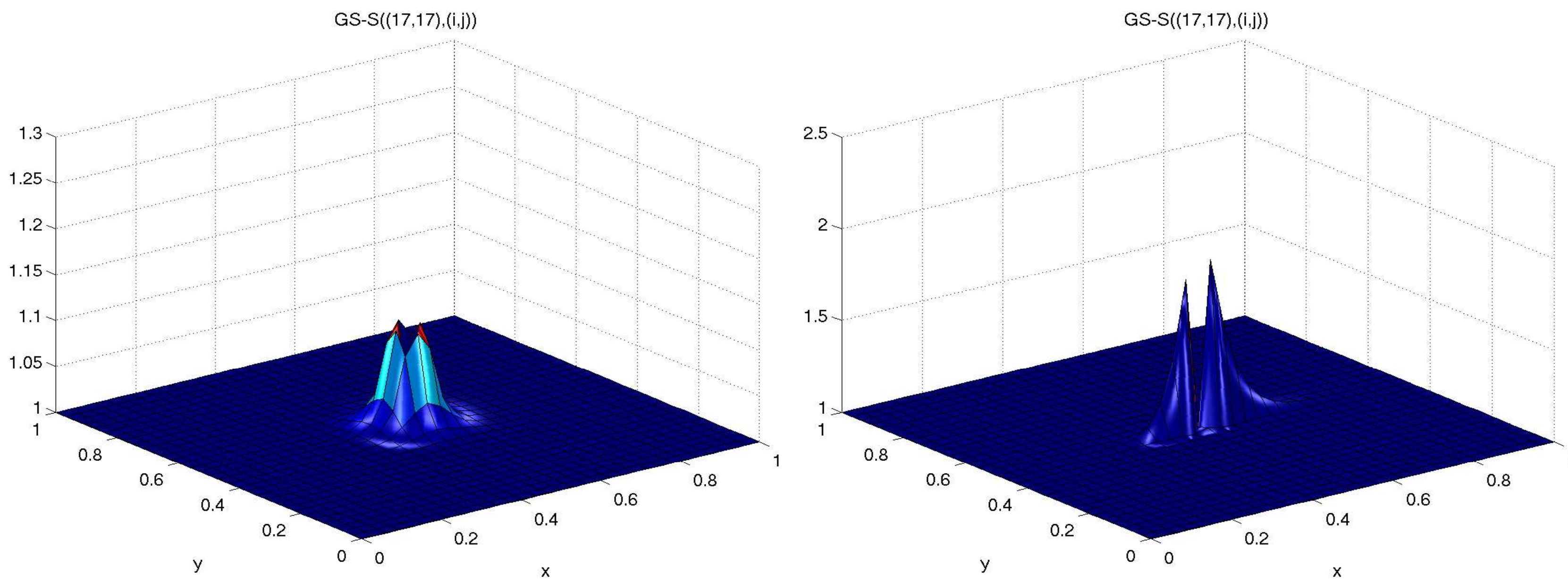
- Can we get useful, local approximations to $(A^h)^{-1}_{ij}$ and, thus, S_{ij}?
- Apply (localized) relaxation to $A^h G^{(i)} = e^{(i)}$
- Gauss-Seidel, 9 steps:





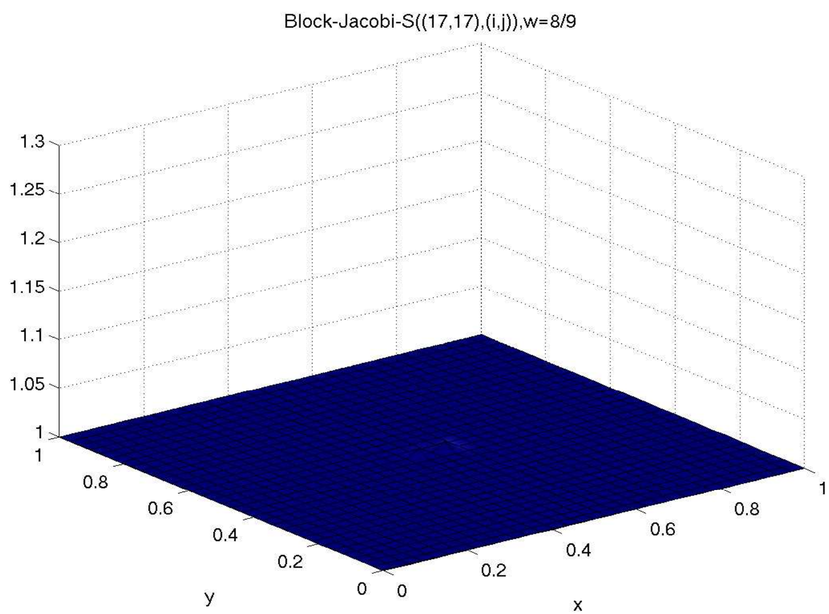
Approximating S_{ij}

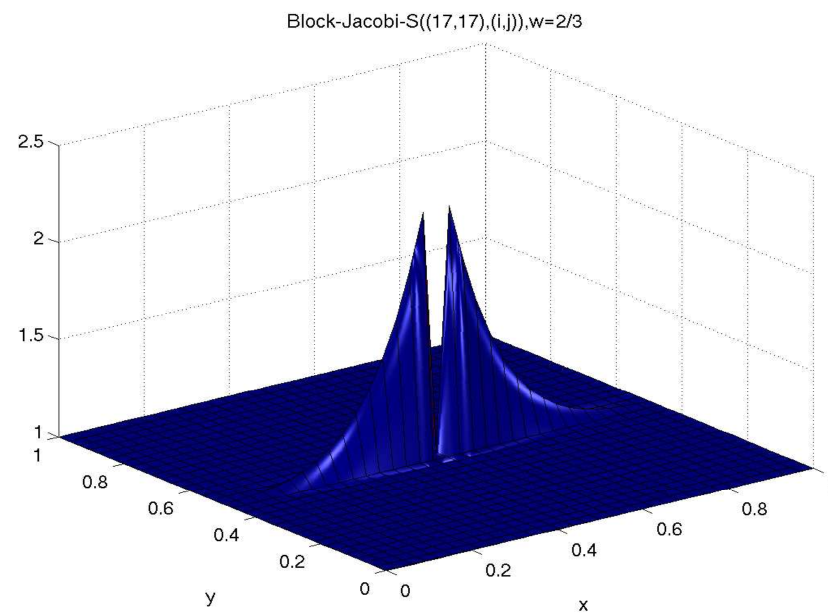
- Can we get useful, local approximations to $(A^h)^{-1}_{ij}$ and, thus, S_{ij}?
- Apply (localized) relaxation to $A^h G^{(i)} = e^{(i)}$
- Gauss-Seidel, 10 steps:



Approximating S_{ij}

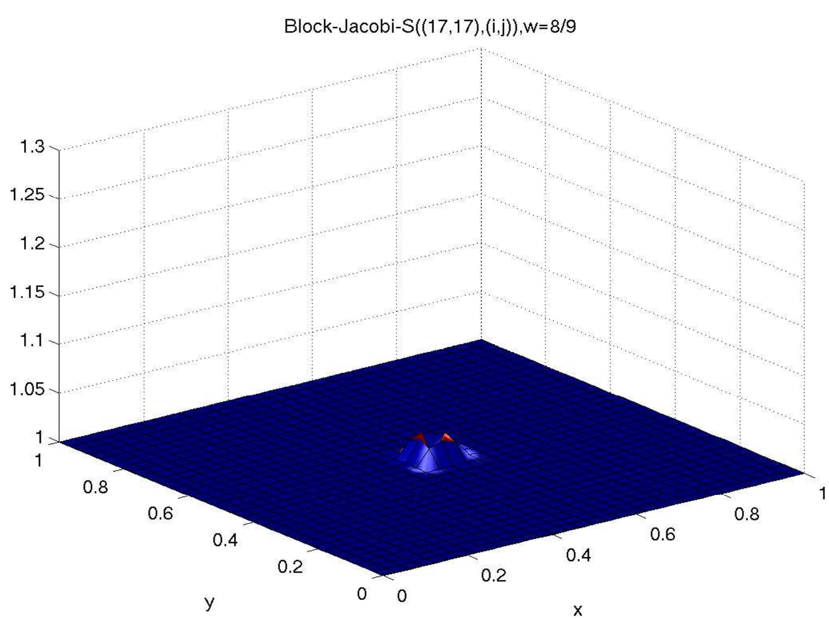
- Can we get useful, local approximations to $(A^h)^{-1}_{ij}$ and, thus, S_{ij}?
- Apply (localized) relaxation to $A^h G^{(i)} = e^{(i)}$
- Weighted Line-Jacobi, 1 step:

