Extension of Stochastic Volatility Equity Models with Hull-White Interest Rate Process

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Abstract

We present an extension of stochastic volatility equity models by a stochastic Hull-White interest rate component while assuming non-zero correlations between the underlying processes. We place these systems of stochastic differential equations in the class of affine jump diffusion - linear quadratic jump-diffusion processes (Duffie, Pan and Singleton [13], Cheng and Scaillet [10]) so that the pricing of European products can be efficiently done within the Fourier cosine expansion pricing framework [14]. We compare the new stochastic volatility Schöbel-Zhu-Hull-White hybrid model with a Heston-Hull-White model [3; 19], and also apply the models to price some hybrid structured derivatives that combine the equity and interest rate asset classes.

Key words: Schöbel-Zhu-Hull-White, Heston-Hull-White, stochastic volatility, hybrid

1 Introduction

In recent years the financial world has focused on accurate pricing of exotic and hybrid products that are based on a combination of underlyings from different asset classes. In this paper we therefore present a flexible multi-factor stochastic volatility (SV) model which includes the term structure of the stochastic interest rates (IR). Our aim is to combine an arbitrage-free Hull-White IR model in which the parameters are consistent with market prices of caps and swaptions. In order to perform efficient option valuation we fit this process in the class of affine jump diffusion (AJD) processes [13] (although jump processes are not included in this work). We specify under which conditions such a general model can fall in the class of AJD processes.

A major step, away from the assumption of constant volatility in derivatives pricing, was made by Hull and White [22], Stein and Stein [39] and Heston [21], who defined the volatility as a diffusion process. This improved the pricing of derivatives under heavy-tailed return distributions significantly and allowed a trader to quantify the uncertainty in the pricing. The stochastic volatility models have become popular for derivative pricing and hedging, see, for example, [16], however financial engineers have developed more complex exotic products, that require additionally the modeling of a stochastic interest rate component. A derivative pricing tool in which all these features are explicitly modeled may have the potential of generating more accurate option prices for hybrid products. These products can be designed to provide capital or income protection, diversification for portfolios and customized solutions for both institutional and retail markets.

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In [14] a highly efficient alternative pricing method was developed based on a Fourier-cosine expansion of the density function, and called COS method. This method can also determine a whole set of option prices in one computation. The COS algorithm relies heavily on the availability of the characteristic function of the price process, which is guaranteed if we stay within the AJD class, see Duffie-Pan-Singleton [13], Lee [25] and Lewis [26]. We examine the effect of correlated processes for asset, stochastic volatility and interest rate on the option prices through a comparison with, for example, the Heston model.

The plan of this paper is as follows: In Section 2 we perform analysis of the Schöbel-Zhu-Hull-White model. In Subsection 2.3, we show that the hybrid model of interest admits a semi-closed form for the characteristic function. In the successive subsection we derive the Heston-Hull-White model with non-zero correlation between the stock and the interest rate. In Section 3 we show how to efficiently price options with a Fourier cosine expansion technique when the characteristic function with stochastic interest rate of the asset process is available. Further, in Section 4 the two hybrid models, Schöbel-Zhu-Hull-White, Heston-Hull-White and the stochastic volatility Heston model are compared in detail with respect to calibration and hybrid product pricing. Section 5 concludes. The lengthy proofs of the lemmas are placed in the appendices.

2 Extension of stochastic volatility equity models

In this section we present a hybrid stochastic volatility equity model which includes a stochastic interest rate process. In particular, we add to the SV model the well-known Hull-White stochastic interest rate process [23], which is a generalization of the Vašiček model [40].

So, we consider a three-dimensional system of stochastic differential equations, of the following form:

$$\begin{cases} \mathrm{d}S_t = & r_t S_t \mathrm{d}t + & \sigma_t^p S_t \mathrm{d}W_t^x, \\ \mathrm{d}r_t = & \lambda(\theta_t - r_t)\mathrm{d}t + & \eta \mathrm{d}W_t^r, \\ \mathrm{d}\sigma_t = & \kappa\left(\overline{\sigma} - \sigma_t\right)\mathrm{d}t + & \gamma\sigma_t^{1-p}\mathrm{d}W_t^\sigma, \end{cases} \tag{2.1}$$

where p is an exponent, κ and λ control the speed of mean reversion, η represents the interest rate volatility, and $\gamma\sigma^{1-p}$ determines the variance of the σ_t process. Parameters $\overline{\sigma}$ and θ_t are the long-run mean of the volatility and the interest rate processes, respectively. W^i are correlated Wiener processes, also governed by an instantaneous covariance matrix,

$$\Sigma = \begin{bmatrix} 1 & \rho_{x,\sigma} & \rho_{x,r} \\ \rho_{\sigma,x} & 1 & \rho_{\sigma,r} \\ \rho_{r,x} & \rho_{r,\sigma} & 1 \end{bmatrix} \mathrm{d}t. \tag{2.2}$$

If we keep r_t deterministic and $p = \frac{1}{2}$, we have the Heston model [21],

$$\begin{cases} \mathrm{d}S_t = & rS_t\mathrm{d}t + & \sqrt{\sigma_t}S_t\mathrm{d}W_t^x, \\ \mathrm{d}\sigma_t = & \kappa^H\left(\overline{\sigma}^H - \sigma_t\right)\mathrm{d}t + & \gamma^H\sqrt{\sigma_t}\mathrm{d}W_t^\sigma. \end{cases} \tag{2.3}$$

For $p = 1$ our model is, in fact, the generalized Stein-Stein [39] model, which is also called the Schöbel-Zhu [36] model,

$$\begin{cases} \mathrm{d}S_t = & rS_t\mathrm{d}t + & \sqrt{v_t}S_t\mathrm{d}W_t^x, \\ \mathrm{d}v_t = & -2\kappa\left(v_t + \sigma_t\overline{\sigma} + \dfrac{\gamma^2}{2\kappa}\right)\mathrm{d}t + & 2\gamma\sqrt{v_t}\mathrm{d}W_t^\sigma, \end{cases} \tag{2.4}$$

in which the squared volatility, $v_t = \sigma_t^2$, represents the variance of the instantaneous stock return.

It was already indicated in [21] and [36] that the plain Schöbel-Zhu model is a particular case of the original Heston model. We can see that, if $\overline{\sigma} = 0$, the Schöbel-Zhu model equals the Heston model in which $\kappa^H = 2\kappa$, $\overline{\sigma}^H = \gamma^2/2\kappa$, and $\gamma^H = 2\gamma$. This relation gives a direct connection between their discounted characteristic functions (see [28]). Finally, if we set r_t constant, $p = 0$ in system of Equations (2.1), and zero correlations, the model collapses to the standard Black-Scholes model.

We will choose the parameters in the equations (2.1), such that we deal with the Schöbel-Zhu-Hull-White (SZHW) model in Subsection 2.3, and with the Heston-Hull-White (HHW) in Subsection 2.4. In [17] and [10] it is was shown that the so-called linear-quadratic jump-diffusion (LQJD) models are equivalent to the AJD models with an augmented state vector.

2.1 Affine jump-diffusion processes

The AJD class refers to a fixed probability space $(\Omega, \mathcal{F}, P)$ and a Markovian n-dimensional affine process $\mathbf{X}_t$ in some space $D \subset \mathbb{R}^n$. The model without jumps can be expressed by the following stochastic differential form:

$$\mathrm{d}\mathbf{X}_t = \mu(\mathbf{X}_t)\mathrm{d}t + \sigma(\mathbf{X}_t)\mathrm{d}\mathbf{W}_t,$$

where $\mathbf{W}_t$ is $\mathcal{F}_t$-standard Brownian motion in $\mathbb{R}^n$, $\mu(\mathbf{X_t}) : D \to \mathbb{R}^n$, $\sigma(\mathbf{X}_t) : D \to \mathbb{R}^{n\times n}$. Moreover, for processes in the AJD class it is assumed that drift $\mu(\mathbf{X_t})$, volatility $\sigma(\mathbf{X_t})\sigma(\mathbf{X_t})^T$ and interest rate component $r(X_t)$ are of the affine form, i.e.

$$\begin{aligned} \mu(\mathbf{X}_t) &= a_0 + a_1\mathbf{X}_t, \text{ for any } (a_0, a_1) \in \mathbb{R}^n \times \mathbb{R}^{n\times n}, && (2.5)\\ \sigma(\mathbf{X}_t)\sigma(\mathbf{X}_t)^T &= (c_0)_{ij} + (c_1)_{ij}^T\mathbf{X}_t, \text{ for arbitrary } (c_0, c_1) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n\times n}, && (2.6)\\ r(\mathbf{X}_t) &= r_0 + r_1^T\mathbf{X}_t, \text{ for } (r_0, r_1) \in \mathbb{R} \times \mathbb{R}^n. && (2.7) \end{aligned}$$

Then, for a state vector, $\mathbf{X}_t$, the discounted characteristic function (CF) is of the following form:

$$\phi(\mathbf{u}, \mathbf{X}_t, t, T) = \mathbb{E}^{\mathbb{Q}}\left(\mathrm{e}^{-\int_t^T r_s \mathrm{d}s + i\mathbf{u}^T\mathbf{X}_T}|\mathcal{F}_t\right) = \mathrm{e}^{A(\mathbf{u},\tau)+\mathbf{B}^{\mathbf{T}}(\mathbf{u},\tau)\mathbf{X}_t},$$

where the expectation is taken under the risk-neutral measure, $\mathbb{Q}$. For a time lag, $\tau := T - t$, the coefficients $A(\mathbf{u}, \tau)$ and $\mathbf{B}^T(\mathbf{u}, \tau)$ have to satisfy certain complex-valued ordinary differential equations (ODEs) [13]:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\tau}A(\mathbf{u}, \tau) = -r_0 + \mathbf{B}^T a_0 + \dfrac{1}{2}\mathbf{B}^T c_0 \mathbf{B}, \\ \dfrac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{B}(\mathbf{u}, \tau) = -r_1 + a_1^T\mathbf{B} + \dfrac{1}{2}\mathbf{B}^T c_1 \mathbf{B}. \end{cases} \tag{2.8}$$

The dimension of the (complex valued) ODEs for $\mathbf{B}(\mathbf{u}, \tau)$ corresponds to the dimension of the state vector, $\mathbf{X}_t$. Typically, multi-factor models, like the SZHW or the HHW model, provide a better fit to the observed market data than the one-factor models. However, as the dimension of SDE system increases, the ODEs to be solved to get the CF become increasingly complex. If an analytical solution to the ODEs cannot be obtained, one can apply well-known numerical ODE techniques instead. This may require substantial computational effort, which essentially makes the model problematic for practical calibration applications. Therefore, in this paper we will set up two models for which an analytic solution to most of the ODEs appearing can be obtained.

2.2 The Hull-White model

Here, as a start, we consider the Hull-White, single-factor, no-arbitrage yield curve model in which the short-term interest rate is driven by an extended Ornstein-Uhlenbeck (OU) mean reverting process,

$$\mathrm{d}r_t = \lambda\left(\theta_t - r_t\right)\mathrm{d}t + \eta \mathrm{d}W_t^r, \tag{2.9}$$

where $\theta_t > 0, t \in \mathbb{R}^+$ is a time-dependent drift term, included to fit the theoretical bond prices to the yield curve observed in the market. Parameter η determines the overall level of volatility and the reversion rate parameter λ determines the relative volatilities. A high value of λ causes short-term rate movements to damp out quickly, so that the long-term volatility is reduced.

In the first part of our analysis we present the derivation for the CF of the interest rate process. Integrating equation (2.9), we obtain, for $t \geq 0$,

$$r_t = r_0\mathrm{e}^{-\lambda t} + \lambda\int_0^t \mathrm{e}^{-\lambda(t-s)}\theta_s \mathrm{d}s + \eta\int_0^t \mathrm{e}^{-\lambda(t-s)}\mathrm{d}W_s^r.$$

It is easy to show that r_t is normally distributed with

$$\mathbb{E}^{\mathbb{Q}}\left(r_t|\,\mathcal{F}_0\right) = r_0\mathrm{e}^{-\lambda t} + \int_0^t \lambda\mathrm{e}^{-\lambda(t-s)}\theta_s\mathrm{d}s,$$

and

$$\mathrm{Var}^{\mathbb{Q}}\left(r_t|\,\mathcal{F}_0\right) = \frac{\eta^2}{2\lambda}\left(1-\mathrm{e}^{-2\lambda t}\right).$$

Moreover, it is known that for θ_t being constant, i.e., $\theta_t \equiv \theta$,

$$\lim_{t\to\infty}\mathbb{E}^{\mathbb{Q}}\left(r_t|\mathcal{F}_0\right) = \theta,$$

which means that for large t the first moment of the process converges to the mean reverting level θ.

In order to simplify the derivations to follow we use the following proposition (see Arnold [4], Oksendal [33]).

Proposition 2.1 (Hull-White decomposition)**.** *The Hull-White stochastic interest rate process (2.9) can be decomposed into $r_t = \widetilde{r}_t + \psi_t$, where*

$$\psi_t = \mathrm{e}^{-\lambda t}r_0 + \lambda\int_0^t \mathrm{e}^{-\lambda(t-s)}\theta_s\mathrm{d}s,$$

and

$$\mathrm{d}\widetilde{r}_t = -\lambda\widetilde{r}_t\mathrm{d}t + \eta\mathrm{d}W_t^{\mathbb{Q}},\ \textit{with}\ \widetilde{r}_0 = 0.$$

Proof. The proof is straightforward by Itô's lemma. □

The advantage of this transformation is that the stochastic process $\widetilde{r}_t$ is now a basic OU mean reverting process, determined only by λ and η, independent of function ψ_t. It is easier to analyze than the original Hull and White model [22].

