# New Directions in Multigrid Homogenization

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# Porous Media Flow

- Interested in simulating flow in a reservoir
- Modeling saturated flow via Darcy's Law:

$$
\begin{aligned}
u(x,y) &= -D(x,y)\nabla p(x,y) \\
\nabla \cdot u(x,y) &= Q(x,y)
\end{aligned}
$$

- Simulation domain may be on the order of $10^3$ meters in length
- Fine scale changes in material properties are on the order of $10^{-3}$ meters
- Naive discretizations require too many degrees of freedom (DOFs) to be computationally feasible

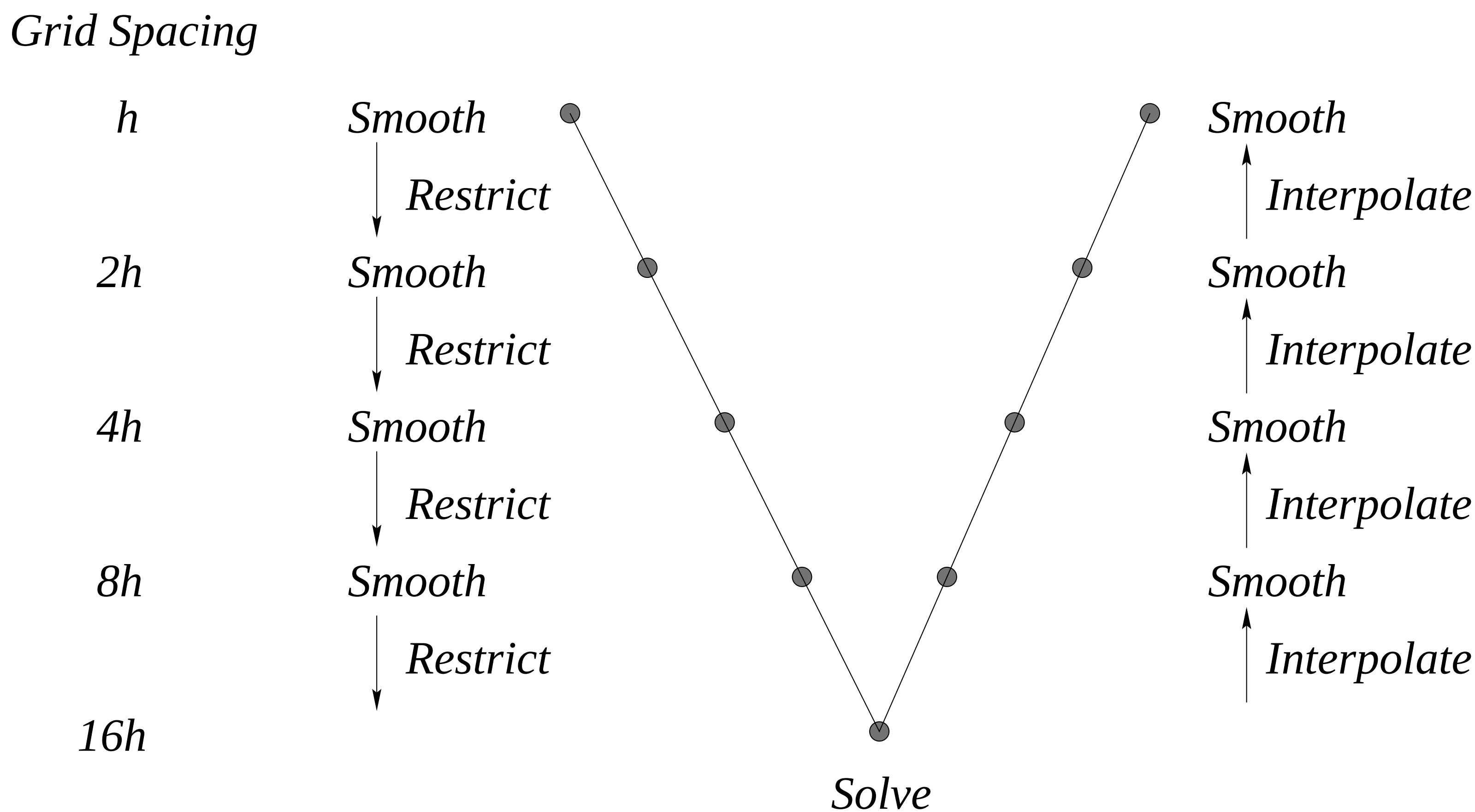
# The Need for Upscaling

- We must accurately account for the influence of fine-scale variation in the material properties if we hope to obtain physically meaningful solutions
- In general, we cannot directly account for the influence of fine-scale variation in material properties in a coarse-scale discretization
- The goal of upscaling and homogenization techniques is to derive effective, coarse-scale material properties to use in coarse-scale models and discretizations
- “Black-box” multilevel solvers provide a natural setting for numerical upscaling