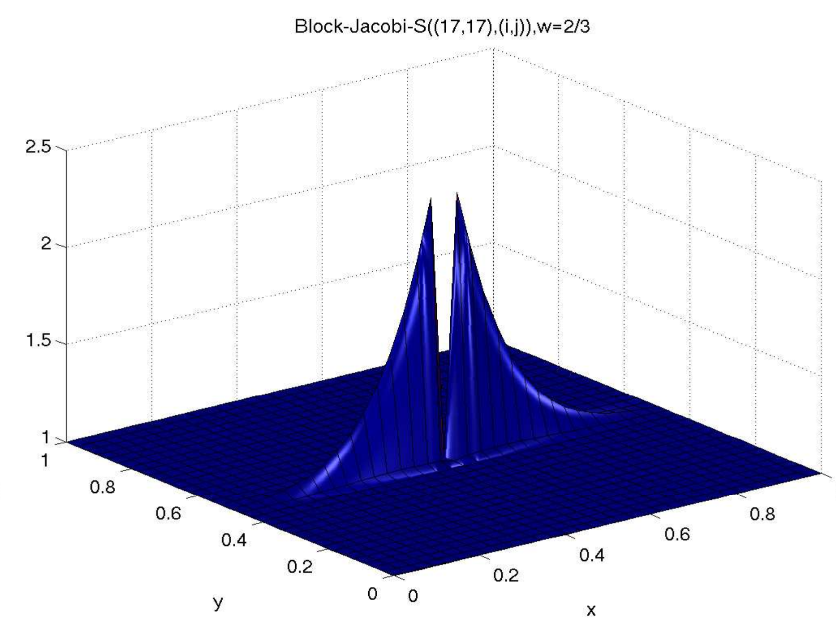




Approximating S_{ij}

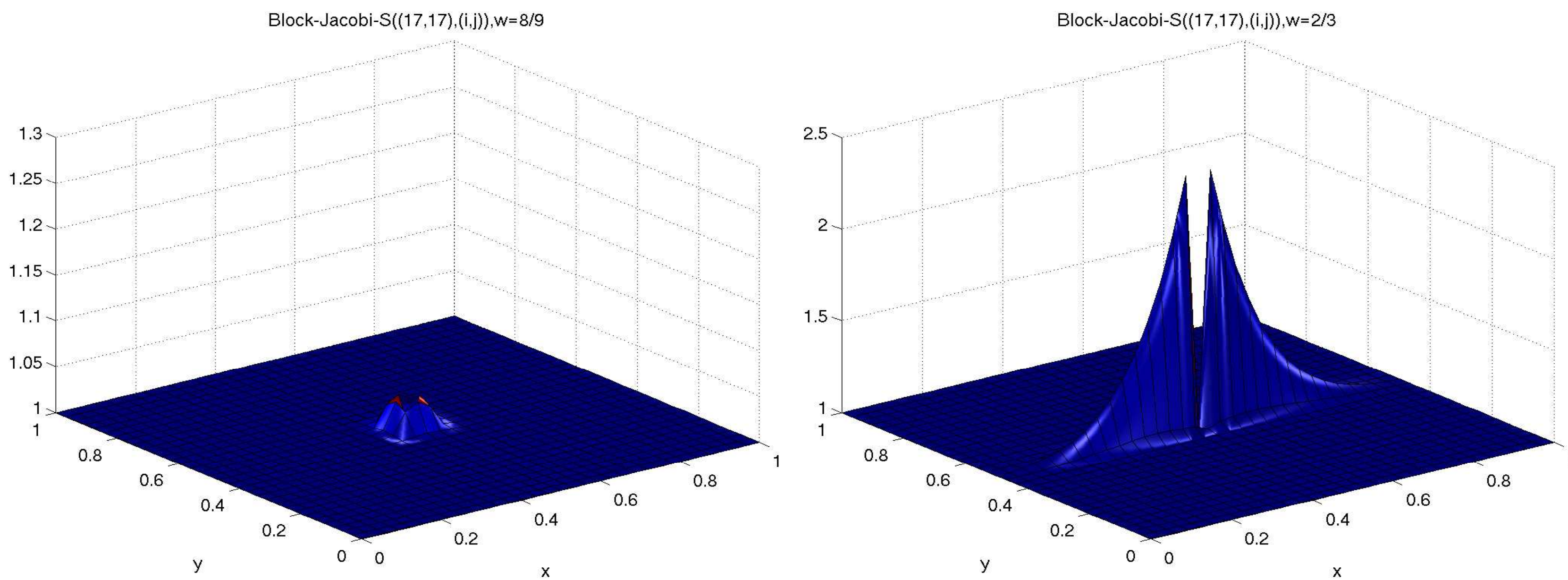
- Can we get useful, local approximations to $(A^h)^{-1}_{ij}$ and, thus, S_{ij}?
- Apply (localized) relaxation to $A^h G^{(i)} = e^{(i)}$
- Weighted Line-Jacobi, 2 steps:





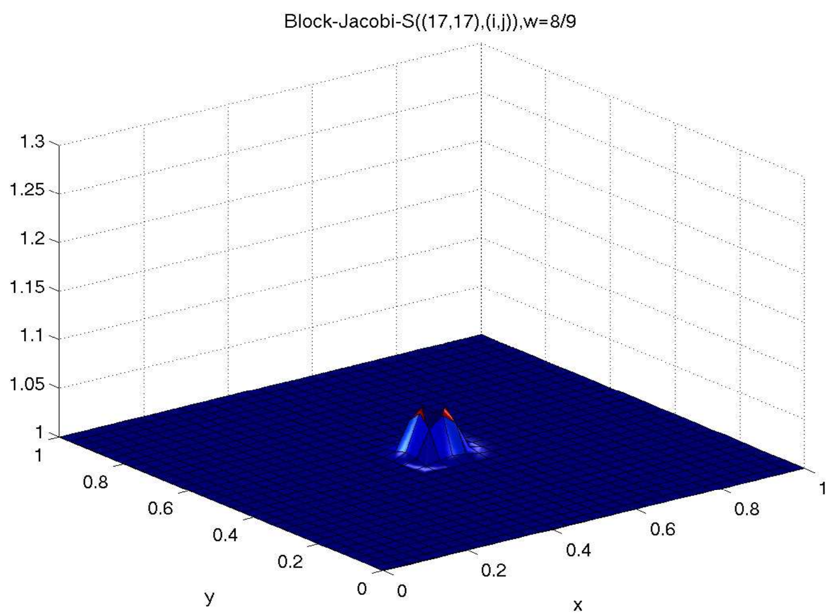
Approximating S_{ij}

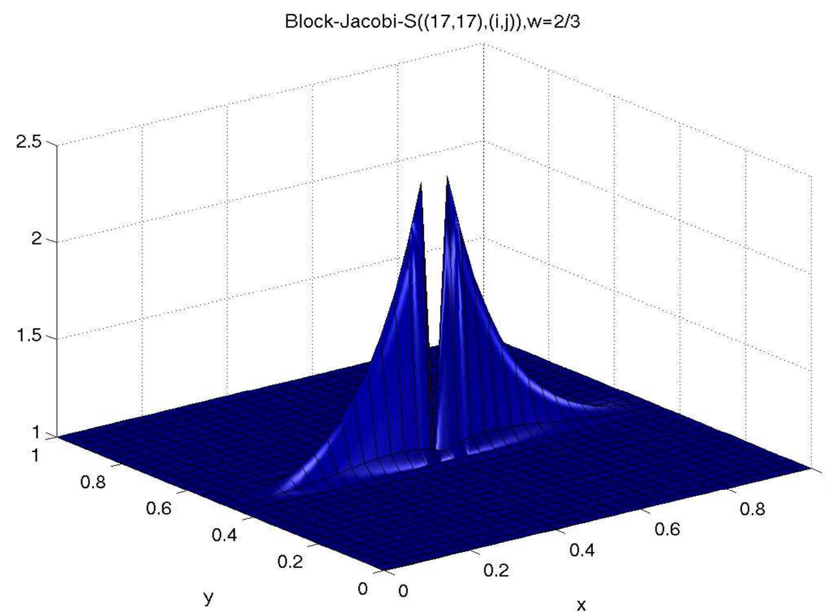
- Can we get useful, local approximations to $(A^h)^{-1}_{ij}$ and, thus, S_{ij}?
- Apply (localized) relaxation to $A^h G^{(i)} = e^{(i)}$
- Weighted Line-Jacobi, 3 steps:



Approximating S_{ij}

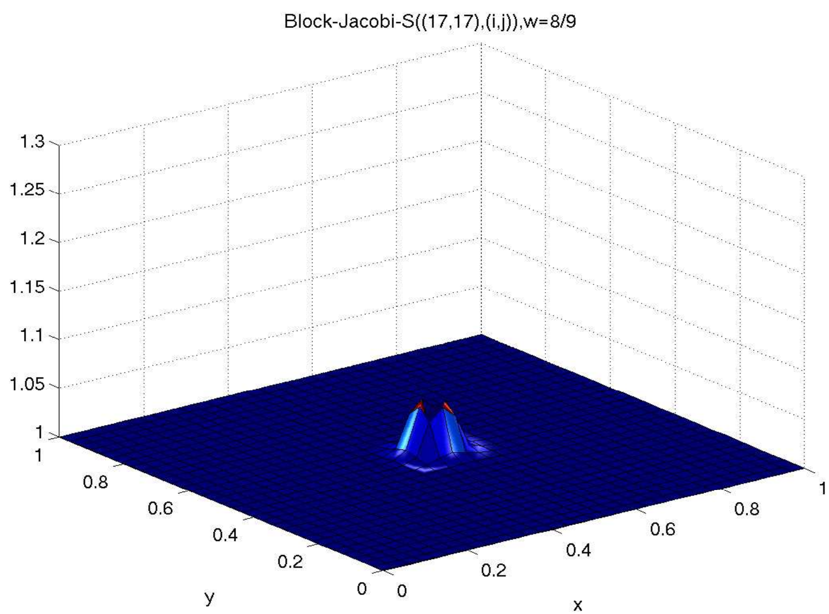
- Can we get useful, local approximations to $(A^h)^{-1}_{ij}$ and, thus, S_{ij}?
- Apply (localized) relaxation to $A^h G^{(i)} = e^{(i)}$
- Weighted Line-Jacobi, 4 steps:

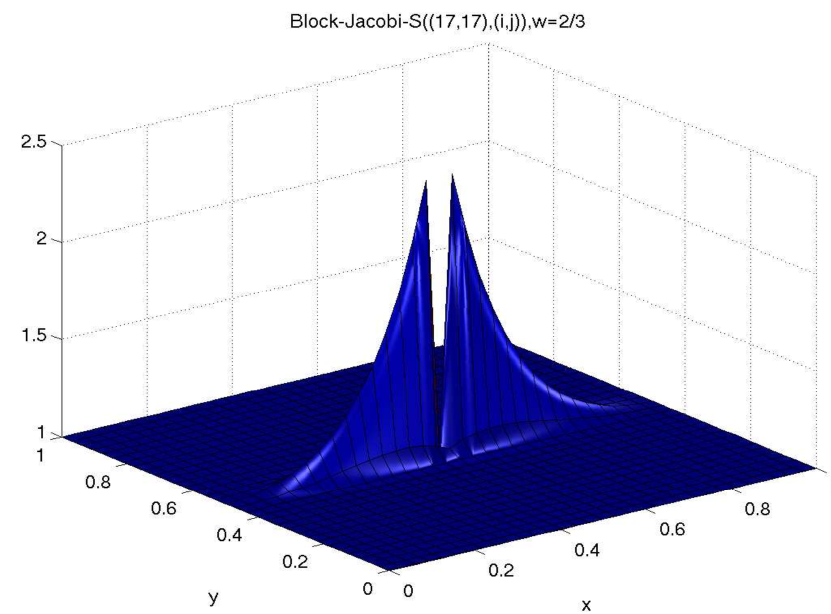




Approximating S_{ij}

- Can we get useful, local approximations to $(A^h)^{-1}_{ij}$ and, thus, S_{ij}?
- Apply (localized) relaxation to $A^h G^{(i)} = e^{(i)}$
- Weighted Line-Jacobi, 5 steps:





Choosing C

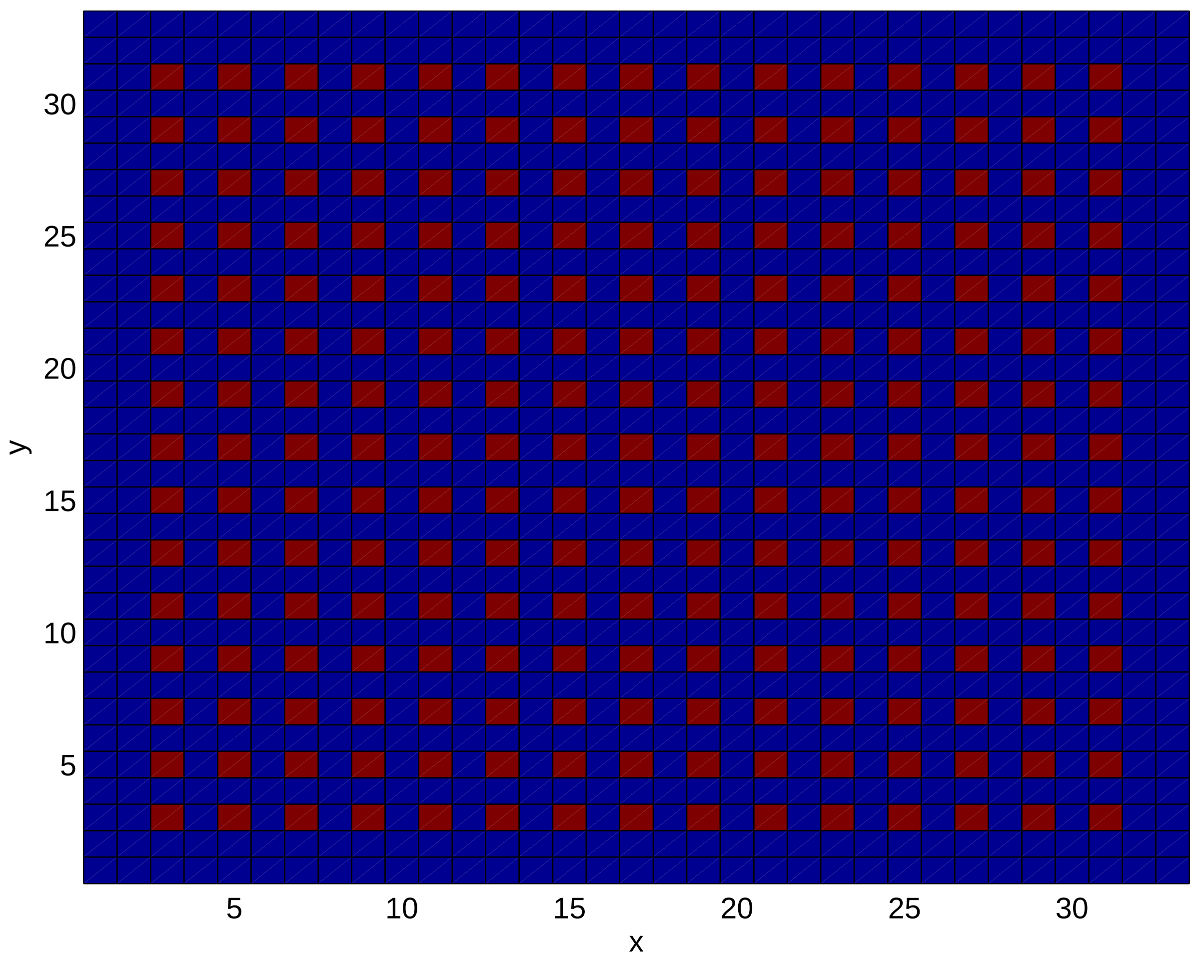
- For point i, $\{S_{ij}\}$ are now measures of strengths of connection
- We now say i strongly depends on j if $(A^h)_{ij} \neq 0$ and

$$S_{ij} - 1 \geq \theta \max_{k \neq i} \{S_{ik} - 1\}$$

- For now, $\theta = 0.25$ seems to work fine
- Coarse grid selection now accomplished by taking a maximal independent subset of the graph of strong connections

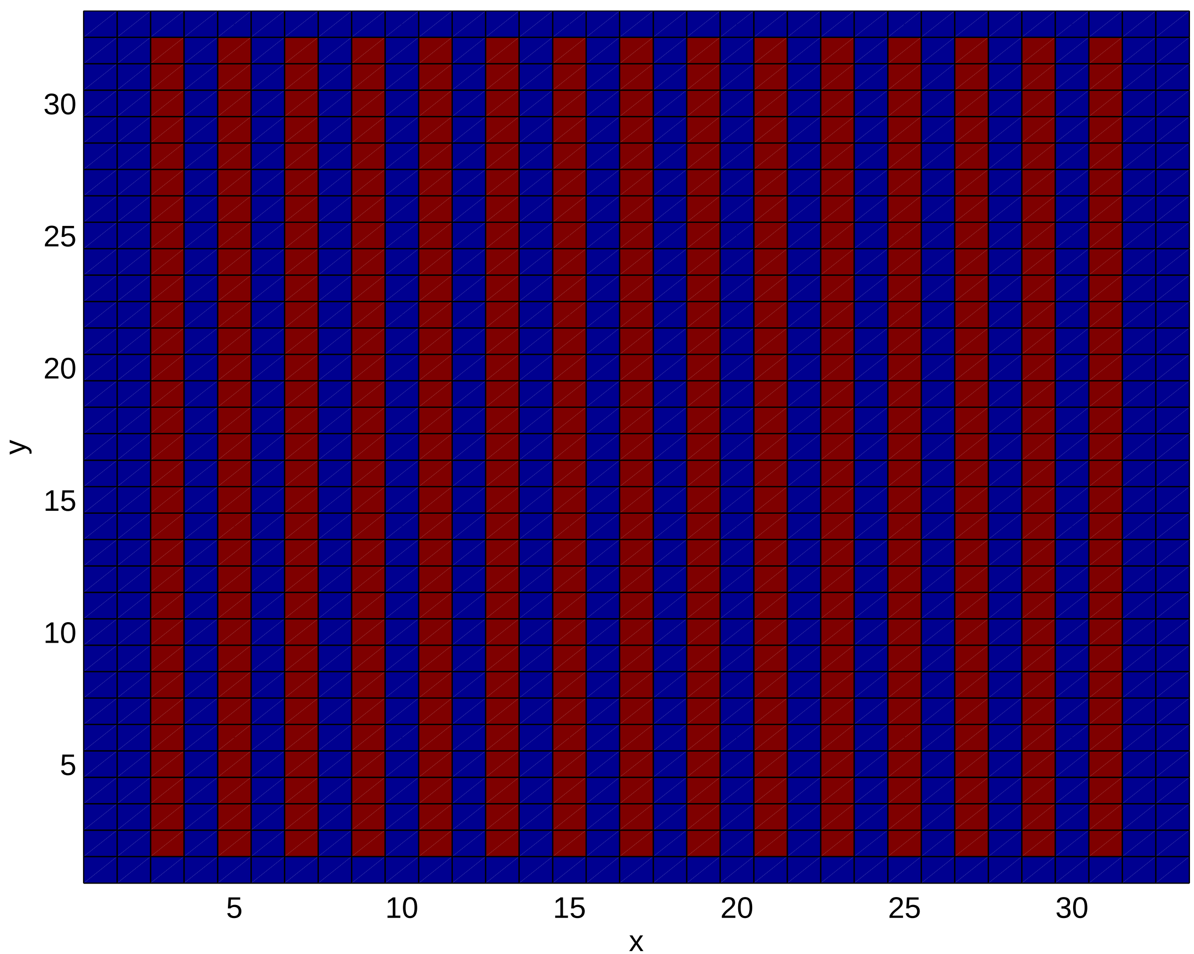
Choices of coarse grids

- $-u_{xx} - u_{yy} = f$, Dirichlet BCs
- 32×32 bilinear finite element grid
- 5 Steps Jacobi-Preconditioned CG to determine S_i



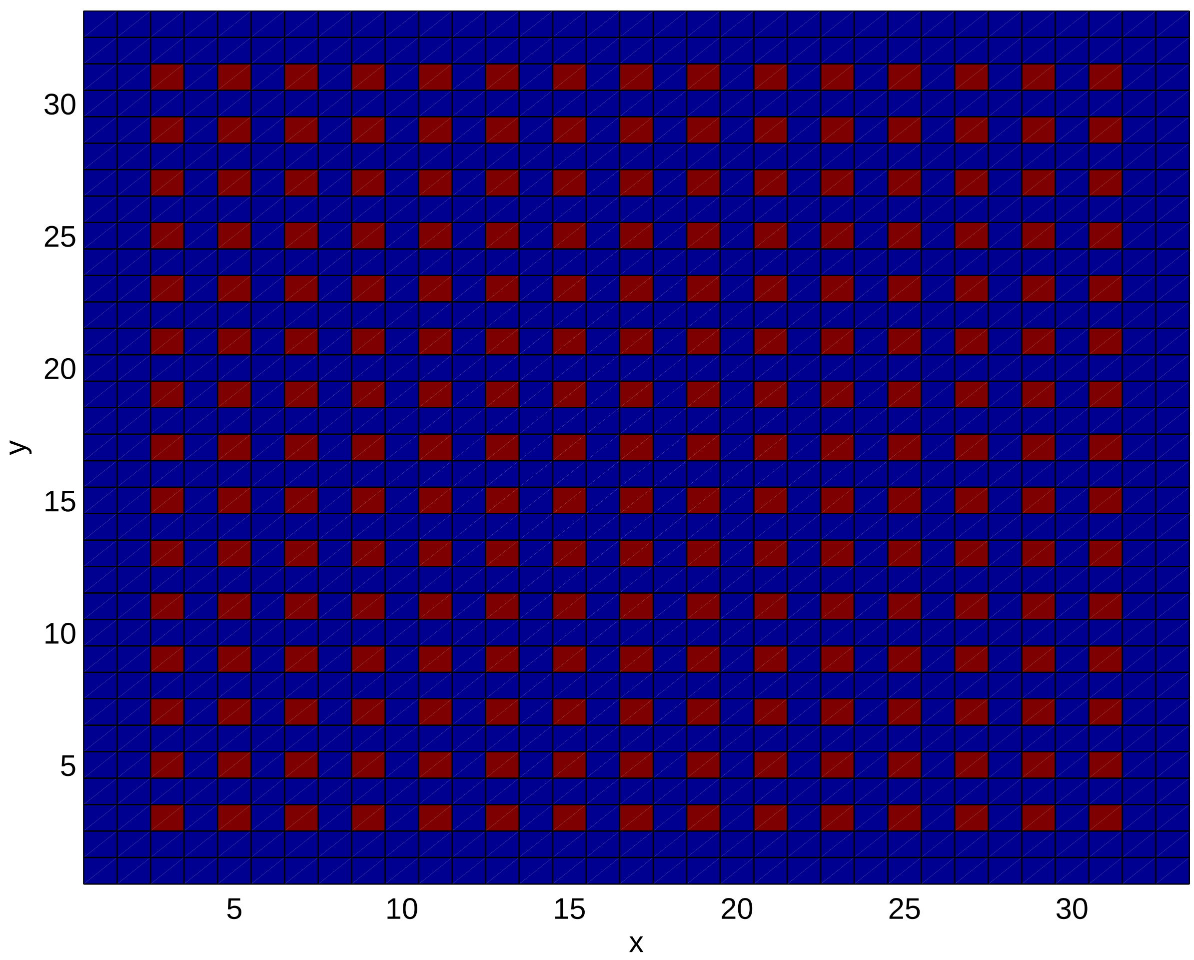
Choices of coarse grids

- $-u_{xx} - 0.01 u_{yy} = f$, Dirichlet BCs
- 32×32 bilinear finite element grid
- 5 Steps Jacobi-Preconditioned CG to determine S_i