We investigate the discounted conditional characteristic function (CF) of spot interest rate r_t,

$$\begin{aligned}\phi_{\mathrm{HW}}(u,r_t,t,T) &= \mathbb{E}^{\mathbb{Q}}\left(\mathrm{e}^{-\int_t^T r_s\mathrm{d}s+iur_T}|\mathcal{F}_t\right)\\ &= \mathbb{E}^{\mathbb{Q}}\left(\mathrm{e}^{-\int_t^T \psi_s\mathrm{d}s+iu\psi_T}\cdot\mathrm{e}^{-\int_t^T \widetilde{r}_s\mathrm{d}s+iu\widetilde{r}_T}|\mathcal{F}_t\right)\\ &= \mathrm{e}^{-\int_t^T \psi_s\mathrm{d}s+iu\psi_T}\cdot\phi_{\mathrm{HW}}(u,\widetilde{r}_t,t,T),\end{aligned}\tag{2.10}$$

and see that process $\widetilde{r}_t$ is affine. Hence according to [13] the discounted CF for the affine interest rate model for $u\in\mathbb{C}$ is of the following form:

$$\phi_{\mathrm{HW}}(u,\widetilde{r}_t,\tau) = \mathbb{E}^{\mathbb{Q}}\left(\mathrm{e}^{-\int_t^T \widetilde{r}_s\mathrm{d}s+iu\widetilde{r}_T}|\mathcal{F}_t\right) = \mathrm{e}^{A(u,\tau)+B(u,\tau)\widetilde{r}_t},\tag{2.11}$$

with $\tau = T-t$. The necessary boundary condition accompanying (2.11) is $\phi_{\mathrm{HW}}(u,\widetilde{r}_t,0) = \mathrm{e}^{iu\widetilde{r}_t}$, so that, $A(u,0)=0$ and $B(u,0)=iu$. The solutions for $A(u,\tau)$ and $B(u,\tau)$ are provided by the following lemma:

Lemma 2.2 (Coefficients for discounted CF for the Hull-White model)**.** *The functions $A(u,\tau)$ and $B(u,\tau)$ in (2.11) are given by:*

$$\begin{aligned}A(u,\tau) &= \frac{\eta^2}{2\lambda^3}\left(\lambda\tau-2\left(1-\mathrm{e}^{-\lambda\tau}\right)+\frac{1}{2}\left(1-\mathrm{e}^{-2\lambda\tau}\right)\right)-iu\frac{\eta^2}{2\lambda^2}\left(1-\mathrm{e}^{-\lambda\tau}\right)^2\\ &\quad -\frac{1}{2}u^2\frac{\eta^2}{2\lambda}\left(1-\mathrm{e}^{-2\lambda\tau}\right),\\ B(u,\tau) &= iu\mathrm{e}^{-\lambda\tau}-\frac{1}{\lambda}\left(1-\mathrm{e}^{-\lambda\tau}\right).\end{aligned}\tag{2.12}$$

Proof. The proof can be found in [7] pp. 75. □

By simply taking $u = 0$, we obtain the risk-free pricing formula for a zero coupon bond $P(t, T)$:

$$\begin{aligned}\phi_{\mathrm{HW}}(0, r_t, \tau) &= \mathbb{E}^{\mathbb{Q}}\left(\mathrm{e}^{-\int_t^T r_s \mathrm{d}s} \cdot 1|\mathcal{F}_t\right) \\ &= \exp\left(-\int_t^T \psi_s \mathrm{d}s + A(0,\tau) + B(0,\tau)\widetilde{r}_t\right). \end{aligned} \tag{2.13}$$

Moreover, we see that a zero coupon bond can be written as the product of a deterministic factor and the bond price in an ordinary Vašiček model with zero mean, under the risk-neutral measure $\mathbb{Q}$. We recall that process $\widetilde{r}_t$ at time $t = 0$ is equal to 0, so

$$P(0,T) = \exp\left(-\int_0^T \psi_s \mathrm{d}s + A(0,T)\right), \tag{2.14}$$

which gives

$$\psi_T = -\frac{\partial}{\partial T}\log P(0,T) + \frac{\partial}{\partial T}A(0,T) = f(0,T) + \frac{\eta^2}{2\lambda^2}\left(1 - \mathrm{e}^{-\lambda T}\right)^2, \tag{2.15}$$

where $f(t, T)$ is an instantaneous forward rate.

This result shows that ψ_T can be obtained from the initial forward curve, $f(0, T)$. The other time-invariant parameters, λ and η, have to be estimated using market prices of, in particular, interest rate caps. Now from Proposition 2.1 we have $\theta_t = \frac{1}{\lambda}\frac{\partial}{\partial t}\psi_t + \psi_t$ which reads,

$$\theta_t = f(0,t) + \frac{1}{\lambda}\frac{\partial}{\partial t}f(0,t) + \frac{\eta^2}{2\lambda^2}\left(1 - \mathrm{e}^{-2\lambda t}\right). \tag{2.16}$$

Moreover, the CF, $\phi_{\mathrm{HW}}(u, r_t, \tau)$, for the Hull-White model can be simply obtained by integration of ψ_s over the interval $[t, T]$.

2.3 Schöbel-Zhu-Hull-White hybrid model

In this section we derive an analytic pricing formula in (semi-)closed form for European call options under the Schöbel-Zhu-Hull-White (SZHW) asset pricing model with a full matrix of correlations, defined by (2.2). The work on the SZHW hybrid model was initiated by Pelsser [30] and resulted (later) in a working paper [20].

For the state vector $\mathbf{X}_t = [S_t, r_t, \sigma_t]^T$ let us fix a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\mathcal{F} = \{\mathcal{F}_t : t \geq 0\}$, which satisfies the usual conditions. Furthermore, $\mathbf{X}_t$ is assumed to be Markovian relative to $\mathcal{F}_t$. The Schöbel-Zhu-Hull-White hybrid model can be expressed by the following 3D system of SDEs

$$\begin{cases} \mathrm{d}S_t = & r_t S_t \mathrm{d}t + & \sigma_t S_t \mathrm{d}W_t^x, \\ \mathrm{d}r_t = & \lambda\left(\theta_t - r_t\right)\mathrm{d}t + & \eta \mathrm{d}W_t^r, \\ \mathrm{d}\sigma_t = & \kappa(\overline{\sigma} - \sigma_t)\mathrm{d}t + & \gamma \mathrm{d}W_t^\sigma, \end{cases} \tag{2.17}$$

with the parameters as in Equations (2.1), for $p = 1$, and the correlations:

$$\begin{cases} \mathrm{d}W_t^x \cdot \mathrm{d}W_t^\sigma &= \rho_{x,\sigma}\mathrm{d}t, \\ \mathrm{d}W_t^x \cdot \mathrm{d}W_t^r &= \rho_{x,r}\mathrm{d}t, \\ \mathrm{d}W_t^r \cdot \mathrm{d}W_t^\sigma &= \rho_{r,\sigma}\mathrm{d}t. \end{cases} \tag{2.18}$$

By extending the space vector (as in [10]) with another stochastic process, defined by

$$v_t := \sigma_t^2,$$

and choosing $x_t = \log S_t$, we obtain the following 4D system of SDEs,

$$\begin{cases} \mathrm{d}x_t = & \left(\widetilde{r}_t + \psi_t - \frac{1}{2}v_t\right)\mathrm{d}t + & \sqrt{v_t}\mathrm{d}W_t^x, \\ \mathrm{d}\widetilde{r}_t = & -\lambda\widetilde{r}_t\mathrm{d}t + & \eta\mathrm{d}W_t^r, \\ \mathrm{d}v_t = & \left(-2v_t\kappa + 2\kappa\sigma_t\overline{\sigma} + \gamma^2\right)\mathrm{d}t + & 2\gamma\sqrt{v_t}\mathrm{d}W_t^\sigma, \\ \mathrm{d}\sigma_t = & \kappa(\overline{\sigma} - \sigma_t)\mathrm{d}t + & \gamma\mathrm{d}W_t^\sigma, \end{cases} \tag{2.19}$$

where we also used $r_t = \widetilde{r}_t + \psi_t$, as in Subsection 2.2. Note that θ_t is now included in ψ_t. We see that model (2.19) is indeed affine in the state vector $\mathbf{X}_t = [x_t, \widetilde{r}_t, v_t, \sigma_t]^T$. By the extension of the vector space we have obtained an affine model which enables us to apply the results from [13]. In order to simplify the calculations, we introduce a variable $x_t := \widetilde{x}_t + \Psi_t$ where $\Psi_t = \int_0^t \psi_s \mathrm{d}s$ and

$$\mathrm{d}\widetilde{x}_t = \left(\widetilde{r}_t - \frac{1}{2}v_t\right)\mathrm{d}t + \sqrt{v_t}\mathrm{d}W_t^x. \tag{2.20}$$

According [13] the discounted CF for $\mathbf{u} \in \mathbb{C}^4$ is of the following form,

$$\begin{aligned} \phi_{\mathrm{SZHW}}(\mathbf{u}, \mathbf{X}_t, t, T) &= \mathbb{E}^{\mathbb{Q}}\left(\mathrm{e}^{-\int_t^T r_s \mathrm{d}s}\mathrm{e}^{i\mathbf{u}^T\mathbf{X}_T}|\mathcal{F}_t\right) && (2.21)\\ &= \mathrm{e}^{-\int_t^T \psi_s\mathrm{d}s + i\mathbf{u}^T[\Psi_T, \psi_T, 0, 0]^T}\cdot\mathbb{E}^{\mathbb{Q}}\left(\mathrm{e}^{-\int_t^T \widetilde{r}_s\mathrm{d}s + i\mathbf{u}^T\mathbf{X}_T^*}|\mathcal{F}_t\right) && (2.22)\\ &= \mathrm{e}^{-\int_t^T \psi_s\mathrm{d}s + i\mathbf{u}^T[\Psi_T, \psi_T, 0, 0]^T}\cdot\mathrm{e}^{A(\mathbf{u},\tau) + \mathbf{B}^T(\mathbf{u},\tau)\mathbf{X}_t^*} && (2.23) \end{aligned}$$

where $\mathbf{X}_t^* = [\widetilde{x}_t, \widetilde{r}_t, v_t, \sigma_t]^T$ and $\mathbf{B}(\mathbf{u}, \tau) = [B_x(\mathbf{u}, \tau), B_r(\mathbf{u}, \tau), B_v(\mathbf{u}, \tau), B_\sigma(\mathbf{u}, \tau)]^T$. Now we set $\mathbf{u} = [u, 0, 0, 0]^T$, so that at time T we obtain the obvious boundary condition:

$$\phi_{\mathrm{SZHW}}(\mathbf{u}, \mathbf{X}_T^*, T, T) = \mathbb{E}^{\mathbb{Q}}\left(\mathrm{e}^{i\mathbf{u}^T\mathbf{X}_T^*}|\mathcal{F}_T\right) = \mathrm{e}^{i\mathbf{u}^T\mathbf{X}_T^*} = \mathrm{e}^{iu\widetilde{x}_T},$$

(as the price at time T is known deterministically). This boundary condition for $\tau = 0$ gives $B_x(u, 0) = iu$, $A(u, 0) = 0$, $B_r(u, 0) = 0$, $B_\sigma(u, 0) = 0$, $B_v(u, 0) = 0$. The following lemmas define the ODEs, from (2.8), and detail their solution.