# Multigrid Basics

- Need a solver whose performance doesn’t significantly degrade as problem size increases
- Multigrid methods obtain optimal efficiency through complementarity
- Use a smoothing process (such as Gauss-Seidel) to eliminate oscillatory errors
- Use a coarse grid correction process to eliminate smooth errors
- Obtain optimal efficiency through recursion

# The V-Cycle


Grid Spacing
h
Smooth
Restrict
2h
Smooth
Restrict
4h
Smooth
Restrict
8h
Smooth
Restrict
16h
Solve
Smooth
Interpolate
Smooth
Interpolate
Smooth
Interpolate
Smooth
Interpolate

# Multigrid Operators

- Multigrid V-Cycle requires transfers of residuals and corrections from one grid to the next
- Accomplished through Interpolation (Prolongation) and Restriction operators
- Often pick a form of interpolation ($P$) and take restriction $R = P^T$ (theoretical benefits)
- Smoothing on coarse grids requires operators on those grids
- These operators must well-approximate the fine grid operator
- If present, we can often use geometric or physical information to choose these operators

# Geometric vs. Algebraic Multigrid

- “Smooth” error can be represented using fewer DOFs
- Choosing the set of coarse DOFs must be done so that the error left after relaxation can be accurately corrected by the form $Pw$
- If geometric information is known, can choose the coarse-grid based on removing points in a geometrically regular manner
- If geometry is unknown or complicated, choose the coarse-grid based on heuristics to ensure good algebraic correction

# Variational Multigrid

- Multigrid with $R = P^T$ and $A_c = RAP$ is called a *variational formulation*
- Terminology comes from minimization form of $Lu = f$:

$$F(v) = \frac{1}{2}\langle Lv, v\rangle - \langle f, v\rangle$$
$$u = \arg\min_{v \in \mathcal{H}} F(v)$$

- Given an approximation $v$ to the solution on the fine level, it can be shown that the optimal coarse-grid correction $Pw$ solves

$$(P^T AP)w = P^T(f - Lv)$$

# Multilevel Flow Calculations

- Once we have created all of the components of a multigrid solver, we can use them to perform coarse-scale flow calculations
- By restricting the fine-scale sources to a given scale and solving the coarse-scale system $A_c p_c = f_c$, we get a coarse-scale representation of the pressure
- This pressure can then be interpolated to the fine-scale, where we can compute the flux
- Using operator-induced interpolation techniques allows accurate, coarse-scale flow calculations

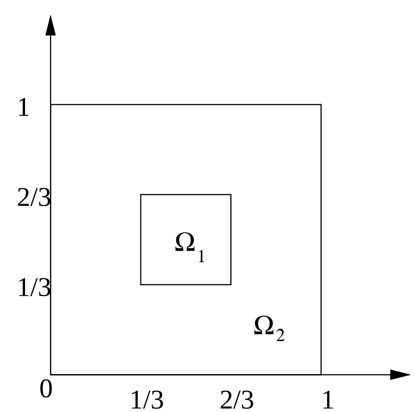
# Numerical Results

- We consider the PDE $-\nabla \cdot D\nabla p = 0$ on $[0,1]^2$
- Discretization is by bilinear Finite Elements
- We impose no flow boundary conditions $(D\nabla p) \cdot n = 0$ on the top and bottom boundaries
- We impose pressures $p = 1$ at the left boundary and $p = 0$ at the right boundary
- Solve the system using the Black Box Multigrid Algorithm (BoxMG), developed by Dendy for discontinuous-coefficient diffusion problems
- BoxMG chooses interpolation in a manner which preserves the continuity of normal flux, and then uses a variational formulation for the rest of the multigrid operators