Choices of coarse grids

- $-u_{xx} - u_{yy} = f$, Dirichlet BCs
- 32×32 bilinear finite element grid, $A^h \rightarrow DA^hD$, $d_{ii} = 10^{5r_i}$
- 5 Steps Jacobi-Preconditioned CG to determine S_i



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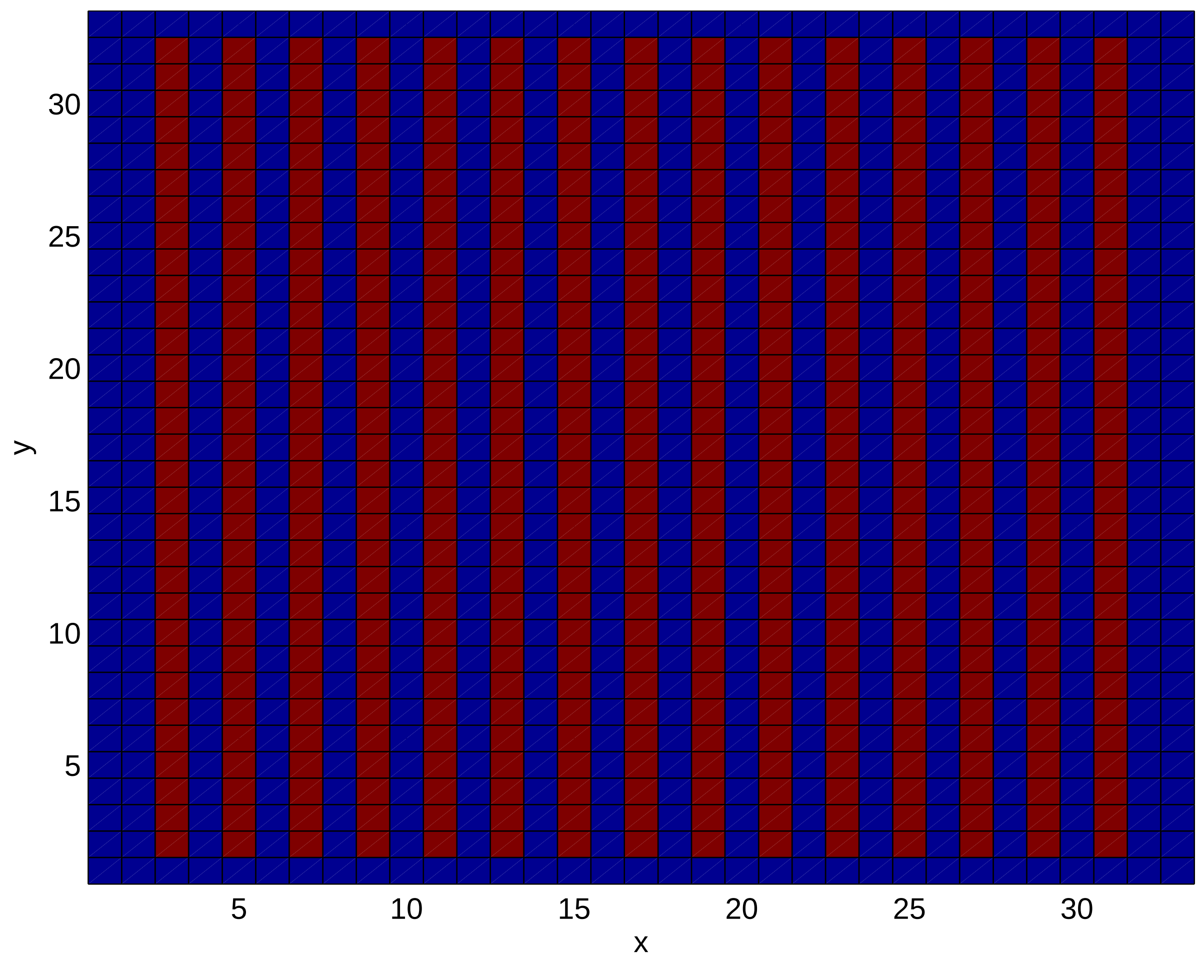
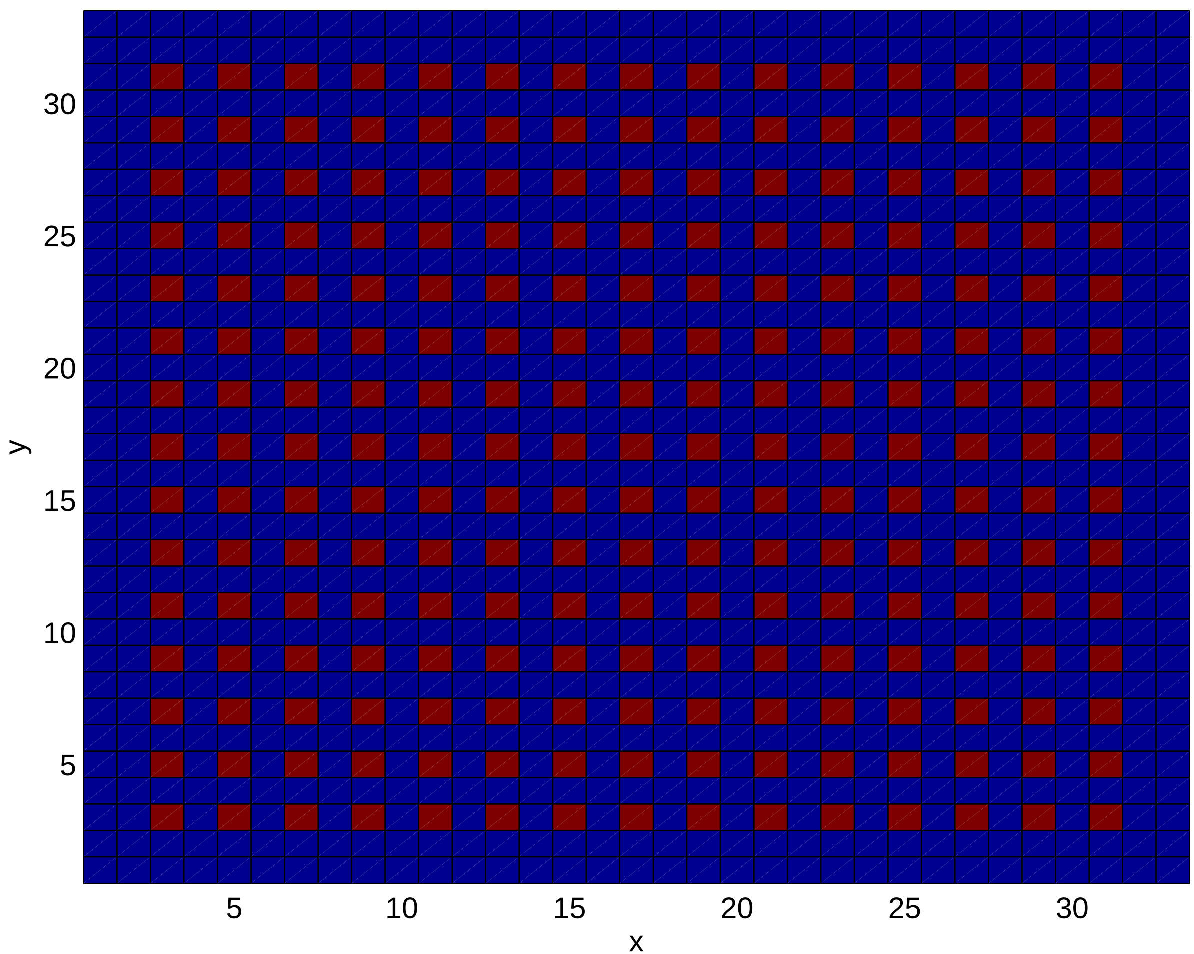


Choices of coarse grids

- $-u_{xx} - u_{yy} = f$, Dirichlet BCs
- 32×32 bilinear finite element grid
- 2 Steps Line-Jacobi to determine S_i