Lemma 2.3 (Schöbel-Zhu-Hull-White ODEs)**.** *The functions $A(u, \tau)$, $B_x(u, \tau)$, $B_\sigma(u, \tau)$, $B_v(u, \tau)$, $B_r(u, \tau)$, $u \in \mathbb{R}$, in (2.23) satisfy the following system of ODEs:*

$$\begin{aligned} \frac{\mathrm{d}B_x}{\mathrm{d}\tau} &= 0, \\ \frac{\mathrm{d}B_r}{\mathrm{d}\tau} &= -1 + B_x - \lambda B_r, \\ \frac{\mathrm{d}B_v}{\mathrm{d}\tau} &= \frac{1}{2}B_x(B_x - 1) + 2\left(\gamma\rho_{x,v}B_x - \kappa\right)B_v + 2\gamma^2B_v^2, \\ \frac{\mathrm{d}B_\sigma}{\mathrm{d}\tau} &= \left(2\kappa\bar{\sigma}B_v + \eta\rho_{x,r}B_xB_r + 2\gamma\eta\rho_{r,v}B_rB_v\right) + \left(2\gamma^2B_v - \kappa + \gamma\rho_{x,\sigma}B_x\right)B_\sigma, \\ \frac{\mathrm{d}A}{\mathrm{d}\tau} &= \gamma^2B_v + \frac{1}{2}\eta^2B_r^2 + \left(\kappa\bar{\sigma} + \frac{1}{2}\gamma^2B_\sigma + \gamma\eta\rho_{r\sigma}B_r\right)B_\sigma. \end{aligned}$$

Proof. The proof can be found in Appendix A.1. □

Lemma 2.4. *The solution to the system of ODEs, specified in Lemma 2.3 is given by:*

$$\begin{aligned} B_x(u, \tau) &= iu, \\ B_r(u, \tau) &= \frac{1}{\lambda}(iu - 1)(1 - \Theta(-2\lambda)), \\ B_v(u, \tau) &= \frac{1}{4\gamma^2}\cdot\left(\frac{1 - \Theta(-2d)}{1 - g\Theta(-2d)}\right)(\beta - d), \\ B_\sigma(u, \tau) &= f_0\left(f_1 + \frac{1}{\lambda}(iu - 1)\left(\eta\rho_{x,r}iu\cdot(f_2 - f_3) + \frac{\eta\rho_{r,v}}{2\gamma}(\beta - d)\cdot(f_4 + f_5)\right)\right), \\ A(u, \tau) &= f_6 - \frac{1}{2\gamma^2}\log\left(\frac{g\Theta(-2d) - 1}{g - 1}\right) - \frac{1}{2\lambda^3}\cdot f_7 + \Gamma(u, \tau), \end{aligned}$$

where:

$$\Gamma(u,\tau) = \int_0^\tau B_\sigma(u,s)\left(\kappa\bar{\sigma}+\frac{1}{2}\gamma^2 B_\sigma(u,s)+\eta\rho_{r,\sigma}\gamma B_r(u,s)\right)\mathrm{d}s, \tag{2.24}$$

with

$$\begin{aligned}
f_0 &= \frac{\Theta(d)}{\Theta(2d)-g},\\
f_1 &= \frac{16\kappa\bar{\sigma}}{4\gamma^2 d}(\beta-d)\cdot\sinh^2\left(\frac{\tau d}{4}\right),\\
f_2 &= \frac{2}{d}\left((\Theta(d)-1)+g(\Theta(-d)-1)\right),\\
f_3 &= \frac{2\left(\Theta(d-2\lambda)-1\right)}{d-2\lambda}-\frac{2g\left(1-\Theta(2\lambda-d)\right)}{d+2\lambda},\\
f_4 &= \frac{2}{d-2\lambda}-\frac{4}{d}+\frac{2}{d+2\lambda},\\
f_5 &= \Theta\left(-2\lambda-d\right)\left(\frac{2}{d}\Theta(2\lambda)(1+\Theta(2d))-\frac{2\Theta(2d)}{d-2\lambda}-\frac{2}{d+2\lambda}\right),\\
f_6 &= \frac{1}{4\gamma^2}(\beta-d)\tau,\\
f_7 &= (iu-1)^2(3+\Theta(-4\lambda)-4\Theta(-2\lambda)-2\tau\lambda),
\end{aligned}$$

and $\beta = 2\left(\kappa-\rho_{x,v}\gamma u i\right)$, $d=\sqrt{\beta^2-8\alpha\gamma^2}$, *with* $\alpha=-\frac{1}{2}u(i+u)$, $g=\frac{\beta-d}{\beta+d}$, $\Theta(x)=\exp\left(\frac{x\tau}{2}\right)$.

Proof. The proof is presented in Appendix A.2. □

Now, since we have found expressions for the coefficients $A(u,\tau)$ and $\mathbf{B}^T(u,\tau)$ we return to equation (2.21) and derive a representation in which the term structure is included. It is known that the price of a zero coupon bond can be obtained from the characteristic function by taking $\mathbf{u}=[0,0,0,0]^T$. So,

$$\phi_{\mathrm{SZHW}}(0,\mathbf{X}_t,\tau)=\exp\left(-\int_t^T\psi_s\mathrm{d}s\right)\cdot\phi_{\mathrm{SZHW}}(0,\mathbf{X}_t^*,\tau). \tag{2.25}$$

Since $\widetilde{r}_0=0$ we find,

$$P(0,T)=\exp\left(-\int_0^T\psi_s\mathrm{d}s+A(0,\tau)+B_x(0,\tau)x_0+B_v(0,\tau)v_0+B_\sigma(0,\tau)\sigma_0\right),$$

with boundary conditions $B_x(0,T)=0, B_v(0,T)=0, B_\sigma(0,T)=0$ and

$$A(0,T)=\frac{1}{2}\eta^2\int_0^T B_r(0,s)^2\mathrm{d}s=\frac{\eta^2}{4\lambda^3}\left(1+2\lambda T-\left(\mathrm{e}^{-\lambda T}-2\right)^2\right). \tag{2.26}$$

We thus find,

$$P(0,T)=\exp\left(-\int_0^T\psi_s\mathrm{d}s+A(0,T)\right).$$

By combining the results from the previous lemmas, we can prove the following lemma.

Lemma 2.5. *In the Schöbel-Zhu-Hull-White model, the discounted characteristic function,* $\phi_{SZHW}(u,\mathbf{X}_t,t,T)$ *for* $\log S_T$, *is given by*

$$\phi_{SZHW}(u,\mathbf{X}_t,t,T)=\exp\left(\widetilde{A}(u,\tau)+B_x(u,\tau)x_t+B_r(u,\tau)\widetilde{r}_t+B_v(u,\tau)v_t+B_\sigma(u,\tau)\sigma_t\right),$$

where $B_x(u,\tau)$, $B_r(u,\tau)$, $B_v(u,\tau)$, $B_\sigma(u,\tau)$ and $A(u,\tau)$ are given in Lemma 2.4, and

$$\widetilde{A}(u,\tau) \quad = \quad A(u,\tau) + (iu-1)\int_t^T \psi_s \mathrm{d}s = A(u,\tau) + \Upsilon(u,t,T), \tag{2.27}$$

with

$$\Upsilon(u,t,T) = (1-iu)\left\{\log\left(\frac{P(0,T)}{P(0,t)}\right) + \frac{\eta^2}{2\lambda^2}\left(\tau + \frac{2}{\lambda}\left(\mathrm{e}^{-\lambda T} - \mathrm{e}^{-\lambda t}\right) - \frac{1}{2\lambda}\left(\mathrm{e}^{-2\lambda T} - \mathrm{e}^{-2\lambda t}\right)\right)\right\}. \tag{2.28}$$

Proof. The proof is straightforward from the definition of the discounted CF. □

2.3.1 Numerical integration for the SZHW hybrid model

Lemma 2.4 indicates that many terms in the CF for the SZHW hybrid model can be obtained analytically, except the $\Gamma(u,\tau)$-term (2.24), which requires numerical integration of the hypergeometric function ${}_2F_1$ [31]. For a given partitioning

$$0 = s_1 \leq s_2 \leq \ldots s_{N'-1} \leq s_{N'} = \tau,$$

we calculate following integral approximation of (2.24):

$$\Gamma(u,\tau) \approx \sum_{i=0}^{N'} B_\sigma(u,s_i)\left(\kappa\bar{\sigma} + \frac{1}{2}\gamma^2 B_\sigma(u,s_i) + \eta\rho_{r,\sigma}\gamma B_r(u,s_i)\right)\delta_{s_i}, \tag{2.29}$$

with the functions $B_r(u,s_i)$ and $B_\sigma(u,s_i)$ as in (2.24). In Table 1 we present the numerical convergence results for two basic quadrature rules for one particular (representative) example of (2.29). It shows that both integration routines – the composite trapezoidal and the composite Simpson rule – converge very satisfactory with only a small number of grid points, N'. Convergence with the trapezoidal rule is of second order, and with Simpson's rule of fourth order, as expected. Simpson's rule is superior in terms of the ratio between time and absolute error. We therefore continue with the Simpson rule, setting $N' = 2^6$.

Table 1: CPU time, absolute error, and the convergence rate for different numbers of integration points N' for evaluating function $\Gamma(u,\tau)$. The time to maturity is set to $\tau = 1$ and $u = 5$ and the remaining parameters for the model are $\lambda = 0.5$, $\kappa = 1$, $\eta = 0.1$, $\bar{\sigma} = 0.3$, $\gamma = 0.5$, $\rho_{x,v} = -0.5$, $\rho_{x,r} = 0.3$, $r_0 = 0.05$, $\sigma_0 = 0.256$ and $\rho_{r,v} = -0.9$.

$(N' = 2^{n'})$	Trapezoidal rule		Simpson's rule	
n'	time (sec)	|error|	time (sec)	|error|
2	1.5e-4	1.5e-4	1.5e-4	7.3e-6
4	2.6e-4	6.0e-6	2.7e-4	2.3e-8
6	3.4e-4	3.4e-7	3.5e-4	1.3e-10
8	6.6e-4	2.1e-8	6.7e-4	6.0e-13

2.4 Heston-Hull-White hybrid model

It is known, for example from [32], that it is not possible to formulate the so-called Heston-Hull-White (HHW) hybrid process, with a full matrix of correlations, so that it belongs to the class of AJD processes. For this, restrictions regarding the parameters or the correlation structure have to be introduced. One possible restriction is to assume that the interest rate process, r_t, evolves independently of the stock price, S_t, and the volatility process, σ_t, while the other correlation is not equal to zero, i.e., $\mathrm{d}W_t^x \cdot \mathrm{d}W_t^r = 0$, $\mathrm{d}W_t^\sigma \cdot \mathrm{d}W_t^r = 0$ and $\mathrm{d}W_t^x \cdot \mathrm{d}W_t^\sigma = \rho_{x,\sigma}\mathrm{d}t$. Another option is to solve the problem under the assumption that $\mathrm{d}W_t^\sigma \cdot \mathrm{d}W_t^r = 0$ and additionally that

$\gamma^2/4 = \kappa\bar{\sigma}$, see [32]. It may, however, be difficult to apply this latter model in practice, as the economical meaning of the parameter relationship is difficult to interpret.

Since for the HHW model with a full matrix of correlations between the processes, the affinity of the model is lost, the aim is to reformulate the HHW model, so that affinity is preserved while the correlations are included to some extent. Giese in [19] has introduced the following HHW-type model:

$$\begin{cases} \mathrm{d}S_t = & r_t S_t \mathrm{d}t + & \sqrt{\sigma_t} S_t \mathrm{d}W_t^x + & \Delta_{S,r} S_t \mathrm{d}W_t^r, \\ \mathrm{d}r_t = & \lambda(\theta_t - r_t)\mathrm{d}t + & \eta \mathrm{d}W_t^r, & \\ \mathrm{d}\sigma_t = & \kappa(\bar{\sigma} - \sigma_t)\mathrm{d}t + & \gamma\sqrt{\sigma_t}\mathrm{d}W_t^\sigma, & \end{cases} \tag{2.30}$$

with

$$\begin{cases} \mathrm{d}W_t^x \cdot \mathrm{d}W_t^\sigma & = & \rho_{x,\sigma}\mathrm{d}t, \\ \mathrm{d}W_t^x \cdot \mathrm{d}W_t^r & = & 0, \\ \mathrm{d}W_t^r \cdot \mathrm{d}W_t^\sigma & = & 0. \end{cases} \tag{2.31}$$

Since the interest rate, r_t, is uncorrelated with the other driving processes, the reformulated HHW model stays (in the log-space for equity) in the class of affine processes. By taking $\Delta_{S,r} = 0$ the model collapses to well-known Heston-Hull-White model with independent interest rate. We see that by $\Delta_{S,r} \neq 0$ one controls, indirectly, the correlation between the equity and interest rate processes.

Now, by log-transform of the stock process, $x_t = \log S_t$, and using $r_t = \widetilde{r}_t + \psi_t$, and $x_t = \widetilde{x}_t + \Psi_t$, we obtain:

$$\begin{cases} \mathrm{d}\widetilde{x}_t = & \left(\widetilde{r}_t - \frac{1}{2}\left(\sigma_t + \Delta_{S,r}^2\right)\right)\mathrm{d}t + & \sqrt{\sigma_t}\mathrm{d}W_t^x + & \Delta_{S,r}\mathrm{d}W_t^r, \\ \mathrm{d}\widetilde{r}_t = & -\lambda\widetilde{r}_t\mathrm{d}t + & \eta\mathrm{d}W_t^r, & \\ \mathrm{d}\sigma_t = & \kappa(\bar{\sigma} - \sigma_t)\mathrm{d}t + & \gamma\sqrt{\sigma_t}\mathrm{d}W_t^\sigma. & \end{cases} \tag{2.32}$$

As in the case of the SZHW hybrid model, the next step of the analysis is to find the corresponding discounted CF. Since for constant $\Delta_{S,r}$ the system is already affine the CF for $\mathbf{u} \in \mathbb{C}^3$ is of the following form:

$$\phi_{\mathrm{HHW}}(\mathbf{u}, \mathbf{X}_t, t, T) = \mathbb{E}^{\mathbb{Q}}\left(\mathrm{e}^{-\int_t^T r_s \mathrm{d}s} \cdot \mathrm{e}^{i\mathbf{u}^T \mathbf{X}_T} | \mathcal{F}_t\right) \tag{2.33}$$

$$= \mathrm{e}^{-\int_t^T \psi_s \mathrm{d}s + i\mathbf{u}^T[\Psi_T, \psi_T, 0]^T} \cdot \mathrm{e}^{A(\mathbf{u},\tau) + \mathbf{B}^T(\mathbf{u},\tau)\mathbf{X}_t^*}, \tag{2.34}$$

where $\mathbf{X}_t^* = [\widetilde{x}_t, \widetilde{r}_t, \sigma_t]^T$ and $\mathbf{B}(\mathbf{u}, \tau) = [B_x(\mathbf{u}, \tau), B_r(\mathbf{u}, \tau), B_\sigma(\mathbf{u}, \tau)]^T$. As before by setting $\mathbf{u} = [u, 0, 0]^T$, we find the corresponding ODEs and their solutions.

