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Efficient Pricing of European-Style Asian Options under Exponential Lévy Processes Based on Fourier Cosine Expansions*

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Abstract. We propose an efficient pricing method for arithmetic and geometric Asian options under exponential Lévy processes based on Fourier cosine expansions and Clenshaw–Curtis quadrature. The pricing method is developed for both European-style and American-style Asian options and for discretely and continuously monitored versions. In the present paper we focus on the European-style Asian options. The exponential convergence rates of Fourier cosine expansions and Clenshaw–Curtis quadrature reduces the CPU time of the method to milliseconds for geometric Asian options and a few seconds for arithmetic Asian options. The method's accuracy is illustrated by a detailed error analysis and by various numerical examples.

Key words. arithmetic Asian options, exponential Lévy asset price processes, Fourier cosine expansions, Clenshaw–Curtis quadrature, exponential convergence

AMS subject classifications. 65C30, 60H35, 65T50

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1. Introduction.

Asian options, introduced in 1987, belong to the class of path-dependent options. Their payoff is typically based on a geometric or arithmetic average of underlying asset prices at monitoring dates before maturity. The number of monitoring dates can be finite (*discretely monitored*) or infinite (*continuously monitored*). Volatility inherent in an asset is reduced due to the averaging feature, leading to cheaper options compared to plain vanilla option equivalents.

For geometric Asian options a closed-form solution under the Black–Scholes model has been presented in [18]. Other asset models driven by an exponential Lévy process have been studied in [15], resulting in an efficient valuation method based on the fast Fourier transform (FFT).

For arithmetic Asian options the prices have to be approximated numerically. Monte Carlo methods have been applied for this task, for example, in [18]. An efficient PDE method for arithmetic Asian options, which works particularly well for short maturities, has been presented in [21].

Advanced pricing methods for options on the arithmetic average are based on a recursive integration procedure in which the transitional probability density function of the log-return of the sum of asset prices is approximated; see [7, 3, 8, 17, 15, 14]. In [7, 3] an FFT and inverse

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FFT have been incorporated into the procedure to approximate the governing densities. The study in [7] was focused on log-normally distributed underlying processes and required a fine grid to approximate the probability density function. This method is extended to more general densities in [3], where the size of the grid was reduced by recentering the probability densities at each monitoring step, resulting in reduced CPU time. A recent contribution in this direction was presented in [8], where discretely sampled Asian options were priced via backward price convolutions. Another pricing approach can be found in [17], where the governing densities were computed by a special Laplace inversion, for guaranteed return rate products, which can be seen as generalized discretely sampled Asian options.

In [15] the FFT was used to approximate the density of the increments under Lévy processes between consecutive monitoring dates, in combination with a recursive Gaussian quadrature procedure. The total computational complexity in [15] was $O(Mn^2)$, with M the number of monitoring dates and n the number of points used in the quadrature. The method in [15] is improved in [14], in which it is shown that the Asian option value can be derived by a price recursion or density recursion procedure. It is transformed into a complex-valued frequency-domain representation via the z-transform. The z-transform can be seen as a discrete-time equivalent of the Laplace transform. The Asian option value is then determined via an inverse z-transform, in combination with a quadrature rule as in [1], which converges exponentially. For each quadrature point, however, an algebraically converging quadrature rule is used for approximation. Another contribution in [14] is that via an Euler acceleration scheme, the number of integral equations that need to be solved remains bounded, so that the computational cost does not increase significantly when the number of monitoring dates exceeds a certain level.

Finally, explicit formulas for upper and lower bounds of the Asian option prices have been derived, for example, in [19] for exponential Lévy processes. The results in [19] are shown to be more accurate than existing bounds.

In this paper we propose a different pricing method for Asian options and call it the *ASCOS* (ASian COSine) method, as it is related to the COS method from [12, 13]. The method is also inspired by the work in [15], but there are significant differences. Instead of recursively recovering the transitional probability density function of the logarithm of the sum of asset prices, as in [15], we *recover the corresponding characteristic function* by means of Fourier cosine expansions. The transitional density function is then in turn approximated in terms of the conditional characteristic function by a Fourier cosine expansion. The characteristic function for an exponential Lévy process is known analytically, and a Fourier cosine expansion most often exhibits exponential convergence. Furthermore, the Clenshaw–Curtis quadrature rule is applied in the ASCOS method to approximate certain integrals appearing. We will perform an extensive error analysis to confirm exponential convergence for Asian options.

The ASCOS pricing method can thus be seen as an efficient alternative to the FFT and convolution methods in [7, 15, 3, 19, 8, 14]. The Asian option prices obtained from the ASCOS pricing method converge at a reliable convergence rate when the number of monitoring dates, M, increases.

In section 2, the ASCOS method to price geometric Asian options under exponential Lévy asset price processes (discretely and continuously monitored) is presented. The pricing algorithm for arithmetic Asian options is then detailed in section 3. An error analysis is given in section 4, and numerical results are presented in section 5. We compare our results to those presented in [15].

The ASCOS method is extended to pricing American-style Asian options in another paper [23]. What is key here is that instead of recovering the density function, like in [7, 15, 3, 19, 14], the characteristic function is recovered, which enables us to also price American-style Asian options.

Here we focus on fixed-strike Asian options. The extension to floating-strike Asian options follows directly from the symmetry between floating-strike and fixed-strike Asian options, as explained in [16, 11].

2. ASCOS method for European-style geometric Asian options. The ASCOS pricing technique for geometric and arithmetic Asian options is described in sections 2 and 3, respectively. The characteristic function of the geometric or arithmetic mean value of the underlying is recovered, which is then used to calculate the Asian option value by Fourier cosine expansions. For geometric Asian options, the characteristic function of the logarithm of the geometric average of the underlying asset at the monitoring dates is known analytically for exponential Lévy processes, as we will see below.

2.1. Introduction to the COS method. The starting point for pricing plain vanilla European options by the COS method is the risk-neutral option valuation formula (the discounted expected payoff approach), i.e.,

$$v(x,t_0) = e^{-r\Delta t}\int_{-\infty}^{\infty} v(y,T) f(y|x)\,dy, \tag{2.1}$$

where $v(x,t_0)$ is the present option value, r is the interest rate, $\Delta t = T - t_0$, and x, y can be any monotone functions of the underlying asset at initial time t_0 and the expiration date T, respectively. Payoff function $v(y,T)$ is known for European options, but the transitional density function, $f(y|x)$, typically is not. Based on (2.1), the transitional density function is approximated on a truncated domain $[a,b]$ by a truncated Fourier cosine series expansion, with N terms, based on the conditional characteristic function (see [12]), as follows:

$$f(y|x) \approx \frac{2}{b-a}{\sum_{k=0}^{N-1}}{}' \,\mathrm{Re}\left(\phi\left(\frac{k\pi}{b-a};x\right)\exp\left(-i\frac{ak\pi}{b-a}\right)\right)\cos\left(k\pi\frac{y-a}{b-a}\right), \tag{2.2}$$

where $\phi(u;x)$ is the conditional characteristic function of $f(y|x)$, a, b determine the integration interval, and Re means taking the real part of the argument. The prime at the sum symbol indicates that the first term in the expansion is multiplied by one-half. The appropriate size of the integration interval can be determined with the help of the cumulants [12].[1]

Replacing $f(y|x)$ by its approximation (2.2) in (2.1) and interchanging integration and summation gives the COS formula for the computation of the price of a European plain vanilla option:

$$\hat{v}(x,t_0) = e^{-r\Delta t}{\sum_{k=0}^{N-1}}{}' \,\mathrm{Re}\left(\phi\left(\frac{k\pi}{b-a};x\right)e^{-ik\pi\frac{a}{b-a}}\right)V_k, \tag{2.3}$$

[1]This is so that $\left|\int_{\mathbb{R}} f(y|x)\,dy - \int_a^b f(y|x)\,dy\right| < TOL$.

where $\hat{v}(x, t_0)$ indicates the approximate option value, and

$$V_k := \frac{2}{b-a}\int_a^b v(y,T)\cos\left(k\pi\frac{y-a}{b-a}\right)dy$$

are the Fourier cosine coefficients of $v(y,T)$, available in closed form for several payoff functions.