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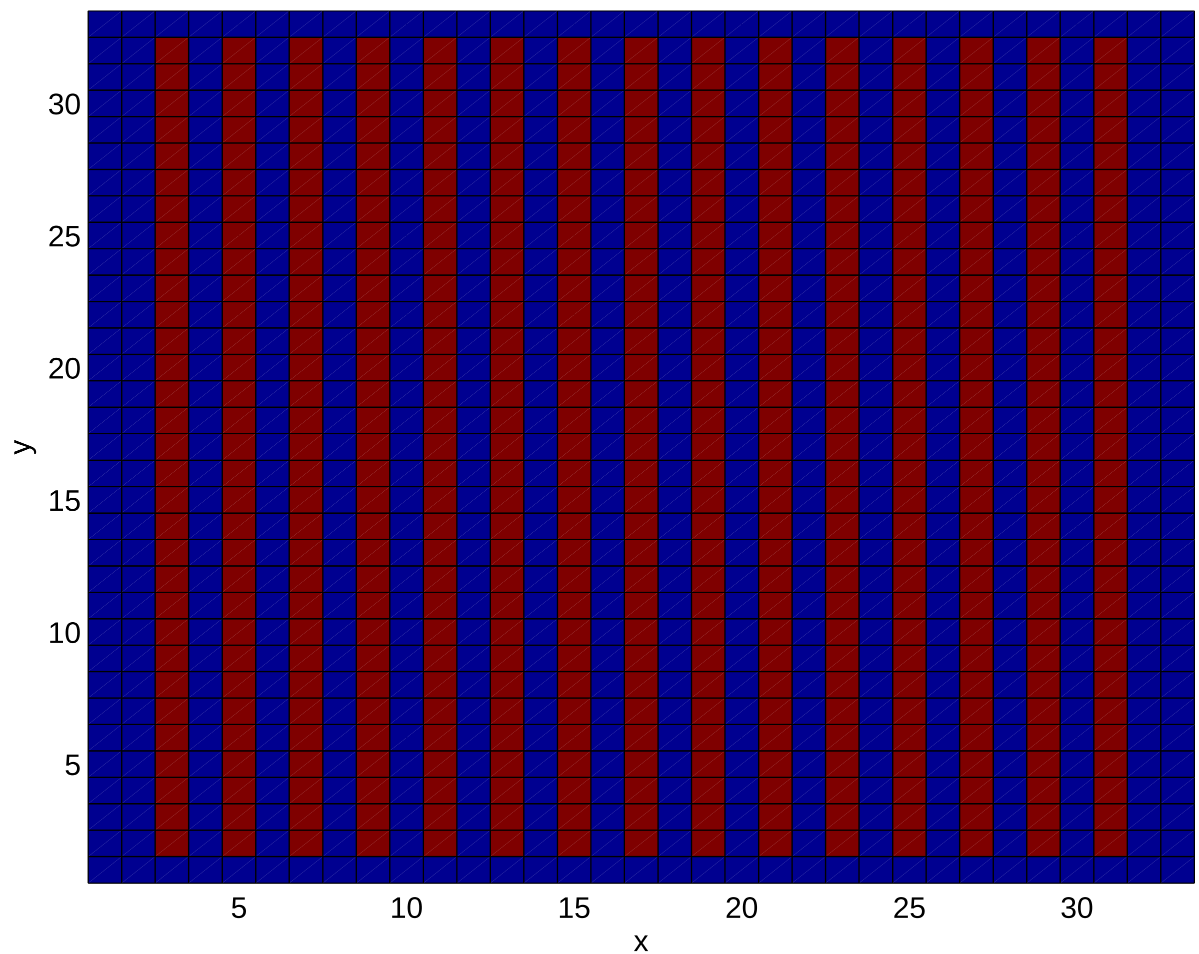

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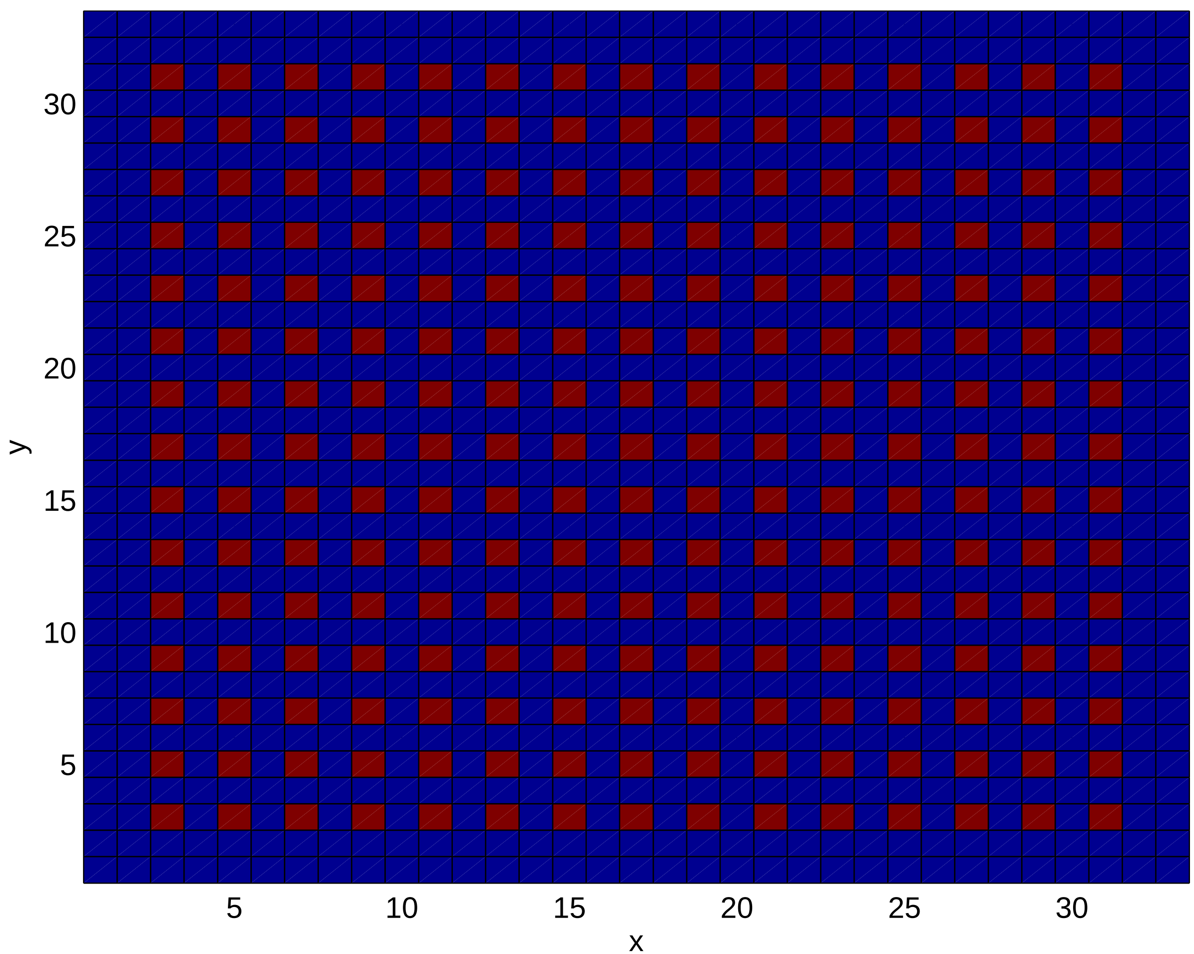

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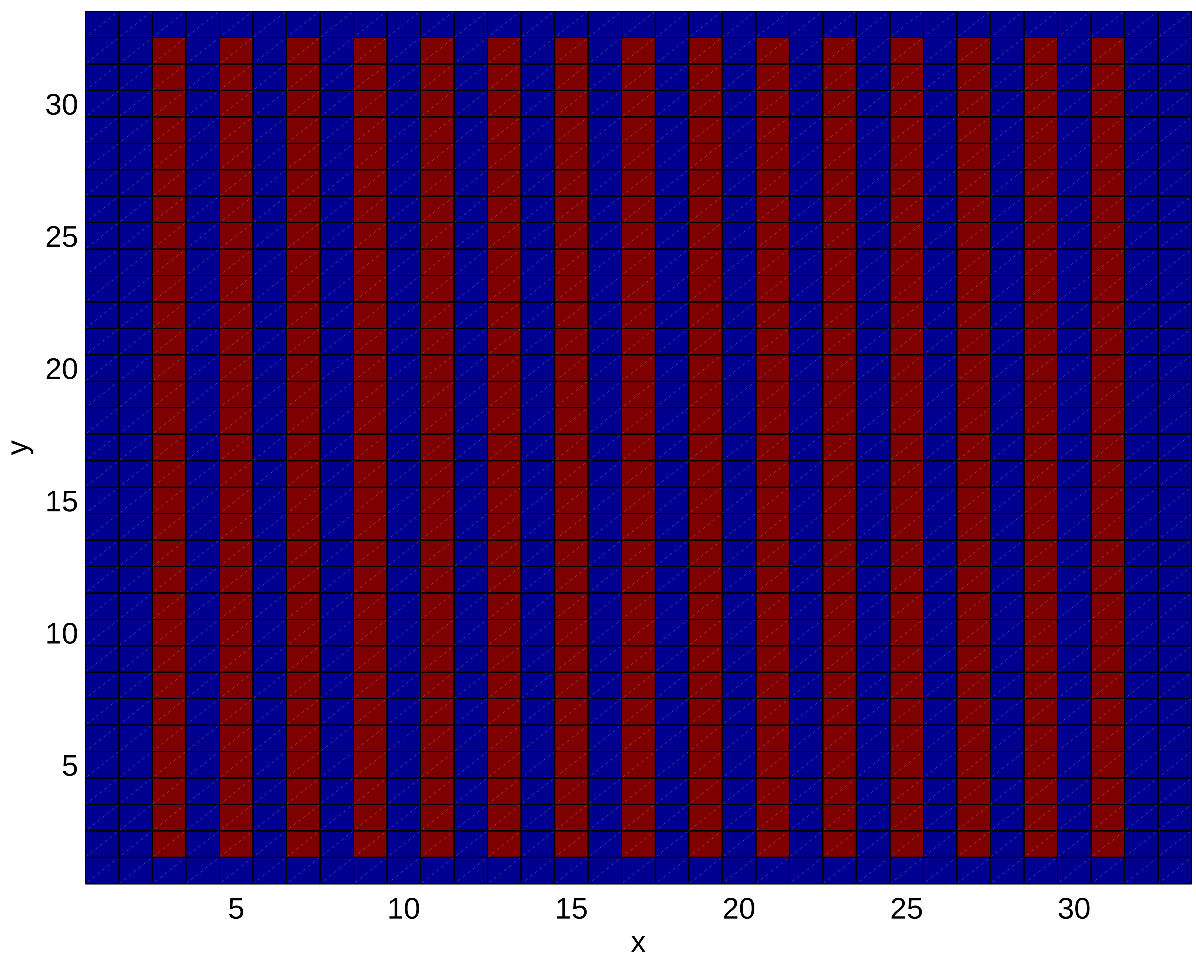

Influence of Relaxation

- Stronger relaxation (GS, Block Relaxation) exposes connections faster
- Strong connections needed for coarsening change with block relaxation
- Want to resolve strength for relaxation applied to A^h
- i strongly depends on j if $(A^h)^{-1}_{ij}$ is large compared to $(A^h)^{-1}_{ik}$
- If relaxation is $I - M^{-1}A^h$, want

$$(M^{-1}A^h)^{-1}_{ij} \text{ large compared to } (M^{-1}A^h)^{-1}_{ik}$$

- Compute i^{th} row of $(M^{-1}A^h)^{-1}$
  - i^{th} column of $(M^{-1}A^h)^{-T}$

$$M^T(A^h)^{-1}e^{(i)} = M^T G^{(i)}$$

Relaxation-induced S_{ij}

- Use relaxation to compute $G^{(i)}$
- Apply transpose of relaxation: $\hat{G}^{(i)} = M^T G^{(i)}$
- Compute $S_{ij} = \dfrac{\|\hat{G}^{(i)} - \hat{G}^{(i)}_j e^{(j)}\|_{A^h}}{\|\hat{G}^{(i)}\|_{A^h}}$