# Sand/Shale Problem

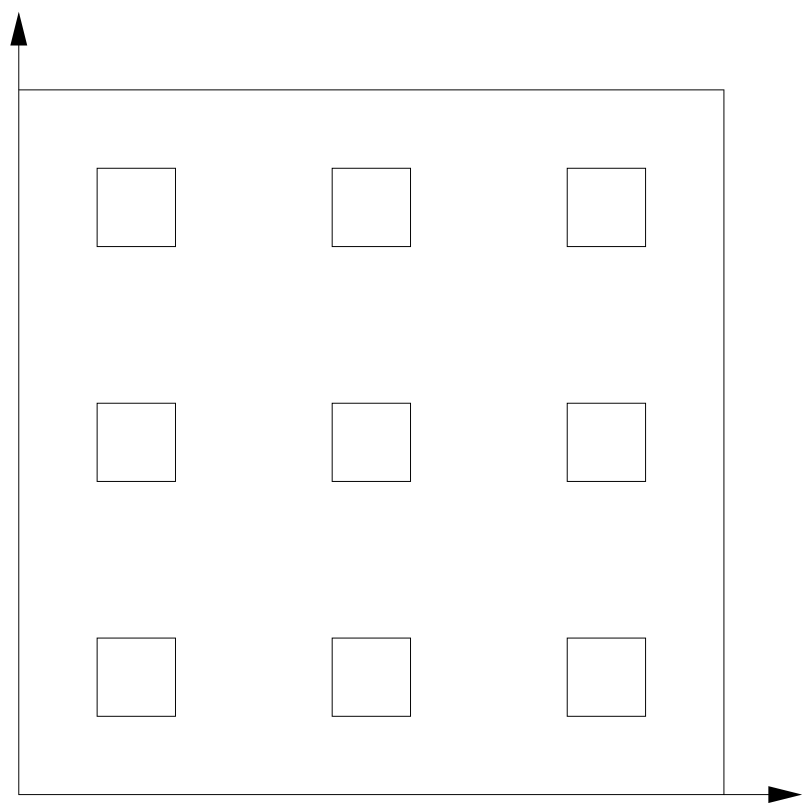
- We choose a piecewise constant $D(x,y)$ to simulate a sand/shale problem
- Start with a box-in-a-box example with

$$D(x,y) = \begin{cases} 10 & (x,y) \in [\frac{1}{3}, \frac{2}{3}]^2 \\ 1 & \text{otherwise} \end{cases}$$



# Sand/Shale Problem

- We choose a piecewise constant $D(x, y)$ to simulate a sand/shale problem
- Then create a 3x3 tiling to simulate finer structure