With integration interval $[a,b]$ chosen sufficiently wide, it was found that the series truncation error dominates the overall error. For transitional density functions $f(y|x) \in C^\infty([a,b] \times \mathbb{R})$, the method converges exponentially; otherwise, convergence is algebraic [12, 13].

2.2. European-style geometric Asian options. The payoff function of a geometric Asian option with M monitoring dates and a fixed strike reads as

$$v(S,T) \equiv g(S) = \begin{cases} \max\left(\left(\prod_{j=0}^{M} S_j\right)^{\frac{1}{M+1}} - K, 0\right) & \text{for a call,} \\ \max\left(K - \left(\prod_{j=0}^{M} S_j\right)^{\frac{1}{M+1}}, 0\right) & \text{for a put.} \end{cases}$$

Here S, K, $g(S)$ denote the stock price, the strike price, and the payoff function, respectively, and $M = 1, 2, \ldots$.

For geometric Asian options, the characteristic function of the geometric mean can be calculated directly. The underlying process is transformed to the logarithm domain, and we use the following notation:

$$y := \log\left(\left(\prod_{j=0}^{M} S_j\right)^{\frac{1}{M+1}}\right) = \frac{1}{M+1}\sum_{j=0}^{M}\log(S_j) =: \frac{1}{M+1}\sum_{j=0}^{M} x_j. \tag{2.4}$$

In order to use the Fourier cosine expansion, we need to determine the conditional characteristic function of y given x_0. Lévy processes have independent and stationary increments, which implies that the increments $x_1 - x_0$, $x_2 - x_1$, ..., $x_M - x_{M-1}$ are identically distributed and all independent of x_0.

Denote the (identical) characteristic functions of these increments by $\varphi(u,\tau)$, i.e.,

$$\varphi(u,\tau) := \mathbb{E}(\exp(iu\log(S_{t+\tau}/S_t))) = \mathbb{E}(\exp(iu(x_{t+\tau} - x_t))) \quad \forall t, \tau \geq 0, \tag{2.5}$$

and $\varphi(u,\tau)$ is known analytically for different Lévy processes, for which we refer the reader to [13]. The characteristic function of y given x_0 is given by

$$\phi(u; x_0) = e^{iux_0}\prod_{j=1}^{M}\varphi\left(u\frac{M+1-j}{M+1}, \frac{T-t_0}{M}\right). \tag{2.6}$$

For the derivation of the characteristic function for the geometric mean of an exponential Lévy process, we refer the reader to [15].

Substitution of characteristic function (2.6) into (2.3) results in the *ASCOS pricing formula for European-style geometric Asian options*, with the underlying asset modeled by an exponential Lévy process:

$$v(x_0, t_0) = e^{-r\Delta t} {\sum_{k=0}^{N-1}}' \operatorname{Re}\left\{ \phi\left(\frac{k\pi}{b-a}; x_0\right) e^{-ik\pi\frac{a}{b-a}} \right\} V_k, \tag{2.7}$$

where

$$V_k = \begin{cases} \dfrac{2}{b-a}\left(\chi_k(\log(K), b) - K\psi_k(\log(K), b)\right) & \text{for a call,} \\ \dfrac{2}{b-a}\left(K\psi_k(a, \log(K)) - \chi_k(a, \log(K))\right) & \text{for a put,} \end{cases}$$

with

$$\begin{aligned} \chi_k(x_1, x_2) &:= \int_{x_1}^{x_2} e^y \cos\left(k\pi\frac{y-a}{b-a}\right) dy, \\ \psi_k(x_1, x_2) &:= \int_{x_1}^{x_2} \cos\left(k\pi\frac{y-a}{b-a}\right) dy, \end{aligned} \tag{2.8}$$

which are known analytically.

The computational complexity for deriving the characteristic function for each value of $u = k\pi/b - a$, $k = 0, \ldots, N-1$, is linear in M and the complexity of the work in (2.7) is linear in N, so that the total computational complexity of the method is $O(MN)$.

For geometric Asian options there is no error in deriving the characteristic function by (2.6). The only errors made are due to the COS formula (2.7). Detailed error analysis of the COS method for European options can be found in [12]. The ASCOS pricing method for geometric Asian options under exponential Lévy asset price processes is thus expected to have an exponential convergence rate in the number of cosine terms for all density functions that satisfy $f(y|x) \in C^\infty([a, b] \subset \mathbb{R})$.

3. ASCOS method for arithmetic Asian options.

For arithmetic Asian options, the characteristic function of the arithmetic mean will be derived recursively by Fourier cosine expansions and Clenshaw–Curtis quadrature. The Fourier cosine expansion is used at each time step (i.e., at each monitoring date), whereas the Clenshaw–Curtis quadrature rule is used once, at the beginning of the computation. In subsection 2.2 the characteristic function of the geometric average (2.4) was discussed, which was explicitly a function of $x_0 = \log(S_0)$, so that the characteristic function was naturally written in the form $\phi(u; x_0)$. In the present section, we recover the characteristic function of the logarithm of the sum of exponential Lévy asset price increments, which is independent of x_0. Therefore, we write the characteristic function here in the form $\phi(u)$ rather than $\phi(u; x_0)$.

The payoff function of an arithmetic Asian option reads as

$$v(S,T) \equiv g(S) = \begin{cases} \max\left(\dfrac{1}{M+1}\displaystyle\sum_{j=0}^{M} S_j - K, 0\right) & \text{for a call,} \\ \max\left(K - \dfrac{1}{M+1}\displaystyle\sum_{j=0}^{M} S_j, 0\right) & \text{for a put.} \end{cases} \tag{3.1}$$

We first explain the recursion procedure for recovering the characteristic function of the arithmetic mean value of the underlying. We denote

$$R_j := \log\left(\frac{S_j}{S_{j-1}}\right), \quad j = 1, \ldots, M. \tag{3.2}$$

For exponential Lévy processes, the log-asset returns R_j, $j = 1, \ldots, M$, are identically and independently distributed, so that $R_j \stackrel{d}{=} R$. Then, $\forall u, j$, we can write $\phi_{R_j}(u) = \phi_R(u)$. Characteristic function $\phi_R(u)$ is known in closed form for different Lévy processes.

A stochastic process, Y_j, is introduced, where $Y_1 = R_M$ and for $j = 2, \ldots, M$ we have

$$Y_j := R_{M+1-j} + \log(1 + \exp(Y_{j-1})). \tag{3.3}$$

We denote $Z_j := \log(1 + \exp(Y_j))$ $\forall j$, so that (3.3) can be rewritten as

$$Y_j := R_{M+1-j} + Z_{j-1}. \tag{3.4}$$

In this setting, Y_j admits the form

$$Y_j = \log\left(\frac{S_{M-j+1}}{S_{M-j}} + \frac{S_{M-j+2}}{S_{M-j}} + \cdots + \frac{S_M}{S_{M-j}}\right), \tag{3.5}$$

and we have that

$$\frac{1}{M+1}\sum_{j=0}^{M} S_j = \frac{(1 + \exp(Y_M))S_0}{M+1}. \tag{3.6}$$

Convolution scheme (3.4)–(3.6) is also called the Carverhill–Clewlow–Hodges factorization, which appeared in [7], based on an insight by S. Hodges, and it has been used in [7, 3, 15], in combination with other numerical methods, to recover the transitional probability density function of Y_M. Here, however, we will recover the *characteristic function of* Y_M instead, by a forward recursion procedure, which is then used in turn to recover the transitional density of the European-style arithmetic mean of the underlying process in the risk-neutral formula (3.7). The arithmetic Asian option value is now defined as

$$v(x_0, t_0) = e^{-r\Delta t}\int_{-\infty}^{\infty} v(y,T) f_{Y_M}(y)\,dy. \tag{3.7}$$