Lemma 2.6 (Heston-Hull-White ODEs)**.** *The functions $A(u, \tau)$, $B_x(u, \tau)$, $B_r(u, \tau)$, and $B_\sigma(u, \tau)$, $u \in \mathbb{R}$, in (2.33) satisfy the following system of ODEs:*

$$\begin{aligned} \frac{\mathrm{d}B_x}{\mathrm{d}\tau} &= 0, \\ \frac{\mathrm{d}B_r}{\mathrm{d}\tau} &= -1 + B_x - \lambda B_r, \\ \frac{\mathrm{d}B_\sigma}{\mathrm{d}\tau} &= \frac{1}{2}B_x(B_x - 1) + (\gamma\rho_{x,\sigma}B_x - \kappa)B_\sigma + \frac{1}{2}\gamma^2 B_\sigma^2, \\ \frac{\mathrm{d}A}{\mathrm{d}\tau} &= \frac{1}{2}\Delta_{S,r}^2 B_x(B_x - 1) + \Delta_{S,r}\eta B_x B_r + \frac{1}{2}\eta^2 B_r^2 + \kappa\bar{\sigma}B_\sigma, \end{aligned}$$

with boundary conditions: $B_x(u, 0) = iu$, $B_\sigma(u, 0) = 0$, $B_r(u, 0) = 0$ and $A(u, 0) = 0$.

Proof. The proof can be found in Appendix A.3. □

Lemma 2.7 (CF coefficients for the HHW model)**.** *The solution to the system of ODEs, specified in Lemma 2.6 is given by:*

$$\begin{aligned}
B_x(u,\tau) &= iu,\\
B_r(u,\tau) &= \frac{1}{\lambda}(iu-1)\left(1-\Theta(-2\lambda)\right),\\
B_\sigma(u,\tau) &= \frac{1}{\gamma^2}\cdot\frac{1-\Theta(-2d)}{1-g\Theta(-2d)}\left(\beta-d\right),\\
A(u,\tau) &= \frac{\Theta(-4\lambda)}{2dg\lambda}\left(dg\left(\Gamma_1(\tau)+\Theta(4\lambda)\Gamma_2(\tau)\right)+\Gamma_3(\tau)\log\left(\frac{g\Theta(-2d)-1}{g-1}\right)\right),
\end{aligned}$$

where

$$\begin{aligned}
\Gamma_1(\tau) &= -f_4f_6^2+2f_6(f_3+2f_4f_6)\Theta(2\lambda),\\
\Gamma_2(\tau) &= (-f_6(2f_3+3f_4f_6)+2(f_1+f_2f_5+f_6(f_3+f_4f_6))\lambda\tau),\\
\Gamma_3(\tau) &= 2f_2f_5\Theta(4\lambda)(g-1)\lambda,
\end{aligned}$$

with: $f_1=-\frac{1}{2}\Delta_{S,r}^2u(i+u)$, $f_2=\kappa\bar{\sigma}$, $f_3=\Delta_{S,r}\eta iu$, $f_4=\frac{1}{2}\eta^2$, $f_5=\frac{1}{\gamma^2}\left(\beta-d\right)$, $f_6=\frac{1}{\lambda}(iu-1)$, $d=\sqrt{\beta^2+(iu+u^2)\gamma^2}$, $g=\frac{\beta-d}{\beta+d}$, $\beta=\kappa-\gamma\rho_{x,\sigma}iu$, *and* $\Theta(x)=\exp\left(\frac{x\tau}{2}\right)$.

Proof. The proof requires solving Riccati-type ODEs which are analogous to ones for the SZHW hybrid model. □

Now, by the lemma above the CF for the HHW hybrid model in (2.33) for $\log S_t$ is given by:

$$\Phi_{\text{HHW}}(u,\mathbf{X}_t,t,T)=\exp\left(\widetilde{A}(u,\tau)+B_x(u,\tau)x_t+B_r(u,\tau)\widetilde{r}_t+B_\sigma(u,\tau)\sigma_t\right), \tag{2.35}$$

where

$$\widetilde{A}(u,\tau)=A(u,\tau)+\Upsilon(u,t,T),$$

where $B_x(u,\tau)$, $B_r(u,\tau)$, $B_\sigma(u,\tau)$ and $A(u,\tau)$ are given in Lemma 2.7, and $\Upsilon(u,t,T)$ is given in Equation (2.28).

As already mentioned, the HHW model defined in (2.30) assumes a zero correlation between the equity process S_t and the short-term r_t, i.e., $\mathrm{d}W_t^x\cdot\mathrm{d}W_t^r=0$, as these two processes are indirectly linked via $\Delta_{S,r}$. Now, we discuss the relation between $\Delta_{S,r}$ and the instantaneous correlation $\rho_{x,r}$.

By Itô's lemma and $\mathrm{d}W_t^x\cdot\mathrm{d}W_t^r=0$ we have the instantaneous correlation:

$$\rho_{x,r}=\frac{\text{Cov}(\mathrm{d}S_t,\mathrm{d}r_t)}{\sqrt{\text{Var}(\mathrm{d}S_t)}\cdot\sqrt{\text{Var}(\mathrm{d}r_t)}}=\frac{\eta\Delta_{S,r}S_t\ \mathrm{d}t}{\sqrt{\sigma_tS_t^2\ \mathrm{d}t+\Delta_{s,r}^2S_t^2\ \mathrm{d}t}\cdot\sqrt{\eta^2\ \mathrm{d}t}}=\frac{\Delta_{S,r}}{\sqrt{\sigma_t+\Delta_{S,r}^2}}. \tag{2.36}$$

From (2.36) we find $\Delta_{S,r}$ as a function of $\rho_{x,r}$:

$$\Delta_{S,r}(t)=\frac{\rho_{x,r}\sqrt{\sigma_t}}{\sqrt{1-\rho_{x,r}^2}}.$$

Since $\Delta_{S,r}$ is defined in terms of the stochastic process σ_t it is stochastic as well. The first approach to deal with state-dependent $\Delta_{S,r}$ is to include it in the original system (2.30); however then the system's affinity may become problematic. In this article we therefore adopt the basic approximation for $\Delta_{S,r}$, proposed in [19], i.e.:

$$\Delta_{S,r}\approx\frac{\rho_{x,r}\mathbb{E}\left(\sqrt{\frac{1}{T}\int_0^T\sigma_t\mathrm{d}t}\right)}{\sqrt{1-\rho_{x,r}^2}}, \tag{2.37}$$

which further can be simplified [1] via:

$$\mathbb{E}\left(\sqrt{\frac{1}{T}\int_0^T \sigma_t \mathrm{d}s}\right) \approx \left(\mathbb{E}\left(\frac{1}{T}\int_0^T \sigma_t \mathrm{d}s\right) - \frac{1}{4T}\frac{\mathrm{Var}\left(\int_0^T \sigma_t \mathrm{d}s\right)}{\mathbb{E}\left(\int_0^T \sigma_t \mathrm{d}s\right)}\right)^{\frac{1}{2}}. \tag{2.38}$$

Since σ_t is a Cox-Ingersoll-Ross (CIR) type process the expectations and variance on the RHS in (2.38) can be found analytically.

2.4.1 Limits for the HHW hybrid model

We analyze here the accuracy of the approximation for $\Delta_{S,r}$ introduced in (2.37). With a prescribed correlation, $\rho_{x,r}$, we approximate the effective $\Delta_{S,r}$ in Equation (2.37) and compare it with the correlation, $\widetilde{\rho}_{S,r}$, obtained by a Monte Carlo simulation of (2.30). The results are generated by 50.000 Monte Carlo paths with step-size of 0.01.

Table 2: The error for instantaneous correlation, by a Monte Carlo simulation. The simulation is performed with: $\kappa = 0.35$, $\bar{\sigma} = 0.05$, $\gamma = 0.4$, $\lambda = 0.15$, $\eta = 0.07$, $\rho_{x,\sigma} = -0.7$,$S_0 = 1$, $v_0 = 0.0625$ and $r_0 = 0.02$.

maturity	$\rho_{x,r}$	$\Delta_{S,r}$	$\widetilde{\rho}_{S,r}$	$\rho_{x,r} - \widetilde{\rho}_{x,r}$
$\tau = 2$	30%	0.0646	29.90%	0.100%
	50%	0.1187	48.84%	1.160%
	70%	0.2016	66.90%	3.100%
	90%	0.4247	79.23%	10.77%
$\tau = 12$	30%	0.0587	25.25%	4.750%
	50%	0.1078	38.85%	11.15%
	70%	0.1831	45.89%	24.11%
	90%	0.3857	41.16%	48.81%

Table 2 shows that, although the instantaneous correlation between the equity process and the interest rate can be indirectly included in the HHW model via $\Delta_{S,r}$, some extreme correlations cannot be generated. Moreover, we also see that this effect is more pronounced for long maturities. Thus, with the HHW model we cannot fully control as accurate calibration and pricing especially for high correlations and long maturities is not guaranteed. Often in practice, however we hardly encounter such high correlations. However, since this model admits closed form for the CF, we do not need a numerical integration procedure as in Section 2.3.1 for SZHW.

3 Pricing Methodology

The pricing of plain vanilla options is common practice in the Fourier domain when the CF of the logarithm of the stock price is available.

Recently, an effective pricing method, the COS method, based on Fourier-cosine expansion, has been developed in [14]. This method can also, as the Carr-Madan method [9], compute the option prices for a whole strip of strikes in one computation and also depends on the availability of the CF. Implementation is straightforward. The COS method can achieve an exponential convergence rate for European, Bermudan and barrier options for affine models whose probability density function is in $\mathbb{C}^\infty[a, b]$, with nonzero derivatives [14; 15].

Here, we extend the COS method to include the stochastic interest rate process.

[1]
$$\mathrm{Var}(f(X)) \approx \left(f'(\mathbb{E}(X))\right)^2 \mathrm{Var}(X)$$

We start the description of the pricing method with the general risk-neutral pricing formula:

$$V(t, S_t) \quad = \quad \mathbb{E}^{\mathbb{Q}}\left(\mathrm{e}^{-\int_t^T r_s \mathrm{d}s} V(T, S_T)|\mathcal{F}_t\right) = \int_{\mathbb{R}} V(T, y)\widehat{f}_Y(y|x)dy, \tag{3.1}$$

where $\widehat{f}_Y(y|x) = \int_{\mathbb{R}} \mathrm{e}^z f_{Y,Z}(y, z|x)dz$, with $z = -\int_t^T r_s \mathrm{d}s$.

The claim $V(t, S_t)$ under $\mathbb{E}^{\mathbb{Q}}(\cdot)$ is defined in S_t which may be correlated to r_t. As we assume a fast decay of the density function, the following approximation can be made,

$$V(t, S_t) \quad \approx \quad \int_{\Omega} V(T, y)\widehat{f}_Y(y|x)dy, \tag{3.2}$$

where: $\Omega = [\delta_1, \delta_2]$, and $|\Omega| = \delta_2 - \delta_1$, $\delta_2 > \delta_1$. The discounted CF is now given by:

$$\phi(\mathbf{u}, \mathbf{X}_t, t, T) \quad = \quad \mathbb{E}^{\mathbb{Q}}\left(\mathrm{e}^{-\int_t^T r_s \mathrm{d}s + i\mathbf{u}^T X_T}|\mathcal{F}_t\right), \tag{3.3}$$

which, for $\mathbf{u} = [u, 0, \ldots, 0]^T$ and $\mathbf{X}_T = [S_T, r_T, \ldots]^T$, reads

$$\phi(u, \mathbf{X}_t, t, T) \quad = \quad \iint_{\mathbb{R}} \mathrm{e}^{z+iuy} f_{Y,Z}(y, z|x)dzdy = \int_{\mathbb{R}} \mathrm{e}^{iuy}\widehat{f}_Y(y|x)dy. \tag{3.4}$$

Note that the integration in (3.4) is simply the Fourier transform of $\widehat{f}_Y(y|x)$, which can be approximated on a bounded domain Ω,

$$\phi(u, \mathbf{X}_t, t, T) \quad \approx \quad \int_{\Omega} \mathrm{e}^{iuy} \cdot \widehat{f}_Y(y|x)dy =: \tilde{\phi}(u, X_t, t, T). \tag{3.5}$$

Since we are interested in the pricing of claims of the form (3.2), we link $\widehat{f}_Y(y|x)$ with its CF, via the following result:

Result 3.1. *For a given bounded domain $\Omega = [\delta_1, \delta_2]$, and N a number of terms in the expansion, the probability density function $\widehat{f}_Y(y|x)$ given by (3.2) can be approximated by,*

$$\widehat{f}_Y(y|x) \approx \sum_{n=0}^{N} \theta_n \cos\left(n\pi \frac{(y - \delta_1)}{|\Omega|}\right), \quad \text{where } \theta_n = \frac{2\omega_n}{|\Omega|}\Re\left\{\tilde{\phi}\left(\frac{n\pi}{|\Omega|}\right)\exp\left(-n\pi\frac{i\delta_1}{|\Omega|}\right)\right\},$$

where $\Re$ denotes taking the real part, $\omega_0 = \frac{1}{2}$ and $\omega_n = 1$, $n \in \mathbb{N}^+$.

For a proof we refer to the original paper on the COS method [14].