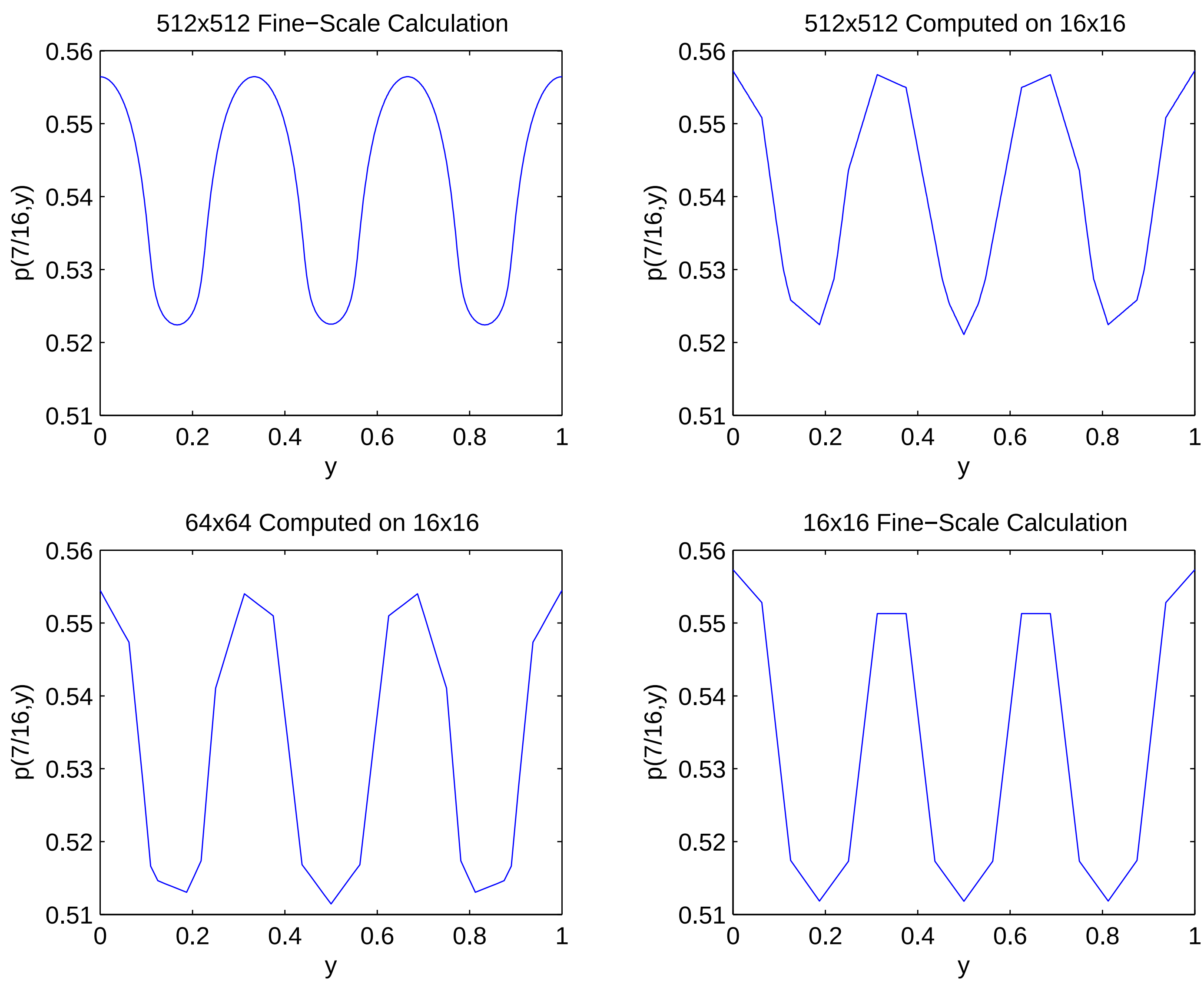


# Sand/Shale Problem Fluxes

- Flux integrated along line $x = 0.5$ from $y = 0$ to $y = 1$
- Flux computed from solving system with $1536^2$ elements is 1.21073 (compared to 1.21083 for $512^2$ elements)

| | $4^2$ | $8^2$ | $16^2$ | $32^2$ | $64^2$ | $128^2$ | $256^2$ | $512^2$ |
|---|---|---|---|---|---|---|---|---|
| $4^2$ | 1.586 | | | | | | | |
| $8^2$ | 1.154 | 1.133 | | | | | | |
| $16^2$ | 1.310 | 1.281 | 1.295 | | | | | |
| $32^2$ | 1.131 | 1.176 | 1.190 | 1.191 | | | | |
| $64^2$ | 1.135 | 1.188 | 1.226 | 1.233 | 1.230 | | | |
| $128^2$ | 1.090 | 1.143 | 1.159 | 1.209 | 1.207 | 1.206 | | |
| $256^2$ | 1.101 | 1.151 | 1.168 | 1.219 | 1.218 | 1.217 | 1.217 | |
| $512^2$ | 1.097 | 1.146 | 1.163 | 1.213 | 1.212 | 1.211 | 1.211 | 1.211 |

# Sand/Shale Problem Pressures



# Random Problem

- We choose an isotropic, random permeability field to simulate a geostatistically determined field
- Given $\zeta$ uniformly distributed in $(0, 1)$, we choose $D(x, y) = \zeta^{-\ln(10)}$
- Theoretically, the resulting field has isotropic homogenized permeability of the geometric mean, so $\hat{D} = 10$
- Our finite realization will introduce some unavoidable error