By (3.6), $v(y,T)$ in (3.7) is of the following form:

$$v(y,T) = \begin{cases} \left(\dfrac{S_0(1+\exp(y))}{M+1} - K\right)^+ & \text{for a call,} \\ \left(K - \dfrac{S_0(1+\exp(y))}{M+1}\right)^+ & \text{for a put.} \end{cases}$$

3.1. Recovery of characteristic function. To recover the characteristic function of Y_M, i.e., $\phi_{Y_M}(u)$, we start with Y_1, for which the characteristic function reads as

$$\phi_{Y_1}(u) = \phi_R(u). \tag{3.8}$$

Then, at time steps t_j, $j = 2, \dots, M$, $\phi_{Y_j}(u)$ can be recovered in terms of $\phi_{Y_{j-1}}(u)$. This is done by application of (3.4) and the fact that Lévy processes have independent increments. This implies that, $\forall j$, R_{M+1-j} and Z_{j-1} are independent, which gives

$$\phi_{Y_j}(u) = \phi_{R_{M+1-j}}(u)\phi_{Z_{j-1}}(u) = \phi_R(u)\phi_{Z_{j-1}}(u). \tag{3.9}$$

From the definition of characteristic function, we have

$$\phi_{Z_{j-1}}(u) = \mathbb{E}[e^{iu\log(1+\exp(Y_{j-1}))}] = \int_{-\infty}^{\infty} (e^x+1)^{iu} f_{Y_{j-1}}(x)dx. \tag{3.10}$$

To apply the Fourier cosine series expansion to *approximate* the characteristic function, we first truncate the integration range, i.e.,

$$\hat\phi_{Z_{j-1}}(u) = \int_a^b (e^x+1)^{iu} f_{Y_{j-1}}(x)dx. \tag{3.11}$$

If we define the error

$$\epsilon_T(X) := \int_{\mathbb{R}\backslash[a,b]} f_X(x)dx,$$

then, as $\forall j, u \in \mathbb{R}$,

$$|(e^x+1)^{iu}| = |\cos(u\log(1+e^x)) + i\sin(u\log(1+e^x)))| = 1, \tag{3.12}$$

the error in (3.11) can be bounded by

$$\left|\int_{\mathbb{R}\backslash[a,b]} (e^x+1)^{iu} f_{Y_{j-1}}(x)dx\right| \le \int_{\mathbb{R}\backslash[a,b]} f_{Y_{j-1}}(x)dx := \epsilon_T(Y_{j-1}). \tag{3.13}$$

We apply the Fourier cosine expansion to approximate $f_{Y_{j-1}}(x)$, giving

$$\begin{aligned}\hat\phi_{Z_{j-1}}(u) = \frac{2}{b-a}{\sum_{l=0}^{N-1}}{}' \mathrm{Re}\left\{\hat\phi_{Y_{j-1}}\left(\frac{l\pi}{b-a}\right)\exp\left(-ia\frac{l\pi}{b-a}\right)\right\}\\ \cdot \int_a^b (e^x+1)^{iu}\cos\left((x-a)\frac{l\pi}{b-a}\right)dx,\end{aligned} \tag{3.14}$$

where $\hat\phi_{Y_{j-1}}$ is an approximation of $\phi_{Y_{j-1}}$.

In this way, $\hat\phi_{Z_{j-1}}$ is recovered in terms of $\hat\phi_{Y_{j-1}}$. Application of (3.9) gives an approximation $\hat\phi_{Y_j}(u)$ for any u. Equation (3.14) can be written in matrix-vector form as

$$\Phi_{j-1} = \mathcal{M} A_{j-1}, \tag{3.15}$$

using

$$\begin{aligned}
\Phi_{j-1} &= (\Phi_{j-1}(k))_{k=0}^{N-1}, \quad \Phi_{j-1}(k) = \hat{\phi}_{Z_{j-1}}(u_k),\\
u_k &= \frac{k\pi}{b-a}, \quad k = 0, \ldots, N-1,\\
\mathcal{M} &= (\mathcal{M}(k,l))_{k,l=0}^{N-1}, \quad \mathcal{M}(k,l) = \int_a^b (e^x+1)^{iu_k}\cos((x-a)u_l)\,dx,\\
A_j &= \frac{2}{b-a}(A_j(l))_{l=0}^{N-1}, \quad A_j(l) = \mathrm{Re}(\hat{\phi}_{Y_{j-1}}(u_l)\exp(-iau_l)).
\end{aligned}$$

By the recursion procedure in (3.9) and (3.15), characteristic function $\phi_{Y_M}(u)$ can be approximated by $\hat{\phi}_{Y_M}(u)$ efficiently. Application of (2.3) in (3.7) finally gives the European-style arithmetic Asian option value:

$$\hat{v}(x,t_0) = e^{-r\Delta t}\sum_{k=0}^{N-1}{}' \mathrm{Re}\left\{\hat{\phi}_{Y_M}\left(\frac{k\pi}{b-a}\right)e^{-ik\pi\frac{a}{b-a}}\right\}V_k, \tag{3.16}$$

in which

$$V_k = \begin{cases} \dfrac{2}{b-a}\left(\dfrac{S_0}{M+1}\chi_k(x^*,b) + \left(\dfrac{S_0}{M+1} - K\right)\psi_k(x^*,b)\right) & \text{for a call,}\\[2ex] \dfrac{2}{b-a}\left(\left(K - \dfrac{S_0}{M+1}\right)\psi(a,x^*) - \dfrac{S_0}{M+1}\chi(a,x^*)\right) & \text{for a put.}\end{cases} \tag{3.17}$$

Functions $\chi_k(x_1,x_2)$ and $\psi_k(x_1,x_2)$ are as in (2.8), and $x^* = \log(\frac{K(M+1)}{S_0} - 1)$.

3.2. Integration range. We explain how to determine integration range $[a,b]$, so that the errors $\epsilon_T(Y_{j-1})$, $j = 2,\ldots,M$, in (3.13), as well as truncation error $\epsilon_T(Y_M)$ in (3.16), can be controlled. In [12, 13], the integration range for each Y_j, $j = 1,\ldots,M$, was determined by means of the cumulants as

$$\left[\zeta_1(Y_j) - L\sqrt{\zeta_2(Y_j) + \sqrt{\zeta_4(Y_j)}},\ \zeta_1(Y_j) + L\sqrt{\zeta_2(Y_j) + \sqrt{\zeta_4(Y_j)}}\right], \tag{3.18}$$

with $\zeta_1(Y_j), \zeta_2(Y_j), \zeta_4(Y_j)$ the first, second, and fourth cumulants of Y_j, respectively. It is rather expensive to determine these cumulants here, and therefore we propose a different integration range, which is very similar to (3.18).

For Y_j, $j = 1,\ldots,M$, as defined in (3.5), we have

$$\begin{gathered}
\zeta_1\left(\left(\left|\frac{S_{M-j+1}}{S_{M-j}}\right)\right)\right) \le \zeta_1(\exp(Y_j)) \le \zeta_1\left(\left(\left|\frac{S_M}{S_{M-j}}\right)\right)\right),\\
0 \le \zeta_2(\exp(Y_j)) \le \zeta_2\left(\left(\left|\frac{S_M}{S_{M-j}}\right)\right)\right),\\
0 \le \zeta_4(\exp(Y_j)) \le \zeta_4\left(\left(\left|\frac{S_M}{S_{M-j}}\right)\right)\right).
\end{gathered}$$

An integration range for e^{Y_j} can be defined as

$$\left[\zeta_1\left(j\frac{S_{M-j+1}}{S_{M-j}}\right) - L\sqrt{\zeta_2\left(j\frac{S_M}{S_{M-j}}\right) + \sqrt{\zeta_4\left(j\frac{S_M}{S_{M-j}}\right)}},\right.$$

$$\left.\zeta_1\left(j\frac{S_M}{S_{M-j}}\right) + L\sqrt{\zeta_2\left(j\frac{S_M}{S_{M-j}}\right) + \sqrt{\zeta_4\left(j\frac{S_M}{S_{M-j}}\right)}}\right]. \tag{3.19}$$