Using the lemma above, we replace the probability density function $\widehat{f}_Y(y|x)$ in (3.2),

$$V(t, S_t) \quad \approx \quad \int_{\Omega} V(T, y) \sum_{n=0}^{N} \theta_n \cos\left(n\pi \frac{(y - \delta_1)}{|\Omega|}\right) dy = \frac{|\Omega|}{2} \sum_{n=0}^{N} \frac{\theta_n \Gamma_n^{\Omega}}{\omega_n}, \tag{3.6}$$

where,

$$\Gamma_n^{\Omega} = \frac{2\omega_n}{|\Omega|} \int_{\Omega} V(T, y) \cos\left(n\pi \frac{(y - \delta_1)}{|\Omega|}\right) dy. \tag{3.7}$$

The equation above provides us with the pricing formula for any stochastically discounted payoff, $V(T, S_T)$, for which the CF is available. We note that, depending on the payoff, the Γ_n^{Ω} in (3.7) change, but a closed form expression is available for the most common payoffs. As the hybrid products will be calibrated to plain vanilla options, we provide the gamma coefficients for the European call options:

Result 3.2. *The Γ_n^Ω coefficients in (3.7) for pricing a call option defined by*

$$V(T,y) = \max\left(K(\mathrm{e}^y - 1), 0\right),$$

with $y = \log\left(\frac{S}{K}\right)$ for a given strike K, is given by

$$\Gamma_n^\Omega \quad = \quad \frac{2K}{|\Omega|}\left(\kappa_n - \psi_n\right), \tag{3.8}$$

where,

$$\kappa_n = \frac{|\Omega|^2}{|\Omega|^2 + (n\pi)^2}\left\{\cos(n\pi)\mathrm{e}^{\delta_2} - \cos\left(-\frac{\delta_1 n\pi}{|\Omega|}\right) + \frac{n\pi}{|\Omega|}\left[\sin(n\pi)\mathrm{e}^{\delta_2} - \sin\left(-\frac{\delta_1 n\pi}{|\Omega|}\right)\right]\right\}, \tag{3.9}$$

and,

$$\psi_n = \begin{cases} \frac{|\Omega|}{n\pi}\left[\sin(n\pi) - \sin\left(-\frac{\delta_1 n\pi}{|\Omega|}\right)\right] & \textit{for} \quad n \neq 0 \\ \delta_2 & \textit{for} \quad n = 0. \end{cases} \tag{3.10}$$

Proof. The proof is straightforward by calculating the integral in (3.7) with the transformed payoff function $V(T,y)$. □

Since the coefficients Γ_n^Ω are available in closed form, the expression in (3.6) can easily be implemented. The availability of such a pricing formula is particularly useful in a calibration procedure, in which the parameters of the stochastic processes need to be approximated. In practice, option pricing models are calibrated to a number of market observed call option prices. It is therefore necessary for such a procedure to be highly efficient and a (semi-)closed form for an option pricing formula is desirable.

The COS method's accuracy is related to the size of the integration domain, Ω. If the domain is chosen too small, we expect a significant loss of accuracy, see [14]. On the other hand if the domain is too wide, a large number of terms in the Fourier expansion, N, has to be used for satisfactory accuracy. In [14] the truncation range was defined in terms of the moments of $\log\left(\frac{S_T}{K}\right)$ of the form:

$$\delta_{1,2} = \mu_1 \pm L\sqrt{\mu_2 + \sqrt{\mu_4}}, \tag{3.11}$$

with the minus sign for δ_1, and the plus sign for δ_2, the μ_i are the corresponding i-th moments, and L is an appropriate constant. In our work, with the moments not directly available, we apply a simplified approximation for the integration range, and use:

$$\delta_{1,2} = 0 \pm L\sqrt{\tau}, \tag{3.12}$$

with τ, the time to maturity. As in [14], we fix $L = 8$ in (3.12).

4 Calibration and pricing under the hybrid model

For exotic financial products that involve more than one asset class, the pricing engine should be based on a stochastic model which takes into account the interactions between the asset classes, like the SZHW or the HHW models presented above. It is therefore interesting to evaluate price differences between the classical models and these hybrid models. For this purpose we consider several hybrid products, treated in subsequent subsections. The pricing is done using a Monte Carlo method.

Before we can price these products, however, we need to calibrate the models, i.e., to find the model parameters so that the models recover the market prices of plain vanilla options. This calibration procedure relies heavily on the characteristic function derived in the previous section and the appendices.

4.1 Calibration of the models

In this section we examine the extended stochastic volatility models and compare their performance to the Heston model. We use financial market data to estimate the model parameters and discuss the effect of the correlation between the equity and interest rate on the estimated parameters. For this purpose we have chosen the CAC40 call option implied volatilities of 17.10.2007. We perform the calibration of the models in two stages. Firstly, we calibrate the parameters for the interest rate process by using caplets and swaptions. Secondly, the remaining parameters, for the underlying asset, the volatility and the correlations, are calibrated to the plain vanilla option market prices. Standard procedures for the Hull-White calibration are employed [7]. Tables 3 and 4 present the estimated parameters and the associated squared sum errors (SSE) defines as,

$$\mathrm{SSE} = \sum_{i=1}^{n} \sum_{j=1}^{m} \left(C\left(T_i, K_j\right) - \hat{C}\left(T_i, T_j\right) \right)^2, \tag{4.1}$$

where $C(T_i, K_i)$ and $\hat{C}(T_i, T_j)$ are the market and the model prices, respectively, T_i is the ith time to maturity and K_j is the jth strike. We have 32 strikes, ($m = 32$), and 20 time points ($n = 20$).

Table 4 shows the calibration results for the Heston, Heston-Hull-White and Schöbel-Zhu-Hull-White models. We see that all the models are reasonably well calibrated with approximately the same error. We have used a two level calibration routine: a global search algorithm (simulated annealing) combined with a local search (Nelder-Mead) algorithm. In order to reduce parameter risk we set the speed of mean reversion of the volatility process, κ, to 0.5 and we have performed the simulation for a number of correlations, $\rho_{x,r}$. For both hybrid models some patterns in the calibrated parameters can be observed (see Table 4). For the SZHW and the HHW models two parameters, $\bar{\sigma}$ and σ_0, are unaffected by changing the correlation $\rho_{x,r}$. For the SZHW model we found $\bar{\sigma} \approx 0.2$, $\sigma_0 \approx 0.1$ and for the HHW model $\bar{\sigma} \approx 0.035$, $\sigma_0 \approx 0.01$. Another pattern we observed is that the the vol-vol parameter γ decreases from 0.08 to 0.02 for the SZHW and from 0.29 to 0.05 for the HHW model with increasing correlation $\rho_{x,r}$ from -70% to 0%. The reverse effect was obtained for positive correlation $\rho_{x,r}$. The correlation $\rho_{x,\sigma}$ between stock S_t and the volatility σ_t remains relatively stable for the HHW model, oscillating around -0.98. For the SZHW model it decreased from -0.31 to -0.99 for $\rho_{x,r}$ varying from -70% to -10% and increased from -0.72 to -0.38 for $\rho_{x,r}$ from 10% to 70%. The correlation $\rho_{r,\sigma}$ in the SZHW model does not show any regularity.

In the next section we use obtained the calibration results and check the impact of the correlation between the equity and interest rate on pricing exotic products.

For the pricing of financial derivatives, Monte Carlo methods are commonly used tools, especially for products like hybrid derivatives for which a closed-form pricing formula is not available. Because of discretization techniques like the Euler-Maruyama or Milstein schemes (see, for example, [37]) a Monte Carlo technique may sometimes give a negative or imaginary variance in the SV models. This is not acceptable. In the literature, improved techniques to perform a simulation of the AJD processes have been developed, see [2],[8]. An analysis of the possible ways to overcome the negative variance problem can be found in [29]. We have chosen the so-called absorption scheme, from [29], where at each iteration step $\max(\sigma_{t+\Delta_t}, 0)$ is taken.

Table 3: Parameters estimated from the market data (Hull-White model), r_0 is assumed to be the earliest forward rate. The interest rate term structure θ_t was found via Equation (2.16).

model	r_0	λ	η	SSE
Hull-White	0.01733	1.12	0.001	1e-3

4.2 Cliquet options

Cliquet options are very popular in the world of equity derivatives [43]. The contracts are constructed to give a protection against downside risk combined with a significant upside potential.

Table 4: Calibration results for the Schöbel-Zhu-Hull-White, Heston-Hull-White and Heston models defined in (2.17) and (2.30). The experiment was done with a priori defined speed of reversion for the volatility $\kappa = 0.5$, and correlation $\rho_{x,r}$ (SZHW and HHW). In the simulation for the Heston model a constant interest rate of $r = 0.0327$ was chosen.

model	$\rho_{x,r}$	$\bar{\sigma}$	γ	$\rho_{x,v}$	$\rho_{r,\sigma}$	σ_0	SSE
SZHW	-70%	0.1929	0.0787	-0.3116	0.4000	0.1000	9.5e-3
	-50%	0.2000	0.0539	-0.3967	0.1190	0.0990	9.1e-3
	-30%	0.2030	0.0400	-0.5699	0.3238	0.1000	9.0e-3
	-10%	0.2049	0.0189	-0.9888	0.3173	0.1002	9.2e-3
	10%	0.2039	0.0315	-0.7167	0.0634	0.0998	9.2e-3
	30%	0.2029	0.0376	-0.6039	0.2407	0.1001	9.0e-3
	50%	0.2018	0.0429	-0.5335	0.2505	0.0980	9.0e-3
	70%	0.1981	0.0576	-0.3822	-0.0776	0.0990	9.2e-3
HHW	-70%	0.0242	0.2905	-0.4157	–	0.0129	7.9e-3
	-50%	0.0309	0.0732	-0.9900	–	0.0104	8.3e-3
	-30%	0.0372	0.0596	-0.9899	–	0.0124	8.3e-3
	-10%	0.0403	0.0543	-0.9900	–	0.0134	8.3e-3
	10%	0.0402	0.0545	-0.9899	–	0.0134	8.3e-3
	30%	0.0370	0.0600	-0.9899	–	0.0123	8.3e-3
	50%	0.0306	0.0740	-0.9900	–	0.0103	8.3e-3
	70%	0.0215	0.1327	-0.8641	–	0.0078	8.3e-3
Heston	–	0.0770	0.3500	-0.6622	–	0.0107	7.8e-3

A cliquet option can be interpreted as a series of forward-starting European options, for which the total premium is determined in advance. The payout on each option can either be paid at the final maturity date, or at the end of a *reset* period. One of the cliquet type structures is a Globally Floored Cliquet with the following payoff:

$$\Pi(t_0 = 0, T) = \mathbb{E}^{\mathbb{Q}}\left(\mathrm{e}^{-\int_0^T r_s \mathrm{d}s} \cdot \max\left(\sum_{i=1}^{M} \min\left(A_{t_i}, \mathrm{LocalCap}\right), \mathrm{MinCoupon}\right) |\mathcal{F}_0\right). \tag{4.2}$$

Here $A_{t_i} = \max\left(\mathrm{LocalFloor}, \frac{S_{t_i}}{S_{t_{i-1}}} - 1\right)$, $t_i = i\frac{T}{M}$, with maturity T. M indicates the number of reset periods. We notice that the term A_{t_i} can be recognized as an ATM forward starting option, which is driven by a forward skew. It has been shown in [18] that the cliquet structures are significantly underpriced under a local volatility model for which forward skews are basically too flat.

Since the forward prices are not known a-priori, we derive the values from the so-called forward characteristic function. If we define $\mathbf{X}_T$ as a state vector at time T then the forward characteristic function, ϕ_F, can be found as

$$\begin{aligned}
\phi_F(\mathbf{u}, \mathbf{X}_T, t^*, T) &= \mathbb{E}^{\mathbb{Q}}\left(\mathrm{e}^{-\int_0^T r_s \mathrm{d}s} \mathrm{e}^{i\mathbf{u}^T(\mathbf{X}_T - \mathbf{X}_{t^*})} |\mathcal{F}_0\right) \\
&= \mathbb{E}^{\mathbb{Q}}\left(\mathrm{e}^{-\int_0^{t^*} r_s \mathrm{d}s - i\mathbf{u}^T \mathbf{X}_{t^*}} \phi\left(\mathbf{u}, \mathbf{X}_T, t^*, T\right) |\mathcal{F}_0\right) \\
&= \mathrm{e}^{A(\mathbf{u}, t^*, T)} \mathbb{E}^{\mathbb{Q}}\left(\mathrm{e}^{-\int_0^{t^*} r_s \mathrm{d}s - i\mathbf{u}^T \mathbf{X}_{t^*} + \mathbf{B}^T(\mathbf{u}, t^*, T)\mathbf{X}_{t^*}} |\mathcal{F}_0\right).
\end{aligned} \tag{4.3}$$

In the case of the plain Heston model, the forward characteristic function, ϕ_{FH}, reads:

$$\phi_{FH}(u, \mathbf{X}_T, t^*, T) = \mathrm{e}^{A(u,\tau^*)} \mathbb{E}^{\mathbb{Q}}\left(\mathrm{e}^{B_\sigma(u,\tau^*) v_{t^*}} |\mathcal{F}_0\right), \tag{4.4}$$

where $\tau^* = T - t^*$ and $A_H(u, \tau^*)$, $B_\sigma(u, \tau^*)$ are the Heston functions as introduced in [21]. The expectation under the risk neutral measure in (4.4) can be recognized as the Laplace transform of the transitional probability density function of a Cox-Ingersoll-Ross model [11], which is given by the following lemma:

Lemma 4.1 (Laplace transform of for Heston volatility process)**.** *The Laplace transform of the equation given by (4.4) for Heston stochastic volatility process has the following form*

$$\mathbb{E}^{\mathbb{Q}}\left(\mathrm{e}^{B_\sigma(u,t^*,T)v_{t^*}}|\mathcal{F}_0\right) = \left(\frac{1}{1-\dfrac{\gamma^2}{2\kappa}\left(1-\mathrm{e}^{-\kappa\tau}\right)B_\sigma(u,t^*,T)}\right)^{\frac{2\kappa\bar{\sigma}}{\gamma^2}} \exp\left(\frac{\mathrm{e}^{\kappa\tau}B_\sigma(u,t^*,T)\sigma_0}{1-\dfrac{\gamma^2}{2\kappa}\left(1-\mathrm{e}^{-\kappa\tau}\right)B_\sigma(u,t^*,T)}\right).$$

Proof. A detailed proof can be found in [38] or [1]. □

Figure 1 shows the performance of all three models applied to the pricing of the cliquet option defined in (4.2). We choose here $T = 3$, LocalCap $= 0.01$, LocalFloor $= -0.01$ and $M = 36$ (the contract measures the monthly performance). For large values of the MinCoupon the values of the hybrid under the three models are identical, which is expected since a large MinCoupon dominates the max operator in (4.2) and the expectation becomes simply the price of a zero coupon bond at time $t = 0$ multiplied by the deterministic MinCoupon. Figure 1 shows the pricing results for two correlations $\rho_{x,r} = -0.7$ and $\rho_{x,r} = 0.7$. In both cases the HHW model generates lower prices than other models. Moreover, the cliquet is priced significantly lower by the SZHW model than by the Heston model for $\rho_{x,r} = 0.7$ and it is priced higher than the Heston model for $\rho_{x,r} = -0.7$.