# Random Problem Fluxes

- Flux integrated along line $x = 0.5$ from $y = 0$ to $y = 1$

| | $4^2$ | $8^2$ | $16^2$ | $32^2$ | $64^2$ | $128^2$ | $256^2$ | $512^2$ |
|---|---|---|---|---|---|---|---|---|
| $4^2$ | 9.098 | | | | | | | |
| $8^2$ | 8.033 | 7.473 | | | | | | |
| $16^2$ | 10.392 | 8.851 | 9.100 | | | | | |
| $32^2$ | 9.260 | 7.614 | 7.698 | 7.016 | | | | |
| $64^2$ | 8.918 | 8.826 | 8.535 | 9.280 | 9.150 | | | |
| $128^2$ | 7.832 | 7.703 | 7.257 | 7.004 | 8.389 | 8.664 | | |
| $256^2$ | 7.458 | 7.348 | 7.676 | 8.072 | 9.237 | 8.374 | 8.610 | |
| $512^2$ | 7.728 | 7.713 | 7.701 | 7.711 | 7.715 | 7.910 | 8.706 | 8.765 |

# Interpretation of Multigrid CGOs

- Consider a fine-scale discretization via finite elements

$$A_{ij} = e_j^T A e_i = \int_\Omega \langle D(x,y) \nabla \phi_i, \nabla \phi_j \rangle d\Omega$$

- Use of Galerkin coarsening means that the coarse grid operator is equivalent to a finite element discretization on that grid

$$\begin{aligned}(A_c)_{ij} &= (P^T A P)_{ij} = (P\hat{e}_j)^T A (P\hat{e}_i) \\ &= (\sum_k p_{kj} e_k^T) A (\sum_l p_{li} e_l) \\ &= \sum_{k,l} p_{kj} p_{li} (e_k^T A e_l)\end{aligned}$$

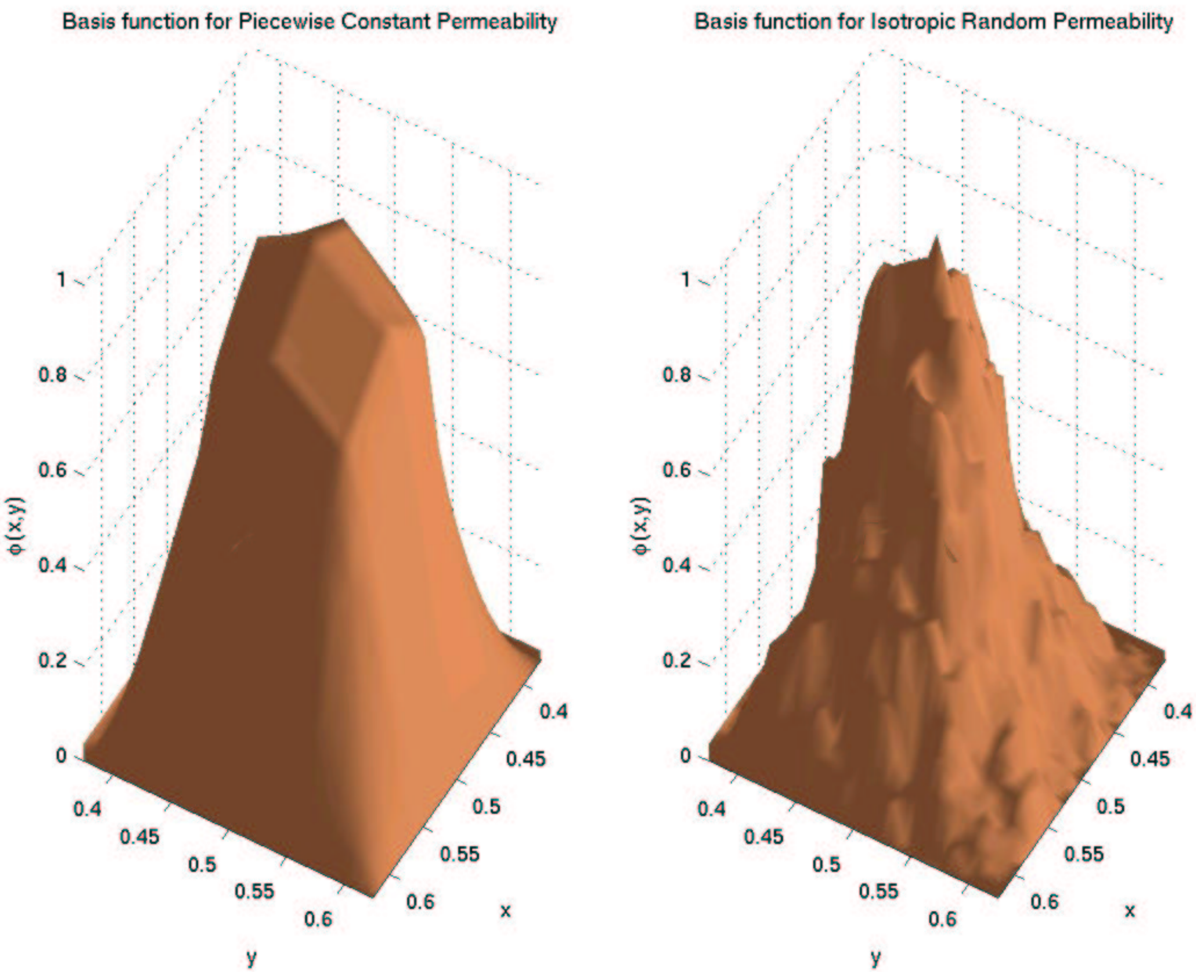
# Interpretation . . .