Denoting

$$a_j := \zeta_1\left(\log\left(j\frac{S_{M-j+1}}{S_{M-j}}\right)\right) - L\sqrt{\zeta_2\left(\log\left(j\frac{S_M}{S_{M-j}}\right)\right) + \sqrt{\zeta_4\left(\log\left(j\frac{S_M}{S_{M-j}}\right)\right)}},$$

$$b_j := \zeta_1\left(\log\left(j\frac{S_M}{S_{M-j}}\right)\right) + L\sqrt{\zeta_2\left(\log\left(j\frac{S_M}{S_{M-j}}\right)\right) + \sqrt{\zeta_4\left(\log\left(j\frac{S_M}{S_{M-j}}\right)\right)}}, \tag{3.20}$$

we can define suitable intervals $[a_j, b_j]$. Note that (3.20) is not strictly derived from (3.19), as $\log(\zeta_n(Z)) \neq \zeta(\log(Z))$, but this does not influence the fact that, as $L \to \infty$, the truncation error goes to zero. The cumulants of $\log(j\frac{S_{M-j+1}}{S_{M-j}})$ and $\log(j\frac{S_M}{S_{M-j}})$ in (3.20) are known in closed form for exponential Lévy asset price processes, since

$$\zeta_1\left(\log\left(j\frac{S_{M-j+1}}{S_{M-j}}\right)\right) = \log(j) + \zeta_1(R) \quad \text{and,} \quad \forall n \geq 2, \quad \zeta_n\left(\log\left(j\frac{S_{M-j+1}}{S_{M-j}}\right)\right) = \zeta_n(R),$$

$$\zeta_1\left(\log\left(j\frac{S_M}{S_{M-j}}\right)\right) = \log(j) + j\zeta_1(R) \quad \text{and,} \quad \forall n \geq 2, \quad \zeta_n\left(\log\left(j\frac{S_M}{S_{M-j}}\right)\right) = j\zeta_n(R),$$

with R the logarithm of the increment of an exponential Lévy process, between any two consecutive time steps. These expressions are based on $\log(jZ) = \log j + \log(Z)$, for random variable Z, and on the fact that for an exponential Lévy asset price process, the cumulants of the log-asset returns, $\log(S_l/S_k)$ $\forall l > k$, are linearly increasing functions of $t := (l - k)\Delta t$.

In order to compute the integration in (3.14) only once, we adopt the following integration range:

$$[a, b] := \left[\min_{j=1,\dots,M} a_j, \max_{j=1,\dots,M} b_j\right] \tag{3.21}$$

for all time steps, so that the truncation errors, $\epsilon_T(Y_j)$ $\forall j$, can be controlled easily.

An exception may be formed by underlying processes exhibiting very fat tails, as then interval (3.21) may result in a wide integration range, so that large N values are required to ensure accuracy. In those cases, it may be more efficient to recenter the range, using (3.20). In the numerical examples we will show in section 5, interval (3.21) can be safely used so that the integration in (3.14) needs to be computed only once.

In accordance with [12, 13], we will use $L = 10-12$ in (3.20) in our numerical experiments.

Remark 3.1 (put-call parity for Asian options). For a call option, the payoff is unbounded, which may lead to large errors when truncating the integration range of the risk-neutral formula. Assuming that the integration range is sufficiently large, so that the expression $(\frac{S_0(1+\exp(b))}{M+1} - K) > 0$, the truncation error, ϵ, based on an integration range $[a, b]$ is given by

$$\begin{aligned}
\epsilon &:= e^{-r\Delta t} \int_{\mathbb{R}\setminus[a,b]} v(y,T) f_{Y_M}(y)dy > e^{-r\Delta t} \int_b^{\infty} v(y,T) f_{Y_M}(y)dy \\
&= e^{-r\Delta t} \int_b^{\infty} \left(\frac{S_0(1+\exp(y))}{M+1} - K \right) f_{Y_M}(y)dy \\
&> e^{-r\Delta t} \left(\frac{S_0(1+\exp(b))}{M+1} - K \right) \int_b^{\infty} f_{Y_M}(y)dy.
\end{aligned}$$

The larger the range of integration, the larger the value of $(\frac{S_0(1+\exp(b))}{M+1} - K)$, which grows exponentially with respect to the upper bound of the range. Therefore, although the value of $\int_b^{\infty} f_{Y_M}(y)dy$ decreases as the integration range increases, the total error may increase. To avoid this, the call option price can be obtained via the put option price by means of the put-call parity relation. It is well known that a put option payoff is bounded, so that the problem described above cannot occur.

Assuming that no dividend is paid and denoting the Asian call and put option prices by $c(S_0, t_0)$ and $p(S_0, t_0)$, respectively, we have

$$\max\left(\frac{1}{M+1}\sum_{j=0}^{M} S_j - K, 0\right) - \max\left(K - \frac{1}{M+1}\sum_{j=0}^{M} S_j, 0\right) = \frac{1}{M+1}\sum_{j=0}^{M} S_j - K.$$

Using the risk-neutral valuation formula gives, for $t_0 < T$,

$$c(S_0, t_0) - p(S_0, t_0) = e^{-rT}\mathbb{E}\left(\frac{1}{M+1}\sum_{j=0}^{M} S_j - K \,\Big|\, \mathcal{F}_0\right) = \frac{S_0 e^{-rT}}{M+1}\sum_{j=0}^{M} e^{rj\Delta t} - Ke^{-rT}.$$

A similar discussion can be found in [14], where the put-call parity relation was used for put option pricing.

In our numerical examples, we can directly use the pricing method for call options and the option values obtained from our method converge to the same values in [15]. However, for deep-in-the-money call options and fat tailed asset price densities, or for call options with a long time to maturity, the put-call parity is advocated.

3.3. Clenshaw–Curtis quadrature. In this section we denote by n_q the number of terms in the Clenshaw–Curtis quadrature (q stands for quadrature). We discuss the efficient computation of matrix $\mathcal{M}$ in (3.15). An important feature is that matrix $\mathcal{M}$ remains constant for all time steps t_j, $j = 1, \ldots, M-1$, so that we need to calculate it only once. Its elements are given by

$$\mathcal{M}(k,l) = \int_a^b (e^x + 1)^{iu_k} \cos((x-a)u_l)dx, \quad k, l = 0, \ldots, N-1, \tag{3.22}$$

which can be rewritten in terms of incomplete Beta functions (see Appendix A). Here (3.22) is approximated numerically by the Clenshaw–Curtis quadrature rule, which is based on an expansion of the integrand in terms of Chebyshev polynomials (as proposed in [10]; more information can be found in [5]).

The Clenshaw–Curtis as well as the Gaussian quadrature rules exhibit an exponential convergence for the integration in (3.22), but the Clenshaw–Curtis quadrature is preferred here, since it is computationally cheaper. The weights and nodes of the Clenshaw–Curtis quadrature are easy to determine. Moreover, Clenshaw–Curtis quadrature is *a nested integration rule*, where the nodes for a small value of N are also nodes for larger N-values.