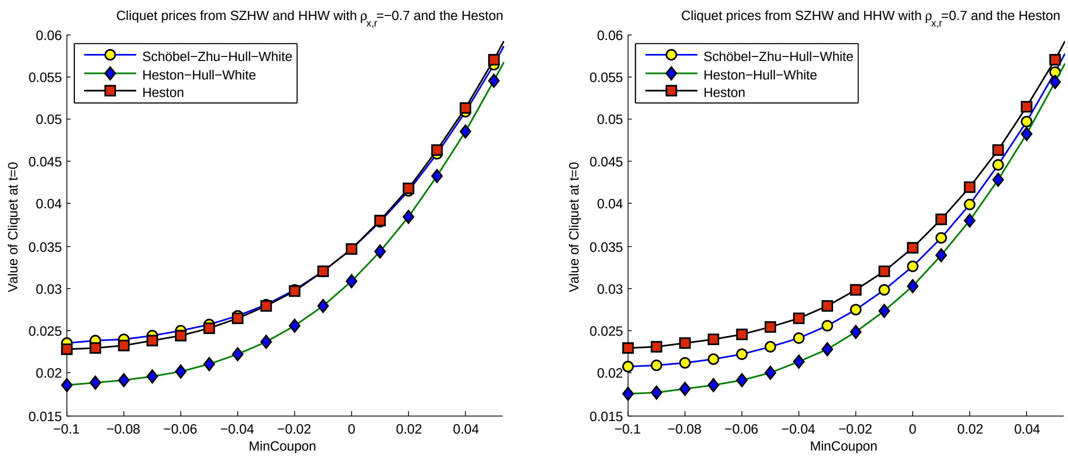

Figure 1: Pricing a cliquet product under the SZHW, the HHW and the Heston models. Both figures present the price of a globally floored cliquet as a function of MinCoupon given by (4.2) for $T = 3$ years and $M = 36$. The remaining parameters are as in Table 4. Left: Pricing with $\rho_{x,r} = -0.7$, Right: Pricing with $\rho_{x,r} = 0.7$.

4.3 A diversification product (performance basket)

Other hybrid products that an investor may use in strategic trading are so-called diversification products. These products, also known as 'performace baskets', are based on sets of assets with different expected returns and risk levels. Proper construction of such products may give reduced risk compared to any single asset, and an expected return that is greater than that of the least risky asset [24]. A simple example is a portfolio with two assets: a stock with a high risk and high return and a bond with a low risk and low return. If one introduces an equity component in a pure bond portfolio the expected return will increase. However, because of a non-perfect correlation between these two assets also a risk reduction is expected. If the percentage of the equity in the portfolio is increased, it eventually starts to dominate the structure and the risk may increase with a higher impact for a low or negative correlation [24]. An example is a financial product, defined in the following way:

$$\Pi(t_0 = 0, T) = \mathbb{E}^{\mathbb{Q}}\left(\mathrm{e}^{-\int_0^T r_s \mathrm{d}s} \cdot \max\left(0, \omega \cdot \frac{S_T}{S_0} + (1-\omega) \cdot \frac{B_T}{B_0}\right)|\mathcal{F}_0\right), \tag{4.5}$$

where S_T is the underlying asset at time T, B_T is a bond, ω represents a percentage ratio. Figure 2 shows the pricing results for the models discussed. The product pricing is performed with the Monte Carlo method and the parameters calibrated from the market data. For $\omega \in [0\%, 100\%]$ the max disappears from the payoff and only a sum of discounted expectations remains. The figure shows that the Heston model generates a significantly higher price, whereas the HHW and the SZHW prices are relatively close. The absolute difference between the models increases with percentage ω.

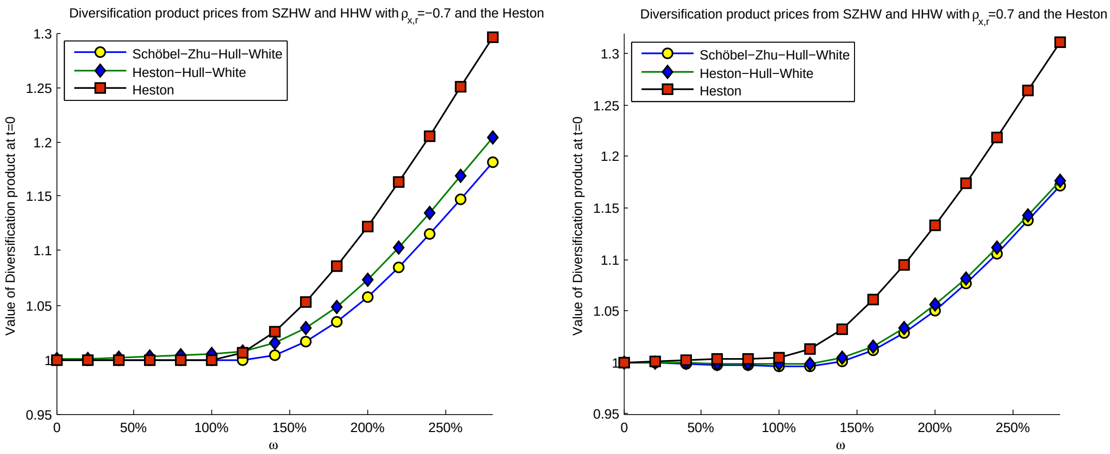

Figure 2: Pricing of a diversification hybrid product under different models. The simulations performed with $\tau = 10$. The remaining parameters are as in Table 4. Left: Pricing with $\rho_{x,r} = -0.7$, Right: Pricing with $\rho_{x,r} = 0.7$,

4.4 Strategic investment hybrid (best-of-strategy)

Suppose that an investor believes that if the price of an asset, S_t^1, goes up, then the equity markets under-perform relative to the interest rate yields, whereas, if S_t^1 drops down, the equity markets over-perform relative to the interest rate [24]. If the prices of S_t^1 are high, the market may expect an increase of the inflation and hence of the interest rates and low S_t^1 prices could have the opposite effect. In order to include such a feature in a hybrid product we define a contract in which an investor is allowed to buy a weighted performance coupon depending on the performance of another underlying. Such a product can be defined as follows,

$$\Pi(t_0 = 0, T) \quad = \quad \mathbb{E}^{\mathbb{Q}}\left(\mathrm{e}^{-\int_0^T r_s \mathrm{d}s} \cdot V_T | \mathcal{F}_0\right), \text{ with} \tag{4.6}$$

$$V_T = \max\left(0, \omega \cdot \frac{L_0}{L_T} + (1-\omega)\frac{S_T}{S_0}\right)\mathbf{1}_{S_T^1 > S_0^1} + \max\left(0, (1-\omega)\frac{L_0}{L_T} + \omega \cdot \frac{S_T}{S_0}\right)\mathbf{1}_{S_T^1 < S_0^1},$$

where $\omega \geq 0$ is a weighting factor related to a percentage, $L_T = \sum_{i=1}^M P(T, t_i)$ with $t_1 = T$ is the T-value of projected liabilities for certain time t_M, with $\omega > 100\% - \omega$.

Figure 3 shows the prices obtained from Monte Carlo simulation of the contract at time $t_0 = 0$ for maturity $T = t_1 = 3$ and time horizon $t_M = 12$ with one year spacing. Since we did not model the second underlying process, S_T^1, we assume that $S_T^1 > S_0^1$. We see that for $\omega \in [0\%, 100\%]$ the max over the sum of performances disappears and the hybrid can be relatively easily priced, i.e., separately for both underlyings (L_0/L_T and S_T/S_0). The difference between the stochastic models becomes more pronounced for $\omega > 0\%$ since then the correlation plays a more important role. The simulations performed for $\rho_{x,r} = -70\%$ and $\rho_{x,r} = 70\%$ show that the absolute difference between the SZHW and the HHW models becomes significant for $\omega > 200\%$. The figure shows that for small ω the prices of the SZHW and the HHW models are relatively close, whereas the Heston model gives lower prices for $\omega > 50\%$.

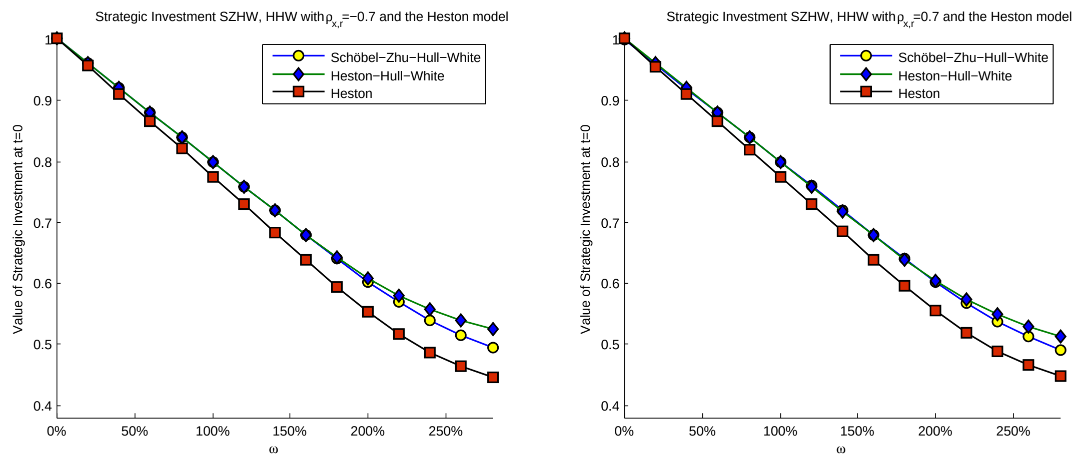

Figure 3: Discounted payoffs of the strategic investment hybrid priced with the SZHW, the HHW and the Heston models in dependence of ω. The remaining parameters are as in Table 4. Left: Pricing with $\rho_{x,r} = -0.7$, Right: Pricing with $\rho_{x,r} = 0.7$

5 Conclusions

In this paper we have presented an extension of the Schöbel-Zhu stochastic volatility model with a Hull-White interest rate process and evaluated it by means of pricing structured hybrid derivative products.

The aim was to define a hybrid stochastic process which belongs to the class of affine jump-diffusion models, as this may lead to efficient calibration of the model. We have shown that the so-called Schöbel-Zhu-Hull-White model belongs to the category of affine jump diffusion processes. No restrictions regarding the choice of correlation structure between the different Wiener processes appearing need to be made.

We also compared the model to the Heston-Hull-White hybrid model with an indirectly implied correlation between the equity and the interest rate. We have found that although the model is very attractive, because of its square root volatility structure it is unable to generate extreme correlations.

Due to the resulting semi-closed (for Schöbel-Zhu-Hull-White) and closed (Heston-Hull-White) characteristic functions we were able to calibrate models in an efficient way by means of the Fourier cosine expansion pricing technique, adapted to stochastic interest rate.

It has been shown by numerical experiments for different hybrid products that under the same plain vanilla prices the extended stochastic volatility models give different prices than the Heston model.

The present hybrid model cannot model a skew in the interest rates, which will be part of our future work.

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A Proofs of various lemmas

In this appendix we have placed the proofs of the various lemmas.