- So,

$$
\begin{aligned}
(A_c)_{ij} &= \sum_{k,l} p_{kj} p_{li} \int_\Omega \langle D(x,y) \nabla \phi_l, \nabla \phi_k \rangle d\Omega \\
&= \int_\Omega \left\langle D(x,y) \nabla \left( \sum_l p_{li} \phi_l \right), \nabla \left( \sum_k p_{kj} \phi_k \right) \right\rangle d\Omega \\
&= \int_\Omega \langle D(x,y) \nabla \hat{\phi}_i, \nabla \hat{\phi}_j \rangle d\Omega
\end{aligned}
$$

- Basis functions on coarse grids come from summing the fine grid basis functions (weighted by the interpolation/restriction operators)

# Coarse-Grid Basis Functions



# Reinterpretation of Multigrid CGOs

- Consider a bilinear discretization in 2-D
- Using a full-coarsening multigrid algorithm (such as BoxMG) results in 9-point operators on all coarse grids
- Any 9-point operator can be written as a linear combination of the bilinear FE operators for $I, \partial_x, \partial_y, \partial_{xx}, \partial_{yy}, \partial_{xy}, \partial_{xxy}, \partial_{xyy}, \partial_{xxyy}$
- If we start with a symmetric, zero row-sum operator, Galerkin coarsening guarantees that the coarse grid operator will also have these properties
- This forces the coarse grid operator to be a linear combination of $\partial_{xx}, \partial_{yy}, \partial_{xy}, \partial_{xxyy}$

# Reinterpretation . . .

- The coarse grid operator can thus be interpreted as the coarse grid discretization of

$$-\nabla \cdot (\hat{D}\nabla u) + \partial_{xy}E(x,y)\partial_{xy}u = \hat{f}$$

- It is possible to recover piecewise constant approximations of the effective $\hat{D}$ and $E$ based on the stencil entries
- That is, we can recover the homogenized permeability tensor directly from the coarse grid operator

# Effects on Upscaling

- Accounting for the regularization term has allowed us to (in some instances) accurately recover the homogenized permeabilities for model problems
- This allows the extension of the work of Dendy, Hyman, and Moulton from periodic BCs to Neumann BCs
- This term also suggests a relationship between Multigrid Homogenization and the closures used in the method of moments approach to ensemble averaging

# The Algebraic Connection

- These techniques and ideas are all applicable in the case of algebraic multigrid methods
- Algebraic multigrid (AMG) has been demonstrated to be an efficient solver for many problems, including Darcy law flow
- AMG does not, however, consider the underlying physics of the discretized PDE
- Recently, there have been significant advances in algebraic multigrid methods, particularly in improving their robustness

# Conclusions and Future Directions

- Improved coarse-scale flux calculations are possible using modern multigrid (multilevel) methods
- Accurately accounting for regularization term can lead to a more reliable numerical homogenization algorithm
- Accurate recovery of coarse-scale properties can be used to improve performance for other discretizations, such as Finite Volumes
- Algebraic multigrid methods show promise both for the ability to handle both rapidly changing coefficients and non-structured discretizations