To use the Clenshaw–Curtis rule for (3.22), we first change the integration interval from $[a,b]$ to $[-1,1]$:

$$\begin{aligned}&\int_a^b (e^x+1)^{iu_k}\cos\left((x-a)u_l\right)dx\\ &= \int_{-1}^{1}\frac{b-a}{2}\left(\exp\left(\frac{b-a}{2}x+\frac{a+b}{2}\right)+1\right)^{iu_k}\cos\left(\left(\frac{b-a}{2}x+\frac{a+b}{2}-a\right)u_l\right)dx.\end{aligned}$$

The integral can then be approximated as follows:

$$\int_a^b (e^x+1)^{iu_k}\cos((x-a)u_l)dx \approx (D^T d)^T y =: w^T y, \tag{3.23}$$

where D is an $(n_q/2+1)\times(n_q/2+1)$-matrix, whose elements read as

$$D(k,n) = \frac{2}{n_q}\cos\left(\frac{(n-1)(k-1)\pi}{n_q/2}\right)\begin{cases}1/2 & \text{if } n=\{1, n_q/2+1\},\\ 1 & \text{otherwise.}\end{cases} \tag{3.24}$$

Vector d and the elements y_n in $y=\{y_n\}_{n=0}^{n_q/2}$ are defined as

$$\begin{aligned} d &:= \left(1, \frac{2}{(1-4)}, \frac{2}{(1-16)}, \ldots, \frac{2}{(1-(n_q-2)^2)}, \frac{1}{(1-n_q^2)}\right)^T,\\ y_n &:= f\left(\cos\left(\frac{n\pi}{n_q}\right)\right) + f\left(-\cos\left(\frac{n\pi}{n_q}\right)\right),\end{aligned} \tag{3.25}$$

where, in our case,

$$f(x) = \frac{b-a}{2}\left(\exp\left(\frac{b-a}{2}x+\frac{a+b}{2}\right)+1\right)^{iu_k}\cos\left(\left(\frac{b-a}{2}x+\frac{a+b}{2}-a\right)u_l\right).$$

$\forall(k,l)$, the vector $w=D^T d$ remains the same, so that it needs to be computed only once $\forall(k,l)$. Because $D^T d$ is a so-called type I discrete cosine transform, the computational complexity is $O(n_q \log_2 n_q)$. Elements y_n must be calculated for each pair (k,l), with complexity $O(n_q)$, and the computational complexity, $\forall(k,l)$, is therefore $O(n_q N^2)$. When using the Clenshaw–Curtis quadrature rule to compute matrix $\mathcal{M}$ (only once, used for all time steps), the total computational complexity is thus $O(n_q \log_2 n_q) + O(n_q N^2)$.

Furthermore, at each time step t_j, we need $O(N^2)$ computations for the matrix-vector multiplication (3.15) and $O(N)$ computations to obtain $\hat{\phi}_{Y_j}$ by (3.8) or (3.9). The computational complexity for this task is thus $O(MN^2)$.

The overall computational complexity of our method for arithmetic Asian options is then $O(n_q \log_2 n_q) + O(n_q N^2) + O(MN^2)$. The number N^2 is in practice much larger than $\log_2 n_q$. The overall complexity is then of order $O((n_q + M)N^2)$.

We will show, in the section on error analysis for arithmetic Asian options, that for most exponential Lévy processes, the Fourier cosine expansion exhibits an exponential convergence rate with respect to N. For the integrand in (3.22) the Clenshaw–Curtis quadrature converges exponentially with respect to n_q. Therefore, the ASCOS pricing method is an efficient alternative to the method proposed in [15], which requires $O(M\bar{N}^2)$ computations ($\bar{N}$ being the number of points used in the quadrature in [15]), with $\bar{N} > n_q$, as well as $\bar{N} > N$, for the same level of accuracy. Our pricing method is especially advantageous when the number of monitoring dates, M, increases. The method is summarized below.

ASCOS Algorithm. Pricing European-style arithmetic Asian options.

Initialization
- Use Clenshaw–Curtis quadrature (3.23) to compute $\mathcal{M} = (\mathcal{M}(k,l))$, $k, l = 0, \dots, N-1$, with $\mathcal{M}$ in (3.15), (3.22).
- Compute $\phi_R(u_k)$, $k = 0, \dots, N-1$.
- Set $\phi_{Y_1}(u_k) = \phi_R(u_k)$.

Main loop to recover $\hat{\phi}_{Y_M}$: For $j = 2$ to M,
- Compute the vector Φ_{j-1} with elements $\hat{\phi}_{Z_{j-1}}(u_k)$, $k = 0, \dots, N-1$, using (3.15).
- Recover $\hat{\phi}_{Y_j}(u_k)$, $k = 0, \dots, N-1$, using (3.9).

Final step:
- Compute $\hat{v}(x_0, t_0)$ by inserting $\hat{\phi}_{Y_M}(u_k)$, $k = 0, \dots, N-1$, into (3.16).

3.4. Extensions. In a series of remarks, we discuss some generalizations of the ASCOS method. The American-style Asian options generalization will be discussed in a separate paper [23].

Remark 3.2 (continuously monitored Asian options). The option values of continuously monitored arithmetic Asian options, with payoff

$$v(S,T) = g(S) = \begin{cases} \left(\dfrac{1}{T}\displaystyle\int_0^T S(t)dt - K\right)^+ & \text{for a call,} \\ \left(K - \dfrac{1}{T}\displaystyle\int_0^T S(t)dt\right)^+ & \text{for a put,} \end{cases}$$

can be obtained from discretely monitored arithmetic Asian option prices by a four-point Richardson extrapolation.

Let $\hat{v}(M)$ denote the computed value of a discretely monitored Asian option with M monitoring dates. The continuously monitored Asian option value, denoted by $\hat{v}_\infty$, can be approximated by a four-point Richardson extrapolation scheme as follows:

$$\hat{v}_\infty(d) = \frac{1}{21}\left(64\hat{v}(2^{d+3}) - 56\hat{v}(2^{d+2}) + 14\hat{v}(2^{d+1}) - \hat{v}(2^d)\right). \tag{3.26}$$

The same technique can be applied for continuously monitored geometric Asian options.

Remark 3.3 (Asian options on the harmonic average). Harmonic Asian options may have their use in the foreign exchange market. For instance, a floating-strike harmonic Asian call option gives the right, but not the obligation, to exchange dollars into euros at an average exchange rate over a certain period. Other applications for harmonic Asian options have been described, for example, in [9].

Asian options with a payoff based on the harmonic average, $M/(\sum_{j=1}^{M} 1/S_j)$, can be priced in a fashion similar to that explained above by the ASCOS method. First, we recover the characteristic function of a variable $y = \log(\sum_{j=1}^{m} S_0/S_j)$ recursively; then we insert the approximation into the COS pricing formula.

We define $\bar{R}_j = \log(S_{j-1}/S_j)$. Starting with $Y_1 = \log(\bar{R}_M)$, we find that, $\forall j, u$,

$$\phi_{\bar{R}_j}(u) = \mathbb{E}\left[e^{iu\log\left(\frac{S_{j-1}}{S_j}\right)}\right] = \mathbb{E}\left[e^{i(-u)\log\left(\frac{S_j}{S_{j-1}}\right)}\right] = \phi_{R_j}(-u), \tag{3.27}$$

with ϕ_{R_j} available in closed form for exponential Lévy processes. For this reason, $\phi_{Y_1}(u)$ is also known analytically.

For $j = 2, \ldots, M$ we then define $Y_j := \bar{R}_{M+1-j} + Z_{j-1}$, where $Z_j := \log(1 + \exp(Y_j))$. In this setting we have $Y_M \equiv \log(\sum_{j=1}^{m} S_0/S_j)$.

Again, $\bar{R}_{M+1-j}$ and Z_{j-1} are independent at each time step, due to the properties of Lévy processes. Therefore

$$\phi_{Y_j}(u) = \phi_{\bar{R}_{M+1-j}}(u)\phi_{Z_{j-1}}(u) \quad \forall u,$$

where $\phi_{\bar{R}_{M+1-j}}(u)$ is known analytically from (3.27) and $\phi_{Z_{j-1}}(u)$ can be recovered, as $\hat{\phi}_{Z_{j-1}}(u)$ from $\hat{\phi}_{Y_{j-1}}(u)$ by Fourier cosine expansions and Clenshaw–Curtis quadrature, as in (3.14). We thus approximate the characteristic function of Y_M, and the fixed-strike Asian option value is then given by

$$\hat{v}(x, t_0) = e^{-r\Delta t} {\sum_{k=0}^{N-1}}' \mathrm{Re}\left\{\hat{\phi}_{Y_M}\left(\frac{k\pi}{b-a}\right) e^{-ik\pi\frac{a}{b-a}}\right\} V_k,$$

in which

$$V_k = \begin{cases} \dfrac{2}{b-a}(MS_0\bar{\chi}_k(x^*, b) - K\psi_k(x^*, b)) & \text{for a call,} \\ \dfrac{2}{b-a}(K\psi(a, x^*) - MS_0\bar{\chi}(a, x^*)) & \text{for a put,} \end{cases}$$

where $x^* = \log(MS_0/K)$, $\bar{\chi}(x_1, x_2) := \int_{x_1}^{x_2} e^{-y}\cos(k\pi\frac{y-a}{b-a})dy$, and $\psi_k(x_1, x_2)$ is defined in (2.8).