A.1 Proof of Lemma 2.3

Proof.
We need to find the solution of:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}A(u,\tau) = -r_0 + \mathbf{B}^T a_0 + \frac{1}{2}\mathbf{B}^T c_0 \mathbf{B}, \tag{A.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{B}(u,\tau) = -r_1 + a_1^T \mathbf{B} + \frac{1}{2}\mathbf{B}^T c_1 \mathbf{B}. \tag{A.2}$$

For the space vector $\mathbf{X}_t^* = [\widetilde{x}_t, \widetilde{r}_t, v_t, \sigma_t]^T$ we have

$$a_0 = [0, 0, \gamma^2, \kappa\overline{\sigma}]^T, \quad a_1 = \begin{bmatrix} 0 & 1 & -\frac{1}{2} & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & -2\kappa & 2\kappa\overline{\sigma} \\ 0 & 0 & 0 & -\kappa \end{bmatrix}, \quad r_0 = 0, \quad r_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

and

$$\Sigma := \sigma(\mathbf{X}_t)\sigma(\mathbf{X}_t)^T = \begin{bmatrix} v & \sigma\eta\rho_{x,r} & 2v\gamma\rho_{x,v} & \sigma\gamma\rho_{x,\sigma} \\ & \eta^2 & 2\eta\sigma\gamma\rho_{r,v} & \eta\gamma\rho_{r,\sigma} \\ & & 4v\gamma^2 & 2\sigma\gamma^2 \\ & & & \gamma^2 \end{bmatrix}.$$

This leads to

$$c_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ & \eta^2 & 0 & \eta\gamma\rho_{r,\sigma} \\ & & 0 & 0 \\ & & & \gamma^2 \end{bmatrix}, \quad c_1 = \begin{bmatrix} (0,0,1,0) & (0,0,0,\eta\rho_{x,r}) & (0,0,2\gamma\rho_{x,v},0) & (0,0,0,\gamma\rho_{x,\sigma}) \\ & (0,0,0,0) & (0,0,0,2\eta\gamma\rho_{r,v}) & (0,0,0,0) \\ & & (0,0,4\gamma^2,0) & (0,0,0,2\gamma^2) \\ & & & (0,0,0,0) \end{bmatrix}.$$

With

$$\frac{1}{2}\mathbf{B}^T c_1 \mathbf{B} = \frac{1}{2}\begin{bmatrix} \sum_{i=1}^4 \sum_{j=1}^4 B_i[s_1(1)]_{i,j}B_j \\ \sum_{i=1}^4 \sum_{j=1}^4 B_i[s_1(2)]_{i,j}B_j \\ \sum_{i=1}^4 \sum_{j=1}^4 B_i[s_1(3)]_{i,j}B_j \\ \sum_{i=1}^4 \sum_{j=1}^4 B_i[s_1(4)]_{i,j}B_j \end{bmatrix},$$

(with $i = 1, \ldots, 4$ representing x, v, r, σ) we obtain the following system

$$\frac{\mathrm{d}A}{d\tau} = [B_x, B_r, B_v, B_\sigma]\begin{bmatrix} 0 \\ 0 \\ \gamma^2 \\ \kappa\overline{\sigma} \end{bmatrix} + \frac{1}{2}[B_x, B_r, B_v, B_\sigma]\begin{bmatrix} 0 & 0 & 0 & 0 \\ & \eta^2 & 0 & \eta\gamma\rho_{r,\sigma} \\ & & 0 & 0 \\ & & & \gamma^2 \end{bmatrix}\begin{bmatrix} B_x \\ B_r \\ B_v \\ B_\sigma \end{bmatrix}, \tag{A.3}$$

$$\frac{\mathrm{d}\mathbf{B}}{d\tau} = \begin{bmatrix} \frac{\mathrm{d}B_x}{d\tau} \\ \frac{\mathrm{d}B_r}{d\tau} \\ \frac{\mathrm{d}B_v}{d\tau} \\ \frac{\mathrm{d}B_\sigma}{d\tau} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & -\lambda & 0 & 0 \\ -\frac{1}{2} & 0 & -2\kappa & 0 \\ 0 & 0 & 2\kappa\overline{\sigma} & -\kappa \end{bmatrix}\begin{bmatrix} B_x \\ B_r \\ B_v \\ B_\sigma \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 \\ 0 \\ S_1 \\ S_2 \end{bmatrix}, \tag{A.4}$$

where

$$S_1 = B_x^2 + 4\gamma\rho_{x,v}B_xB_v + 4\gamma^2 B_v^2, \tag{A.5}$$

$$S_2 = 2\eta\rho_{x,r}B_xB_r + 2\gamma\rho_{x,\sigma}B_xB_\sigma + 4\eta\gamma\rho_{r,v}B_rB_v + 4\gamma^2 B_vB_\sigma. \tag{A.6}$$

Now, simplification of the equations (A.3) and (A.4) finishes the proof. □

A.2 Proof of Lemma 2.4

Proof.
In the 1D case, i.e., $\mathbf{u} = [u, 0, 0, 0]^T$ we start by solving the ODE for dB_r,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}B_r + \lambda B_r = iu - 1.$$

Standard calculations give

$$\int_0^\tau \mathrm{d}\left(\mathrm{e}^{\lambda s}B_r(u,s)\right) = (iu-1)\int_0^\tau \mathrm{e}^{\lambda s}\mathrm{d}s, \text{ i.e.,}$$

$$\mathrm{e}^{\lambda\tau}B_r(u,\tau) - \mathrm{e}^0 B_r(u,0) = (iu-1)\left(\frac{1}{\lambda}\mathrm{e}^{\lambda\tau} - \frac{1}{\lambda}\right).$$

Using the boundary condition, $B_r(u,0) = 0$, gives, $B_r(u,\tau) = \frac{1}{\lambda}(iu-1)\left(1-\mathrm{e}^{-\lambda t}\right)$.

The ODE for B_v now reads (using $B_x = iu$):

$$\frac{\mathrm{d}}{\mathrm{d}\tau}B_v = -\frac{1}{2}u(i+u) + 2\gamma^2 B_v^2 - 2(\kappa - \gamma\rho_{x,v}iu)B_v. \tag{A.7}$$

In order to simplify this equation we introduce the variables $\alpha = -\frac{1}{2}u(i+u)$ and $\beta = 2(\kappa - \gamma\rho_{x,v}iu)$. The ODE can then be presented in the following form:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}B_v = \alpha - \beta B_v + 2\gamma^2 B_v^2. \tag{A.8}$$

Following the calculations for the Heston model the solution of (A.8) reads,

$$B_v(u,\tau) = \frac{\beta - d}{4\gamma^2}\left(\frac{1-\mathrm{e}^{-\tau d}}{1-\mathrm{e}^{-\tau d}\left(\frac{b}{a}\right)}\right),$$

where $a = \beta + d/4\gamma^2$, $b = (\beta - d)/(4\gamma^2)$ and $d = \sqrt{\beta^2 - 8\alpha\gamma^2}$. This solution can be simplified to

$$B_v(u,\tau) = b\left(\frac{1-\mathrm{e}^{-\tau d}}{1-g\mathrm{e}^{-\tau d}}\right),$$

with $g = (\beta - d)/(\beta + d)$.

Next, we solve the ODE for B_σ,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}B_\sigma = (2\kappa\bar{\sigma}B_v + \eta\rho_{x,r}B_x B_r + 2\eta\gamma\rho_{r,v}B_r B_v) + \left(\gamma\rho_{x,\sigma}B_x + 2\gamma^2 B_v - \kappa\right)B_\sigma. \tag{A.9}$$

We introduce the following functions,

$$\zeta(\tau) = 2\kappa\bar{\sigma}B_v + \eta\rho_{x,r}B_x B_r + 2\eta\gamma\rho_{r,v}B_r B_v, \tag{A.10}$$

$$\xi(\tau) = \gamma\rho_{x,\sigma}B_x + 2\gamma^2 B_v - \kappa. \tag{A.11}$$

This leads to the following ODE

$$\frac{\mathrm{d}}{\mathrm{d}\tau}B_\sigma - \xi(\tau)B_\sigma = \zeta(\tau),$$

whose solution follows from,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\mathrm{e}^{-\int_0^\tau \xi(s)\mathrm{d}s}B_\sigma\right) = \zeta(\tau)\exp\left(-\int_0^\tau \xi(s)\mathrm{d}s\right),$$

or

$$\exp\left(-\int_0^\tau \xi(s)\mathrm{d}s\right)B_\sigma = \int_0^\tau \zeta(s)\exp\left(-\int_0^s \xi(k)\mathrm{d}k\right)\mathrm{d}s.$$

So, finally, we need to calculate

$$\begin{cases} B_\sigma(u,\tau) = & \exp\left(\int_0^\tau \xi(s)\mathrm{d}s\right)\int_0^\tau \zeta(s)\exp\left(-\int_0^s \xi(k)\mathrm{d}k\right)\mathrm{d}s. \\ B_\sigma(u,0) = 0. \end{cases} \tag{A.12}$$

For this, we start with the integral for $\xi(k)$:

$$\begin{aligned} \int_0^s \xi(k)\mathrm{d}k &= \int_0^s \left(\gamma\rho_{x,\sigma}iu + 2\gamma^2 B_v - \kappa\right)\mathrm{d}k \\ &= \left(\gamma\rho_{x,\sigma}iu - \kappa + \tfrac{\beta-d}{2g}\right)s + \tfrac{(\beta-d)(g-1)}{2dg}\log\left(\tfrac{\mathrm{e}^{sd}-g}{1-g}\right) \\ &= C_1 s + C_2\log(\tfrac{\mathrm{e}^{sd}-g}{1-g}) \end{aligned} \tag{A.13}$$

where $C_1 = \left(\gamma\rho_{x,\sigma}iu - \kappa + \frac{\beta-d}{2g}\right)$, $C_2 = \frac{(\beta-d)(g-1)}{2dg}$, $\beta = 2(\kappa - \gamma\rho_{x,v}iu)$, $d = \sqrt{\beta^2 - 8\alpha\gamma^2}$ and $g = \frac{\beta-d}{\beta+d}$. After substitution of these quantities, we find that $C_1 = D/2$ and $C_2 = -1$.

Next, we need to calculate the exponent of the integral of ξ:

$$\exp\left(\int_0^s \xi(k)\mathrm{d}k\right) = \exp\left(C_1 s + C_2\log\left(\frac{\mathrm{e}^{sd}-g}{1-g}\right)\right) = \exp\left(\frac{sd}{2}\right)\left(\frac{1-g}{\mathrm{e}^{sd}-g}\right), \tag{A.14}$$

and we can include ζ in the integral,

$$\int_0^\tau \zeta(s)\exp\left(-\int_0^s \xi(k)\mathrm{d}k\right)\mathrm{d}s = \int_0^\tau \left(2\kappa\bar{\sigma}B_v + \eta\rho_{x,r}B_xB_r + 2\eta\gamma\rho_{r,v}B_rB_v\right)\exp\left(-\tfrac{d}{2}s\right)\left(\tfrac{\mathrm{e}^{sd}-g}{1-g}\right)\mathrm{d}s. \tag{A.15}$$

This integral is split into three parts. The first part can be solved analytically,

$$\begin{aligned} \int_0^\tau 2\kappa\bar{\sigma}B_v\mathrm{e}^{-\frac{d}{2}s}\left(\tfrac{\mathrm{e}^{sd}-g}{1-g}\right)\mathrm{d}s &= 2\kappa\bar{\sigma}b\int_0^\tau\left(\tfrac{1-\mathrm{e}^{-sd}}{1-\mathrm{e}^{-sd}g}\right)\mathrm{e}^{-\frac{d}{2}s}\left(\tfrac{\mathrm{e}^{sd}-g}{1-g}\right)\mathrm{d}s \\ &= \tfrac{2\kappa\bar{\sigma}b}{1-g}\int_0^\tau \mathrm{e}^{-\frac{sd}{2}}\left(\mathrm{e}^{sd}-1\right)\mathrm{d}s \\ &= \tfrac{16\kappa\bar{\sigma}b\sinh^2\left(\frac{\tau d}{4}\right)}{(1-g)d} \equiv \frac{f_1}{1-g}. \end{aligned} \tag{A.16}$$