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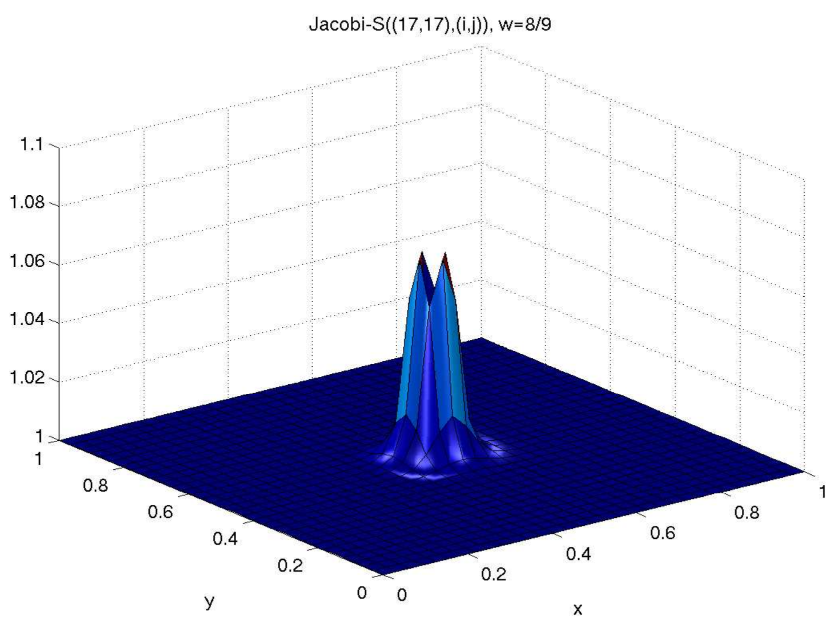
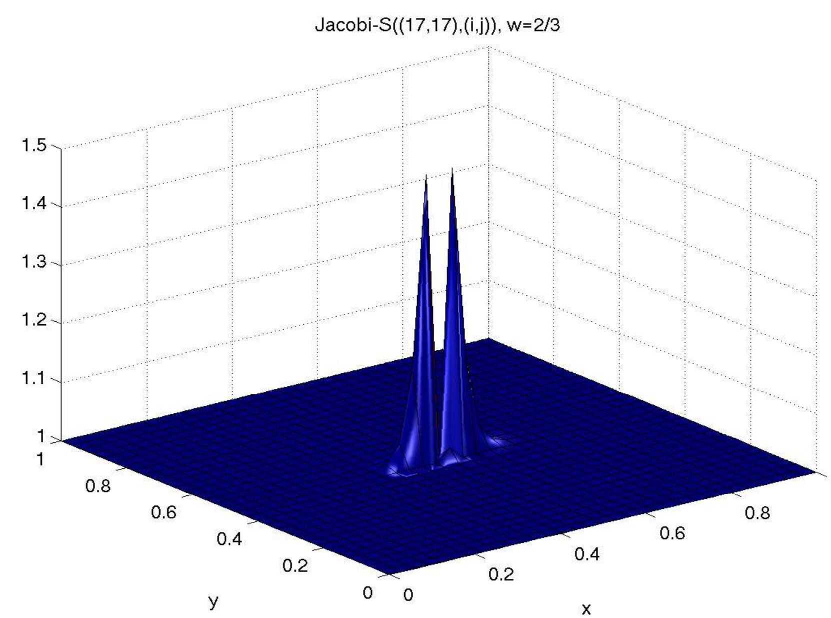
- Jacobi, 10 steps:





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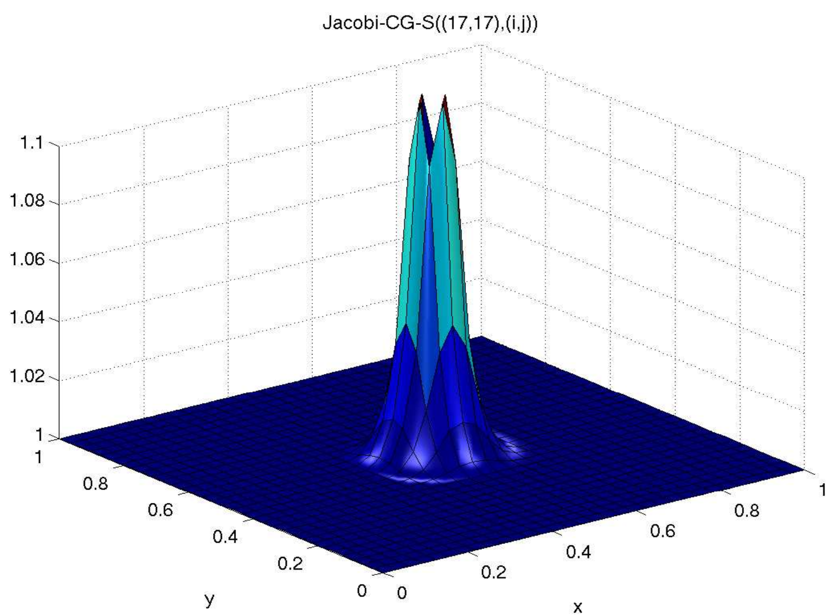

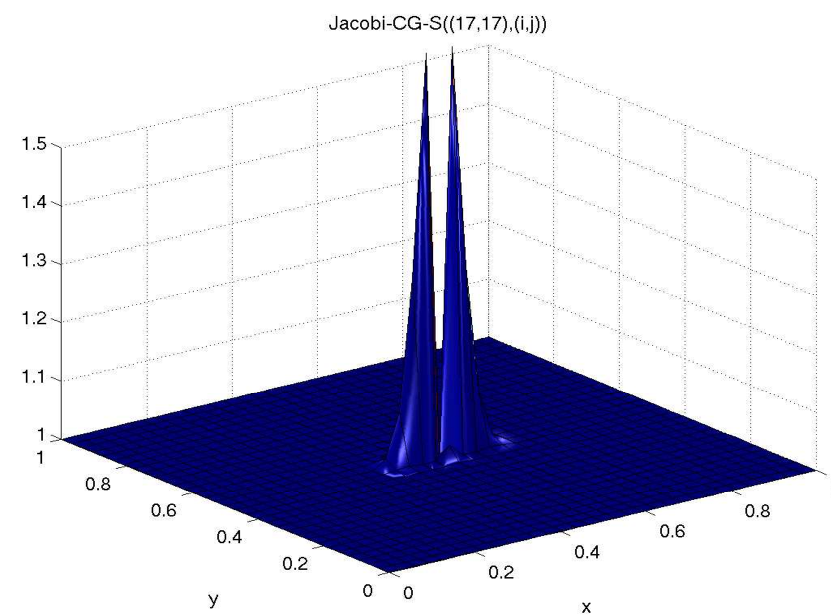

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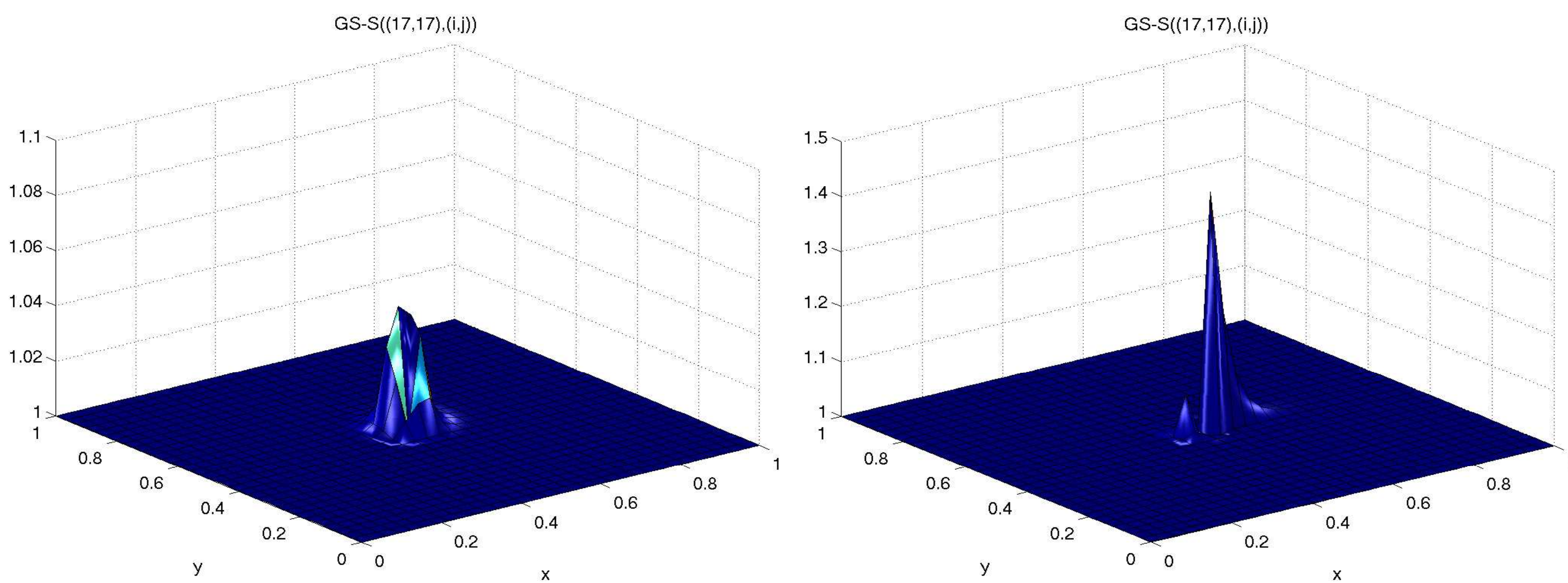
- Gauss-Seidel, 5 steps:



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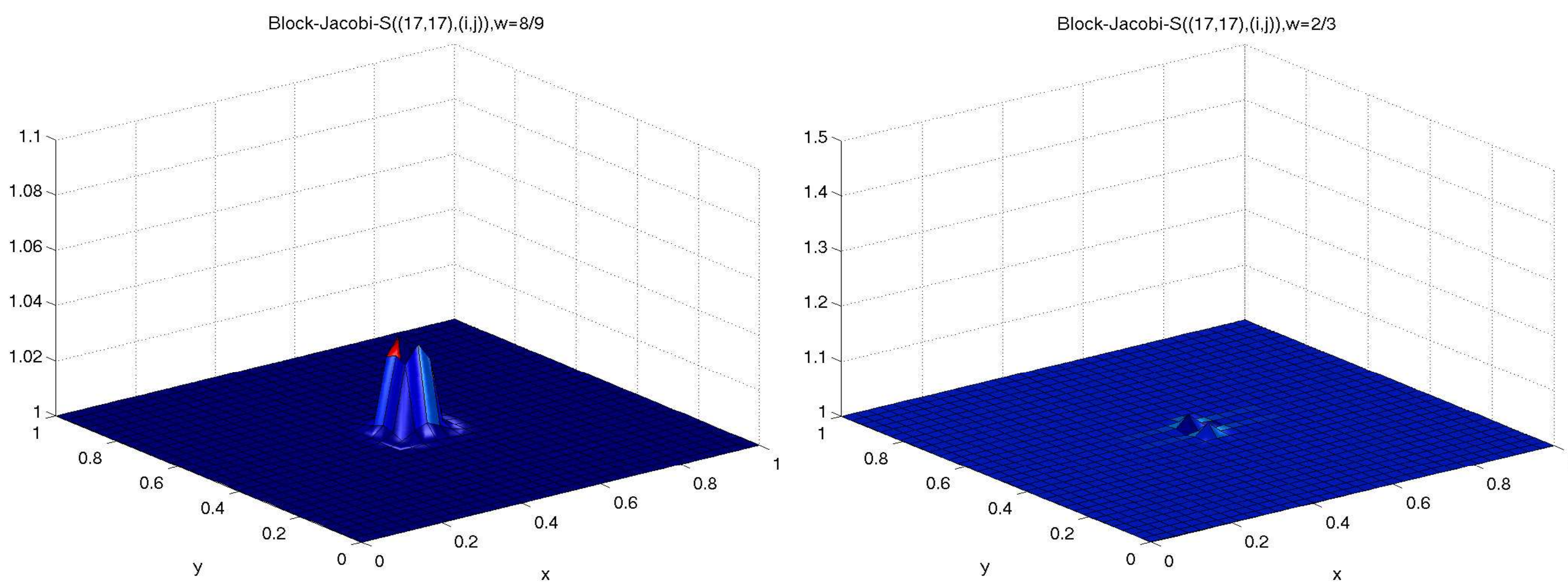
- Weighted Line-Jacobi, 5 steps:



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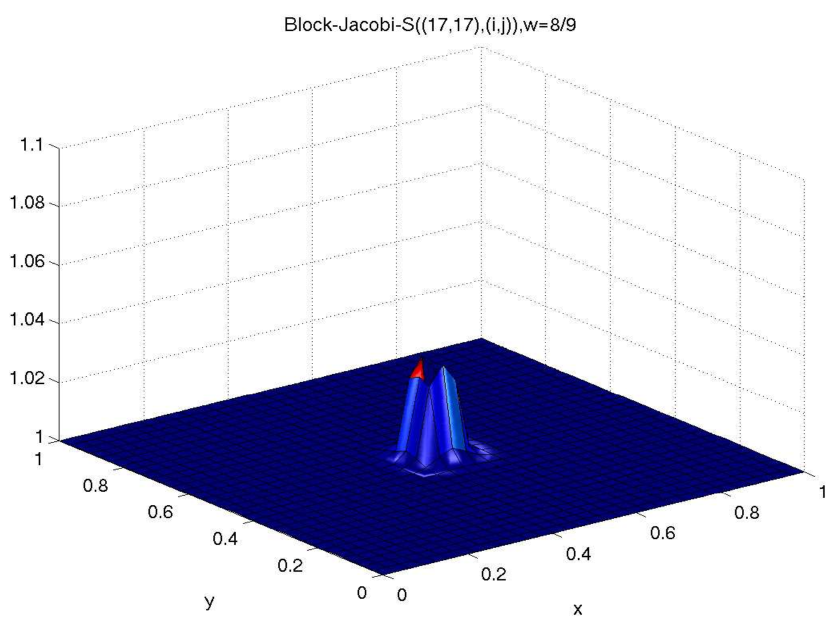

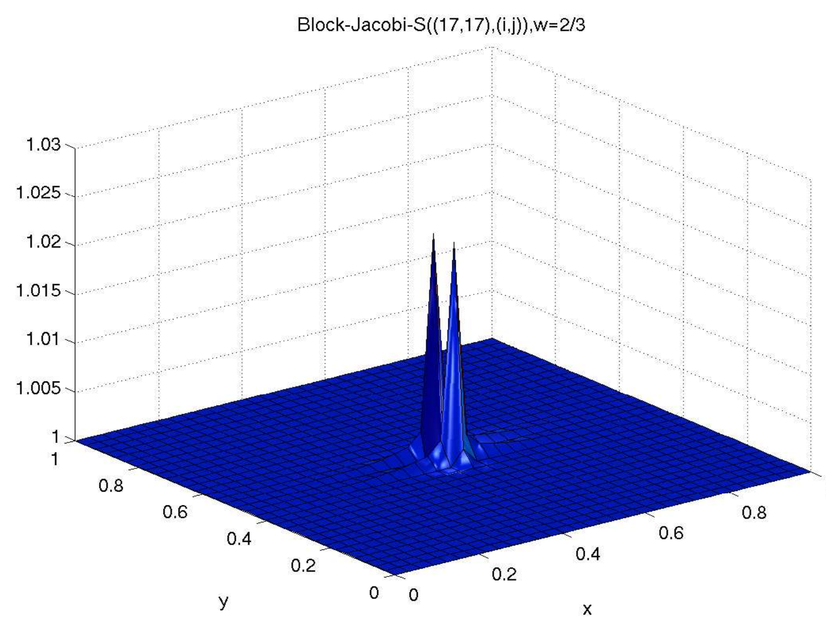

Choices of coarse grids

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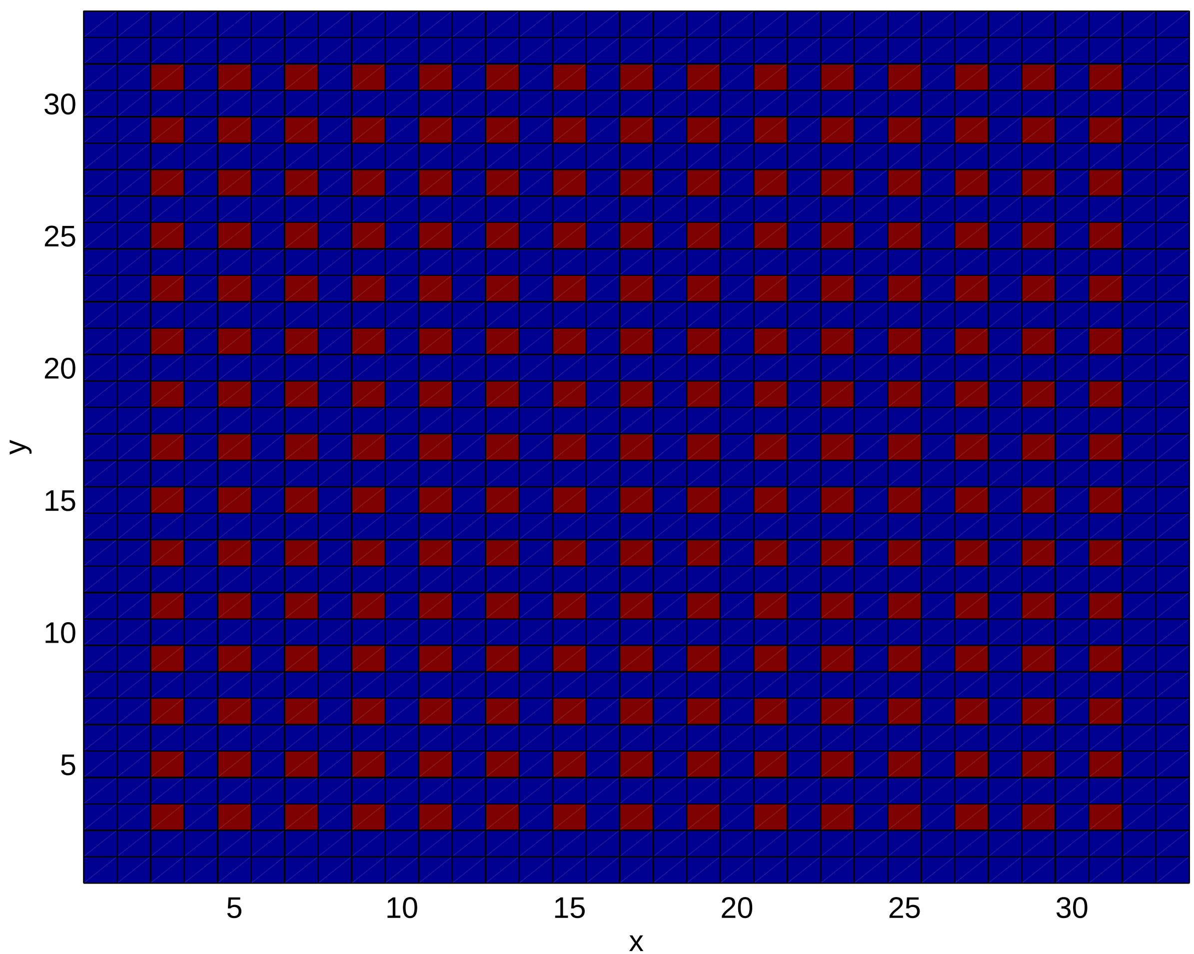