Finally, the symmetry between floating- and fixed-strike Asian options also holds for Asian options on the harmonic average, so that floating-strike options can be valued as well.

Remark 3.4 (a special case: the forward contract). A forward contract, as encountered in commodity markets, may be defined by the payoff:

$$g(S) = \frac{1}{M+1}\sum_{j=0}^{M} S_j - K. \tag{3.28}$$

The contract value then reads as

$$
\begin{aligned}
v(x_0, t_0) &= e^{-r\Delta t}\mathbb{E}\left[\frac{1}{M+1}\sum_{j=0}^{M} S_j - K\right] \\
&= e^{-r\Delta t}\left(\frac{S_0}{M+1}\mathbb{E}[e^{Y_M}] + \left(\frac{S_0}{M+1} - K\right)\right), \qquad (3.29)
\end{aligned}
$$

where the last step follows from (3.6). The expected value of $\exp(Y_M)$ can be obtained by a forward recursion procedure. At each monitoring date, t_j, we have from (3.4) that

$$\mathbb{E}[e^{Y_j}] = \mathbb{E}[e^{R_{M+1-j}}(1 + e^{Y_{j-1}})]. \qquad (3.30)$$

For exponential Lévy processes, R_{M+1-j} and $(1 + \exp(Y_{j-1}))$ are independent and $R_j \stackrel{d}{=} R\ \forall j$, so that (3.30) reads as

$$\mathbb{E}[e^{Y_j}] = \mathbb{E}[e^{R}](1 + \mathbb{E}[e^{Y_{j-1}}]) \quad \forall j, \qquad (3.31)$$

with $\mathbb{E}[e^{Y_1}] = \mathbb{E}[e^{R}]$. The value of $\mathbb{E}[e^{R}]$ reads as

$$\mathbb{E}[e^{R}] = \int_{-\infty}^{\infty} e^{y} f_R(y)dy = {\sum_{k=0}^{N-1}}{}' \operatorname{Re}\left\{\phi_R\left(\frac{k\pi}{b-a}\right)e^{-ik\pi\frac{a}{b-a}}\right\}\chi_k(a,b), \qquad (3.32)$$

where function $\chi_k(x_1, x_2)$ is defined in (2.8) and ϕ_R is the characteristic function of R, which is available for various Lévy processes.

The $\mathbb{E}[e^{R}]$-term needs to be calculated only once, with $O(N)$ complexity. In the recursion procedure to get the forward value, we use (3.31) $M-1$ times and (3.29) once. Therefore, the total computational complexity is $O(N)+O(M)$, and exponential convergence is expected for probability density functions belonging to $C^{\infty}[a,b]$.

With $\mathbb{E}[e^{Y_M}]$ derived recursively, we can also compute the value of the forward price K from (3.29) in such a way that $v(x_0, t_0) = 0$, that is, $K = \frac{S_0}{M+1}(\mathbb{E}[e^{Y_M}] + 1)$.

4. Error analysis for arithmetic Asian options.

Here we give an error analysis of the ASCOS method for arithmetic Asian options. We first discuss, in general terms, three types of error occurring, i.e., the truncation error, ϵ_T, the error of the Fourier cosine expansion, ϵ_F, and the error from the use of the Clenshaw–Curtis quadrature, ϵ_q.

The truncation error is defined as

$$\epsilon_T(Y_j) := \int_{\mathbb{R}\setminus[a,b]} f_{Y_j}(y)dy, \quad j = 1, \ldots, M, \qquad (4.1)$$

and it decreases as interval $[a, b]$ increases. In other words, for a sufficiently large integration range $[a, b]$, this part of the error will not dominate the overall error of the arithmetic Asian option price.

Regarding the error of the Fourier cosine expansions, we know from [12] that, for $f(y|x) \in C^{\infty}[a,b]$, it can be bounded by

$$|\epsilon_F(N, [a, b])| \le P^*(N)\exp(-(N-1)\nu),$$

with $\nu > 0$ a constant and a term $P^*(N)$, which varies less than exponentially with respect to N.

When the transitional probability density function has a discontinuous derivative, the error can be bounded by

$$|\epsilon_F(N,[a,b])| \leq \frac{\bar{P}^*(N)}{(N-1)^{\beta-1}},$$

where $\bar{P}^*(N)$ is a constant and $\beta \geq 1$.

Error ϵ_F thus decays exponentially with respect to N if the density function $f(y|x) \in C^\infty[a,b]$, or algebraically otherwise.

Let us now have a look at the error from the Clenshaw–Curtis quadrature, which we use to approximate

$$I := \int_a^b (e^x+1)^{iu_k} \cos((x-a)u_l)dx, \tag{4.2}$$

by $\hat{I} := w^T y$ in (3.23). In other words, $\epsilon_q = I - \hat{I}$.

According to [20, 22], the Clenshaw–Curtis quadrature rule exhibits an error which can be bounded by $O((2n_q)^{-k}/k)$ for a k-times differentiable integrand. When k is bounded, we have algebraic convergence; otherwise the error converges exponentially with respect to n_q; see also [4]. The integrand in (4.2) belongs to $C^\infty[a,b]$, as all derivatives are continuous on any interval $[a,b]$, confirming that, for the integrand in (4.2), we will have exponential convergence with respect to n_q.

4.1. Error propagation in the characteristic functions. The following lemma is used in the error analysis.

Lemma 4.1. *For any random variable, X, and any $u \in \mathbb{R}$, the characteristic function can be bounded by $|\phi_X(u)| \leq 1$.*

Proof. For any X and u, the characteristic function, $\phi_X(u)$, is defined by

$$\phi_X(u) := \mathbb{E}[e^{iuX}] = \int_{-\infty}^{\infty} e^{iux} f(x)dx.$$

We have

$$|\phi_X(u)| \leq \int_{-\infty}^{\infty} |e^{iux}| f(x)dx,$$

and thus

$$|\phi_X(u)| \leq \int_{-\infty}^{\infty} f(x)dx = 1. \qquad \blacksquare$$

Now we start with the error analysis and denote by $\epsilon(\hat{\phi}_{Y_m}(u))$ and $\epsilon(\hat{\phi}_{Z_m}(u))$, $m = 1,\ldots,M$, the errors in $\hat{\phi}_{Y_m}(u)$ and $\hat{\phi}_{Z_m}(u)$, respectively. From (3.16) the error in the arithmetic Asian option price, denoted by ϵ, is given by

$$
\begin{aligned}
\epsilon &= e^{-r\Delta t}\int_{-\infty}^{\infty} v(y,T) f_{Y_M}(y)\,dy - e^{-r\Delta t}{\sum_{k=0}^{N-1}}{}' \operatorname{Re}\left\{\hat{\phi}_{Y_M}\left(\frac{k\pi}{b-a}\right) e^{-ik\pi\frac{a}{b-a}}\right\} V_k \\
&= e^{-r\Delta t}\int_{-\infty}^{\infty} v(y,T) f_{Y_M}(y)\,dy - e^{-r\Delta t}{\sum_{k=0}^{N-1}}{}' \operatorname{Re}\left\{\phi_{Y_M}\left(\frac{k\pi}{b-a}\right) e^{-ik\pi\frac{a}{b-a}}\right\} V_k \\
&\quad + e^{-r\Delta t}{\sum_{k=0}^{N-1}}{}' \operatorname{Re}\left\{\left[\phi_{Y_M}\left(\frac{k\pi}{b-a}\right) - \hat{\phi}_{Y_M}\left(\frac{k\pi}{b-a}\right)\right] e^{-ik\pi\frac{a}{b-a}}\right\} V_k \\
&= \epsilon_{\cos} + e^{-r\Delta t}{\sum_{k=0}^{N-1}}{}' \operatorname{Re}\left\{\epsilon\left(\hat{\phi}_{Y_M}\left(\frac{k\pi}{b-a}\right)\right) e^{-ik\pi\frac{a}{b-a}}\right\} V_k,
\end{aligned}
$$

where V_k is known analytically and $\epsilon_{\cos}$ is the error resulting from the use of the COS pricing method. From [12] we know that for a sufficiently large range of integration $[a,b]$, we have $\epsilon_{\cos} = O(\epsilon_F)$, and thus

$$
\epsilon = O(\epsilon_F) + e^{-r\Delta t}{\sum_{k=0}^{N-1}}{}' \operatorname{Re}\left\{\epsilon\left(\hat{\phi}_{Y_M}\left(\frac{k\pi}{b-a}\right)\right) e^{-ik\pi\frac{a}{b-a}}\right\} V_k. \tag{4.3}
$$