The second part can be solved analytically as well,

$$\begin{aligned} \int_0^\tau \eta\rho_{x,r}B_xB_r\mathrm{e}^{-\frac{d}{2}s}\left(\tfrac{\mathrm{e}^{sd}-g}{1-g}\right)\mathrm{d}s &= \int_0^\tau \eta\rho_{x,r}\tfrac{1}{\lambda}iu(iu-1)(1-\mathrm{e}^{-\lambda s})\mathrm{e}^{-\frac{sd}{2}}\left(\tfrac{\mathrm{e}^{sd}-g}{1-g}\right)\mathrm{d}s \\ &= \tfrac{\eta\rho_{x,r}iu(iu-1)}{(1-g)\lambda}\int_0^\tau \mathrm{e}^{-\frac{sd}{2}}(1-\mathrm{e}^{-\lambda s})(\mathrm{e}^{sd}-g)\mathrm{d}s \\ &= \tfrac{\eta\rho_{x,r}iu(iu-1)}{(1-g)\lambda}(f_2 - f_3), \end{aligned} \tag{A.17}$$

where

$$f_2 = \frac{2}{d}(\mathrm{e}^{\frac{\tau d}{2}} - 1) + \frac{2g}{d}(\mathrm{e}^{-\frac{\tau d}{2}} - 1), \tag{A.18}$$

$$f_3 = \frac{2\left(\mathrm{e}^{\frac{\tau}{2}(d-2\lambda)} - 1\right)}{d-2\lambda} - \frac{2g\left(1-\mathrm{e}^{-\frac{\tau}{2}(d+2\lambda)}\right)}{d+2\lambda}, \tag{A.19}$$

and the third part reads,

$$\begin{aligned} \int_0^\tau 2\eta\gamma\rho_{r,v}B_rB_v\mathrm{e}^{-\frac{d}{2}s}\left(\tfrac{\mathrm{e}^{sd}-g}{1-g}\right)\mathrm{d}s &= \tfrac{2\eta\gamma\rho_{r,v}}{1-g}\int_0^\tau B_rB_v\mathrm{e}^{-\frac{d}{2}s}\left(\mathrm{e}^{sd}-g\right)\mathrm{d}s \\ &= \tfrac{2\eta\gamma\rho_{r,v}(iu-1)b}{(1-g)\lambda}\int_0^\tau \mathrm{e}^{-\frac{1}{2}s(d+2\lambda)}(\mathrm{e}^{sd}-1)(\mathrm{e}^{s\lambda}-1)\mathrm{d}s \\ &= \tfrac{2\eta\gamma\rho_{r,v}(iu-1)b}{(1-g)\lambda}(f_4 + f_5), \end{aligned} \tag{A.20}$$

where

$$f_4 = \frac{2}{d-2\lambda} - \frac{4}{d} + \frac{2}{d+2\lambda}, \tag{A.21}$$

$$f_5 = \left(\mathrm{e}^{-\frac{1}{2}\tau(d+2\lambda)}\right)\left(\frac{2\mathrm{e}^{\tau\lambda}(1+\mathrm{e}^{d\tau})}{d} - \frac{2\mathrm{e}^{d\tau}}{d-2\lambda} - \frac{2}{d+2\lambda}\right). \tag{A.22}$$

So, finally, we have:

$$\begin{aligned} B_\sigma(u,\tau) &= \exp\left(\int_0^\tau \xi(s)\mathrm{d}s\right)\int_0^\tau \zeta(s)\exp\left(-\int_0^s \xi(k)dk\right)\mathrm{d}s \\ &= f_0\left(f_1 + \tfrac{1}{\lambda}\eta\rho_{x,r}iu(iu-1)(f_2-f_3) + \tfrac{1}{\lambda}2\eta\gamma\rho_{r,v}b(iu-1)(f_4+f_5)\right), \end{aligned} \tag{A.23}$$

with $f_0 = \mathrm{e}^{\frac{d}{2}\tau}/(\mathrm{e}^{\tau d}-g)$, f_2, f_3 from (A.18) and (A.19) respectively , f_4 from (A.21) and f_5 from (A.22).

Now, we solve the ODE for $A(u,\tau)$:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}A = \gamma^2 B_v + \kappa\bar{\sigma}B_\sigma + \frac{1}{2}\eta^2 B_r^2 + \frac{1}{2}\gamma^2 B_\sigma^2 + \eta\gamma\rho_{r,\sigma}B_\sigma B_r, \tag{A.24}$$

with solution,

$$A(u,\tau) - A(u,0) = \gamma^2\int_0^\tau B_v\mathrm{d}s + \kappa\bar{\sigma}\int_0^\tau B_\sigma\mathrm{d}s + \tfrac{1}{2}\eta^2\int_0^\tau B_r^2\mathrm{d}s + \tfrac{1}{2}\gamma^2\int_0^\tau B_\sigma^2\mathrm{d}s + \eta\gamma\rho_{r,\sigma}\int_0^\tau B_\sigma B_r\mathrm{d}s. \tag{A.25}$$

Or,

$$A(u,\tau) = \underbrace{\int_0^\tau\left(\gamma^2 B_v + \frac{1}{2}\eta^2 B_r^2\right)ds}_{A_1(u,\tau)} + \underbrace{\int_0^\tau B_\sigma\left(\kappa\bar{\sigma} + \frac{1}{2}\gamma^2 B_\sigma + \eta\rho_{r,\sigma}\gamma B_r\right)ds}_{\Gamma(u,\tau)} \tag{A.26}$$

In order to find $A(u,\tau)$ we have to evaluate the integrals $A_1(u,\tau)$ and $\Gamma(u,\tau)$. Integral $\Gamma(u,\tau)$ involves a hyper-geometric function (called the ${}_2F_1$ function or simply Gaussian function), which is computed numerically here. For integral $A_1(u,\tau)$ we have two representations,

$$A_1(u,\tau) = -\frac{1}{2\gamma^2}\log\left(\frac{g\mathrm{e}^{-sd}-1}{g-1}\right) + f_6 - \frac{1}{2\lambda^3}\cdot f_7, \text{ or} \tag{A.27}$$

$$A_1(u,\tau) = -\frac{1}{2\gamma^2}\log\left(\frac{\mathrm{e}^{sd}-g}{1-g}\right) + f_6 - \frac{1}{2\lambda^3}\cdot f_7, \text{ where} \tag{A.28}$$

$$f_6 = \frac{1}{4\gamma^2}(\beta-d)\tau \tag{A.29}$$

$$f_7 = (iu-1)^2(3+\mathrm{e}^{-2\tau\lambda} - 4\mathrm{e}^{-\tau\lambda} - 2\tau\lambda). \tag{A.30}$$

Since in $A_1(u,\tau)$ a complex-valued logarithm appears, it should be treated with some care. It turns out that the second formulation gives rise to discontinuities which may cause inaccuracies. According to [28], an easy way to avoid any errors due to complex-valued discontinuities is to apply numerical integration.

We know that the price of a zero coupon bond can be obtained from the characteristic function, $\phi_{\mathrm{SZHW}}(\mathbf{u},\mathbf{X}_t,t,T)$, by setting $\mathbf{u} = [0,0,0,0]^T$. So,

$$\begin{aligned} P(t,T) &= \phi(0,\mathbf{X}_t,\tau) \\ &= \exp\left(-\int_t^T \psi_s ds\right)\exp\left(A(0,\tau) + B_x(0,\tau)x_t + B_r(0,\tau)\widetilde{r}_t + B_v(0,\tau)v_t + B_\sigma(0,\tau)\sigma_t\right). \end{aligned} \tag{A.31}$$

Since $\widetilde{r}_0 = 0$, we have $P(0,T) = \exp\left(-\int_0^T \psi_s ds\right)\exp\left(A(0,\tau) + B_x(0,\tau)x_0 + B_v(0,\tau)v_0 + B_\sigma(0,\tau)\sigma_0\right)$ and it is easy to check that $B_x(0,T) = 0$, $B_v(0,T) = 0$, $B_\sigma(0,T) = 0$, and,

$$A(0,T) = \frac{1}{2}\eta^2\int_0^T B_r(0,s)^2\mathrm{d}s = \frac{\eta^2}{2\lambda^3}\left(-\frac{3}{2} - \frac{1}{2}\mathrm{e}^{-2\lambda T} + 2\mathrm{e}^{-\lambda T} + \lambda T\right). \tag{A.32}$$

Therefore, $P(0,T) = \exp\left(-\int_0^T \psi_s \mathrm{d}s + A(0,T)\right)$, or, $\log\left(P(0,T)\right) = -\int_0^T \psi_s \mathrm{d}s + A(0,T)$, which finally gives us:

$$\psi_T = -\frac{\partial}{\partial T}\log P(0,T) + \frac{\partial}{\partial T}A(0,T) = f(0,T) + \frac{\eta^2}{2\lambda^2}\left(1-\mathrm{e}^{-\lambda T}\right)^2. \tag{A.33}$$

Since $\psi_0 = f(0,0) \equiv r_0$, where r_0 is the initial value of the interest rate process r_t. With $\mathbf{u} = [u,0,0,0]^T$, we find:

$$\phi_{\mathrm{SZHW}}(u,\mathbf{X}_t,t,T) = \exp\left(\widetilde{A}(u,\tau) + B_x(u,\tau)x_t + B_r(u,\tau)\widetilde{r}_t + B_v(u,\tau)v_t + B_\sigma(u,\tau)\sigma_t\right), \tag{A.34}$$

with

$$\begin{aligned}
\widetilde{A}(u,\tau) &= -\int_t^T \psi_s \mathrm{d}s + iu\int_t^T \psi_s \mathrm{d}s + A(u,\tau)\\
&= (iu-1)\int_t^T \left(f(0,s) + \tfrac{\eta^2}{2\lambda^2}\left(1-\mathrm{e}^{-\lambda s}\right)^2\right)\mathrm{d}s + A(u,\tau)\\
&= (1-iu)\int_t^T d\left(\log(P(0,s))\right) + (1-iu)\tfrac{\eta^2}{2\lambda^2}\int_t^T \left(1-\mathrm{e}^{-\lambda s}\right)^2 \mathrm{d}s + A(u,\tau)\\
&= (1-iu)\log\left(\tfrac{P(0,T)}{P(0,t)}\right) + (1-iu)\tfrac{\eta^2}{2\lambda^2}\left((T-t) + \tfrac{2}{\lambda}\left(\mathrm{e}^{-\lambda T}-\mathrm{e}^{-\lambda t}\right) - \tfrac{1}{2\lambda}\left(\mathrm{e}^{-2\lambda T}-\mathrm{e}^{-2\lambda t}\right)\right) + A(u,\tau),
\end{aligned} \tag{A.35}$$

and $A(u,\tau)$ as in (A.26). Now, by setting $\Theta(x) = \exp\left(\frac{x\tau}{2}\right)$ the discounted CF for the Schöbel-Zhu-Hull-White hybrid process is determined and the proof is finished. ☐

A.3 Proof of Lemma 2.6

Proof.
As in case of the SZHW hybrid model we need to find the solution of:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}A(u,\tau) = -r_0 + \mathbf{B}^T a_0 + \frac{1}{2}\mathbf{B}^T c_0 \mathbf{B}, \tag{A.36}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{B}(u,\tau) = -r_1 + a_1^T\mathbf{B} + \frac{1}{2}\mathbf{B}^T c_1 \mathbf{B}. \tag{A.37}$$

For the space vector $\mathbf{X}_t^* = [\widetilde{x}_t, \widetilde{r}_t, \sigma_t]^T$ we have

$$a_0 = [-\frac{1}{2}\Delta_{S,r}^2, 0, \kappa\bar{\sigma}]^T,\quad a_1 = \begin{bmatrix} 0 & 1 & -\frac{1}{2}\\ 0 & -\lambda & 0\\ 0 & 0 & -\kappa \end{bmatrix},\quad r_0 = 0,\quad r_1 = \begin{bmatrix} 0\\ 1\\ 0\end{bmatrix},$$

and

$$\Sigma := \sigma(\mathbf{X}_t)\sigma(\mathbf{X}_t)^T = \begin{bmatrix} \sigma_t + \Delta_{S,r}^2 & \Delta_{S,r}\eta & \rho_{x,\sigma}\gamma\sigma_t\\ & \eta^2 & 0\\ & & \gamma^2\sigma_t \end{bmatrix}.$$

This leads to

$$c_0 = \begin{bmatrix} \Delta_{S,r}^2 & \eta\Delta_{S,r} & 0\\ \eta\Delta_{S,r} & \eta^2 & 0\\ 0 & 0 & 0\end{bmatrix},\quad c_1 = \begin{bmatrix} (0,0,1) & (0,0,0) & (0,0,\rho_{x,\sigma}\gamma)\\ (0,0,0) & (0,0,0) & (0,0,0)\\ (0,0,\rho_{x,\sigma}\gamma) & (0,0,0) & (0,0,\gamma^2)\end{bmatrix}.$$

With

$$\frac{1}{2}\mathbf{B}^T c_1 \mathbf{B} = \frac{1}{2}\begin{bmatrix} \sum_{i=1}^3\sum_{j=1}^4 B_i[s_1(1)]_{i,j}B_j\\ \sum_{i=1}^3\sum_{j=1}^4 B_i[s_1(2)]_{i,j}B_j\\ \sum_{i=1}^3\sum_{j=1}^4 B_i[s_1(3)]_{i,j}B_j \end{bmatrix},$$

(with $i = 1,\ldots,3$ representing x, r, σ) we obtain the following system

$$\frac{\mathrm{d}A}{\mathrm{d}\tau} = [B_x, B_r, B_\sigma]\begin{bmatrix} -\frac{1}{2}\Delta_{S,r}^2\\ 0\\ \kappa\bar{\sigma}\end{bmatrix} + \frac{1}{2}[B_x, B_r, B_\sigma]\begin{bmatrix} \Delta_{S,r}^2 & \eta\Delta_{S,r} & 0\\ \eta\Delta_{S,r} & \eta^2 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix} B_x\\ B_r\\ B_\sigma\end{bmatrix}, \tag{A.38}$$

$$\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}\tau} = \begin{bmatrix} \frac{\mathrm{d}B_x}{\mathrm{d}\tau} \\ \frac{\mathrm{d}B_r}{\mathrm{d}\tau} \\ \frac{\mathrm{d}B_\sigma}{\mathrm{d}\tau} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 1 & -\lambda & 0 \\ -\frac{1}{2} & 0 & -\kappa \end{bmatrix} \begin{bmatrix} B_x \\ B_r \\ B_\sigma \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ S_1 \end{bmatrix}, \tag{A.39}$$

where

$$S_1 = B_x^2 + 2\rho_{x\sigma}\gamma B_x B_\sigma + \gamma^2 B_\sigma^2. \tag{A.40}$$

$$\tag{A.41}$$

Now, simplification of the equations (A.38) and (A.39) finishes the proof. □