Choices of coarse grids

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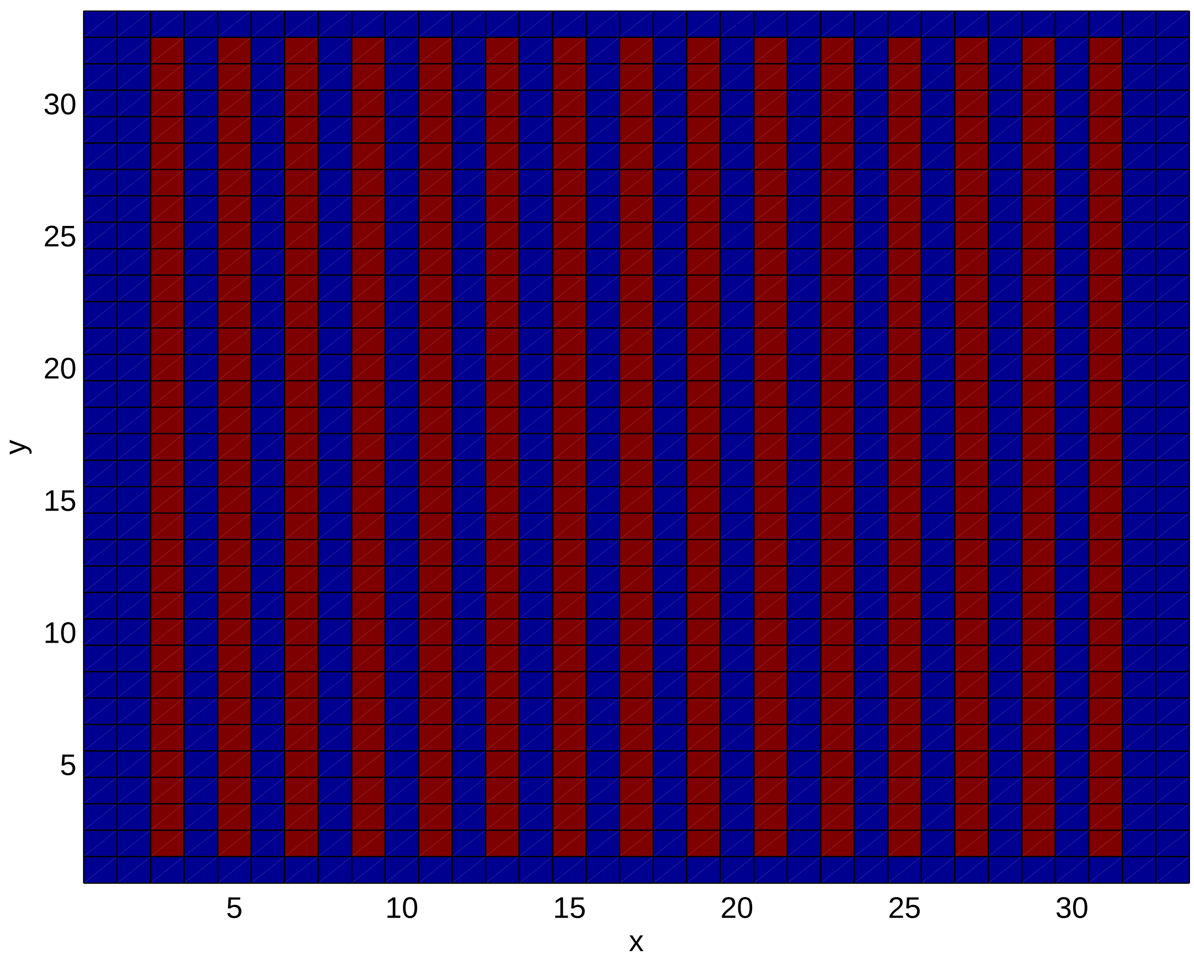

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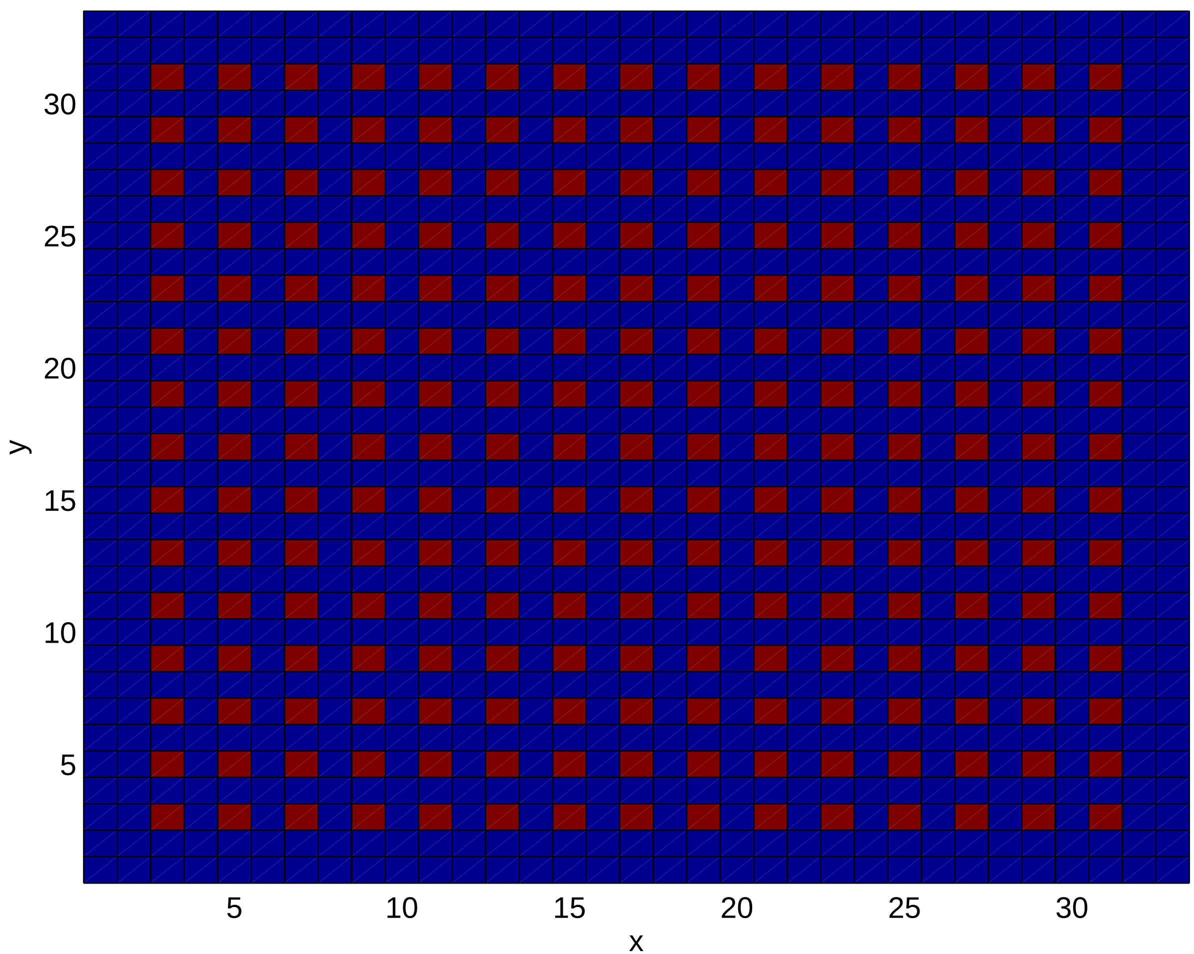

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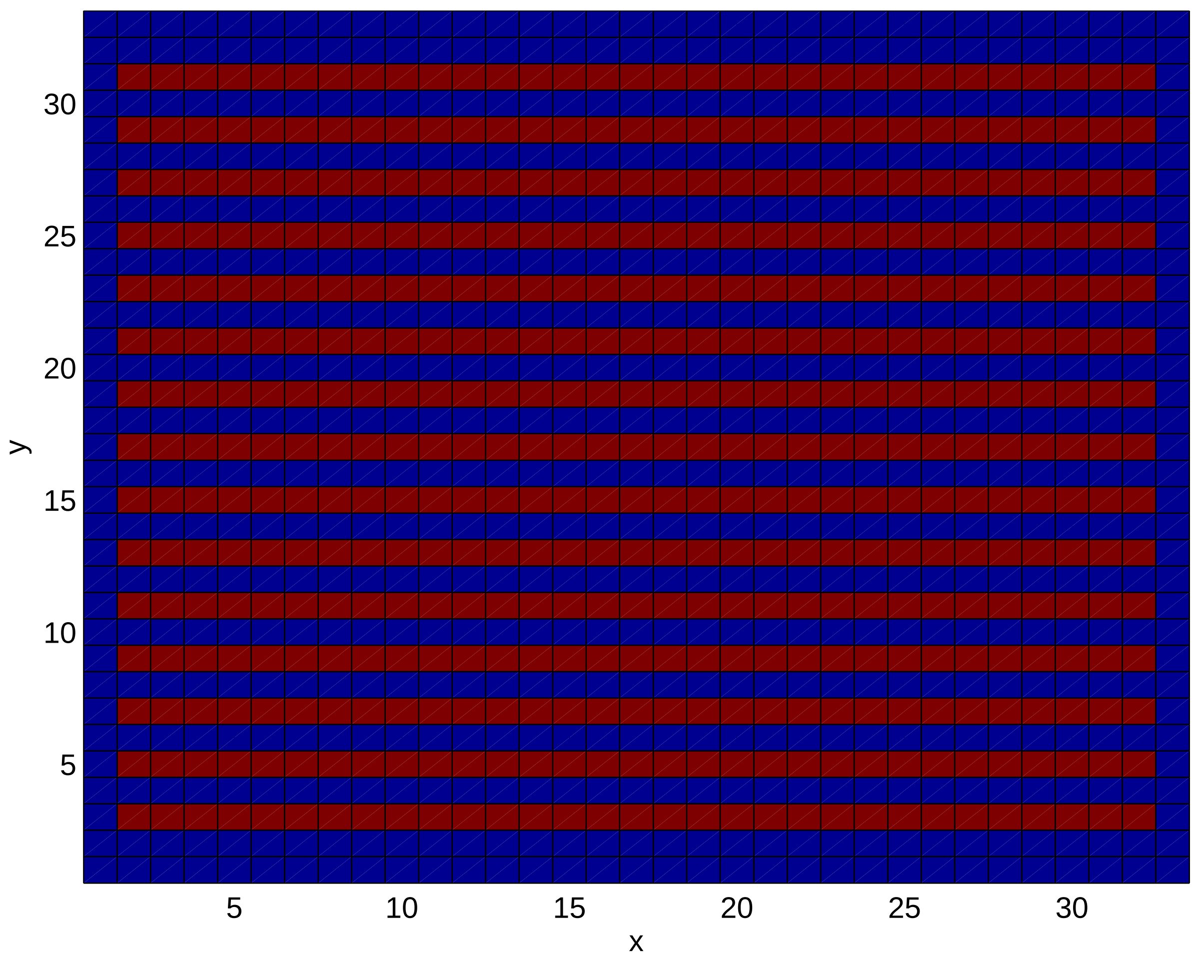

Summary

- Fully Adaptive framework aims to improve robustness of AMG-based algorithms
- FAlosophy: get it right, then make it efficient
- New algebraic measure of strength of connection
- Current work: incorporating relaxation into measure
- Future work: fully study efficiencies and cost implications
- Future work: combine with adaptive AMG for systems of PDEs