The remaining part of the error (4.3) which we need to estimate is $\epsilon(\hat{\phi}_{Y_M}(u))$. This is done by mathematical induction. We first estimate the error in $\hat{\phi}_{Y_1}(u)$ and $\hat{\phi}_{Y_2}(u)$ and then use an induction step to bound the error in $\hat{\phi}_{Y_M}(u)$.

Characteristic function $\phi_{Y_1}(u)$ is known analytically from (3.8), so that $\epsilon(\hat{\phi}_{Y_1}(u)) = 0\ \forall u$.

The error in $\hat{\phi}_{Z_1}(u)$ consists of three parts. The first part is the error due to the truncation of the integration range, as in (3.11). The second part is due to the approximation of $f_{Y_1}(x)$ by the Fourier cosine expansion in (3.14). The third part is due to the use of the Clenshaw–Curtis quadrature rule to approximate the integral in (3.14). Summing up, we have

$$
\begin{aligned}
\epsilon(\hat{\phi}_{Z_1}(u)) &= \int_{-\infty}^{\infty} (e^x+1)^{iu} f_{Y_1}(x)\,dx - \int_a^b (e^x+1)^{iu} f_{Y_1}(x)\,dx \\
&\quad + \int_a^b (e^x+1)^{iu} f_{Y_1}(x)\,dx - \frac{2}{b-a}{\sum_{l=0}^{N-1}}{}' \operatorname{Re}\left\{\phi_{Y_1}\left(\frac{l\pi}{b-a}\right)\exp\left(-ia\frac{l\pi}{b-a}\right)\right\} I \\
&\quad + \frac{2}{b-a}{\sum_{l=0}^{N-1}}{}' \operatorname{Re}\left\{\phi_{Y_1}\left(\frac{l\pi}{b-a}\right)\exp\left(-ia\frac{l\pi}{b-a}\right)\right\} (I - \hat{I}) \\
&= \int_{\mathbb{R}\setminus[a,b]} (e^x+1)^{iu} f_{Y_1}(x)\,dx + \epsilon_F + \frac{2}{b-a}{\sum_{l=0}^{N-1}}{}' \operatorname{Re}\left\{\phi_{Y_1}\left(\frac{l\pi}{b-a}\right)\exp\left(-ia\frac{l\pi}{b-a}\right)\right\} \epsilon_q.
\end{aligned} \tag{4.4}
$$

The lemma below gives an upper bound for the local error.

Lemma 4.2. *We define*

$$
\begin{aligned}
e_j &:= \int_{\mathbb{R}\setminus[a,b]} (e^x+1)^{iu} f_{Y_j}(x)\,dx + \epsilon_F \\
&\quad + \frac{2}{b-a}\sideset{}{'}\sum_{l=0}^{N-1} \mathrm{Re}\left\{\phi_{Y_j}\left(\frac{l\pi}{b-a}\right)\exp\left(-ia\frac{l\pi}{b-a}\right)\right\}\epsilon_q.
\end{aligned} \tag{4.5}
$$

Then, with integration range $[a,b]$ *sufficiently wide, we have*

$$|e_j| \le \bar{P}(N,n_q)\left(|\epsilon_F| + \frac{2}{b-a}N|\epsilon_q|\right) \quad \forall j,$$

where $\bar{P}(N,n_q) > 0$ *varies less than* ϵ_F *and* ϵ_q*, with respect to* N, n_q.

Proof. Application of (3.13) gives us that, $\forall j, u \in \mathbb{R}$,

$$\left|\int_{\mathbb{R}\setminus[a,b]} (e^x+1)^{iu} f_{Y_j}(x)\,dx\right| \le \epsilon_T(Y_j), \tag{4.6}$$

with $\epsilon_T(Y_j)$ defined in (4.1). Substitution into (4.5) results in

$$|e_j| \le |\epsilon_T(Y_j)| + |\epsilon_F| + \frac{2}{b-a}\sideset{}{'}\sum_{l=0}^{N-1}\left|\mathrm{Re}\left\{\phi_{Y_j}\left(\frac{l\pi}{b-a}\right)\exp\left(-ia\frac{l\pi}{b-a}\right)\right\}\right||\epsilon_q|.$$

From Lemma 4.1, it follows that, $\forall j, l$, $|\phi_{Y_j}(l\pi/b-a)| \le 1$, and

$$\left|\exp\left(-ia\frac{l\pi}{b-a}\right)\right| = \left|\cos\left(-a\frac{l\pi}{b-a}\right) + i\sin\left(-a\frac{l\pi}{b-a}\right)\right| = 1 \quad \forall l,$$

so that $\left|\mathrm{Re}\left[\phi_{Y_j}(l\pi/(b-a))\exp(-ial\pi/(b-a))\right]\right| \le 1$ $\forall j, l$.

For $[a,b]$ sufficiently wide, ϵ_F dominates the expression $\epsilon_F + \epsilon_T$, so that we find, $\forall j$, that

$$|e_j| \le \bar{P}(N,n_q)\left(|\epsilon_F| + \frac{2}{b-a}\sideset{}{'}\sum_{l=0}^{N-1}|\epsilon_q|\right) = \bar{P}(N,n_q)\left(|\epsilon_F| + \frac{2}{b-a}N|\epsilon_q|\right), \tag{4.7}$$

where $\bar{P}(N,n_q) > 0$ varies less than ϵ_F and ϵ_q with respect to N, n_q. ∎

Using the notation

$$\epsilon_L := |\epsilon_F| + \frac{2}{b-a}N|\epsilon_q|, \tag{4.8}$$

we can write $|e_j| \le \bar{P}(N,n_q)\epsilon_L$ $\forall j$. Application of Lemma 4.2 and (4.8) to (4.4) gives

$$|\epsilon(\hat{\phi}_{Z_1}(u))| = |e_1| \le \bar{P}(N,n_q)\epsilon_L.$$

We continue with the error in $\hat{\phi}_{Y_2}(u)$. From (3.9) we have that

$$\epsilon(\hat{\phi}_{Y_2}(u)) = \epsilon(\hat{\phi}_{Z_1}(u))\phi_R(u) = e_1\phi_R(u) = e_1\phi_{Y_1}(u) \quad \forall u. \tag{4.9}$$

Applying Lemmas 4.1 and 4.2 to (4.9) results in

$$|\epsilon(\hat{\phi}_{Y_2}(u))| = |e_1||\phi_{Y_1}(u)| \leq |e_1| \leq \bar{P}(N, n_q)\epsilon_L. \tag{4.10}$$

Next, we arrive at the induction step, described in the lemma below.

We use the common notation $\epsilon = O(g(a_1, \ldots, a_n))$ to indicate that a $Q > 0$ exists, so that $|\epsilon| = Q|g(a_1, \ldots, a_N)|$, with Q constant or varying less than function $g(\cdot)$ with respect to parameters $a_1, \ldots, a_N$.

Lemma 4.3. *For $m = 3, \ldots, M$, assuming that*

$$\epsilon(\hat{\phi}_{Y_{m-1}}(u)) = \bar{P}(N, n_q)\sum_{j=1}^{(m-1)-1}\phi_{Y_j}(u)e_{(m-1)-j} \quad \forall u, \tag{4.11}$$

where $\bar{P}(N, n_q)$ is a term which varies less than exponentially with respect to N and n_q, then

$$\epsilon(\hat{\phi}_{Y_m}(u)) = O\left(\sum_{j=1}^{m-1}\phi_{Y_j}(u)e_{m-j}\right) \quad \forall u, \tag{4.12}$$

and thus

$$|\epsilon(\hat{\phi}_{Y_m}(u))| = O(m-1)\epsilon_L. \tag{4.13}$$

Proof. We find that, for $m = 3, \ldots, M$ and $\forall u$,

$$\begin{aligned}
&\epsilon(\hat{\phi}_{Z_{m-1}}(u)) \\
&= \int_{-\infty}^{\infty}(e^x+1)^{iu}f_{Y_{m-1}}(x)dx - \frac{2}{b-a}{\sum_{l=0}^{N-1}}{}'\,\mathrm{Re}\left\{\hat{\phi}_{Y_{m-1}}\left(\frac{l\pi}{b-a}\right)\exp\left(-ia\frac{l\pi}{b-a}\right)\right\}\hat{I} \\
&= \int_{-\infty}^{\infty}(e^x+1)^{iu}f_{Y_{m-1}}(x)dx - \int_a^b(e^x+1)^{iu}f_{Y_{m-1}}(x)dx \\
&\quad + \int_a^b(e^x+1)^{iu}f_{Y_{m-1}}(x)dx - \frac{2}{b-a}{\sum_{l=0}^{N-1}}{}'\,\mathrm{Re}\left\{\phi_{Y_{m-1}}\left(\frac{l\pi}{b-a}\right)\exp\left(-ia\frac{l\pi}{b-a}\right)\right\}I \\
&\quad + \frac{2}{b-a}{\sum_{l=0}^{N-1}}{}'\,\mathrm{Re}\left\{\phi_{Y_{m-1}}\left(\frac{l\pi}{b-a}\right)\exp\left(-ia\frac{l\pi}{b-a}\right)\right\}(I-\hat{I}) \\
&\quad + \frac{2}{b-a}{\sum_{l=0}^{N-1}}{}'\,\mathrm{Re}\left\{\left[\phi_{Y_{m-1}}\left(\frac{l\pi}{b-a}\right) - \hat{\phi}_{Y_{m-1}}\left(\frac{l\pi}{b-a}\right)\right]\exp\left(-ia\frac{l\pi}{b-a}\right)\right\}\hat{I} \\
&= \int_{\mathbb{R}\setminus[a,b]}(e^x+1)^{iu}f_{Y_{m-1}}(x)dx + \epsilon_F + \frac{2}{b-a}{\sum_{l=0}^{N-1}}{}'\,\mathrm{Re}\left\{\phi_{Y_{m-1}}\left(\frac{l\pi}{b-a}\right)\exp\left(-ia\frac{l\pi}{b-a}\right)\right\}\epsilon_q \\
&\quad + \frac{2}{b-a}{\sum_{l=0}^{N-1}}{}'\,\mathrm{Re}\left\{\epsilon\left(\phi_{Y_{m-1}}\left(\frac{l\pi}{b-a}\right)\right)\exp\left(-ia\frac{l\pi}{b-a}\right)\right\}\hat{I}
\end{aligned}$$

$$= e_{m-1} + \frac{2}{b-a}{\sum_{l=0}^{N-1}}{}'\,\mathrm{Re}\left\{\epsilon\left(\phi_{Y_{m-1}}\left(\frac{l\pi}{b-a}\right)\right)\exp\left(-ia\frac{l\pi}{b-a}\right)\right\}\hat{I}. \tag{4.14}$$

Substitution of (4.11) into (4.14) gives

$$
\begin{aligned}
&\epsilon(\hat{\phi}_{Z_{m-1}}(u))\\
&= e_{m-1} + \bar{P}(N, n_q) \sum_{j=1}^{(m-1)-1} \frac{2}{b-a} {\sum_{l=0}^{N-1}}{}' \operatorname{Re}\left\{ \phi_{Y_j}\left(\frac{l\pi}{b-a}\right) e_{(m-1)-j} \exp\left(-ia\frac{l\pi}{b-a}\right)\right\} \hat{I}\\
&= e_{m-1} + \bar{P}(N, n_q) \sum_{j=1}^{(m-1)-1} e_{(m-1)-j} \left(\frac{2}{b-a} {\sum_{l=0}^{N-1}}{}' \operatorname{Re}\left\{ \phi_{Y_j}\left(\frac{l\pi}{b-a}\right) \exp\left(-ia\frac{l\pi}{b-a}\right)\right\} \hat{I}\right)\\
&= e_{m-1} + \bar{P}(N, n_q) \sum_{j=1}^{(m-1)-1} e_{(m-1)-j}\, \hat{\phi}_{Z_j}(u).
\end{aligned}
$$

The error in $\hat{\phi}_{Y_m}(u)$, $\forall u$, is found as

$$
\begin{aligned}
\epsilon(\hat{\phi}_{Y_m}(u)) &= \phi_R(u)\epsilon(\hat{\phi}_{Z_{m-1}}(u))\\
&= \phi_R(u) e_{m-1} + \bar{P}(N, n_q) \sum_{j=1}^{(m-1)-1} e_{(m-1)-j}\, \phi_R(u)\hat{\phi}_{Z_j}(u)\\
&= \phi_R(u) e_{m-1} + \bar{P}(N, n_q) \sum_{j=1}^{(m-1)-1} e_{(m-1)-j}\, \hat{\phi}_{Y_{j+1}}(u)\\
&= \phi_{Y_1}(u) e_{m-1} + \bar{P}(N, n_q) \sum_{j=2}^{m-1} e_{m-j}\, \hat{\phi}_{Y_j}(u)\\
&= O\left(\sum_{j=1}^{m-1} \phi_{Y_j}(u) e_{m-j}\right) + O(e_k e_l), \quad k, l \in 1, \ldots, m-1.
\end{aligned}
$$

From Lemma 4.2 we see that $|e_j| = O(|\epsilon_F| + |\epsilon_q|)$ $\forall j$ if N and n_q increase simultaneously. Error ϵ_F decays exponentially with respect to N, and ϵ_q decays exponentially with respect to n_q, so that e_j decays exponentially and the quadratic term, $e_k e_l$, converges to zero faster than e_j. We thus have that

$$
\epsilon(\hat{\phi}_{Y_m}(u)) = O\left(\sum_{j=1}^{m-1} \phi_{Y_j}(u) e_{m-j}\right),
$$

and application of Lemmas 4.1 and 4.2 gives, $\forall u \in \mathbb{R}$,

$$
\left|\sum_{j=1}^{m-1} \phi_{Y_j}(u) e_{m-j}\right| \leq \sum_{j=1}^{m-1} |\phi_{Y_j}(u)||e_{m-j}| \leq \bar{P}(N, n_q)(m-1)\epsilon_L,
$$

where $\bar{P}(N, n_q)$ varies less than ϵ_F and ϵ_q with respect to N, n_q, respectively. So

$$
|\epsilon(\hat{\phi}_{Y_m}(u))| = O((m-1)\epsilon_L), \tag{4.15}
$$

which concludes the proof. ■

As a result of the lemma above, we have, $\forall u$,

$$\epsilon(\hat{\phi}_{Y_M}(u)) = O\left(\sum_{j=1}^{M-1} \phi_{Y_j}(u) e_{m-j}\right) \tag{4.16}$$

and

$$|\epsilon(\hat{\phi}_{Y_M}(u))| = O((M-1)\epsilon_L). \tag{4.17}$$

Remark 4.1 (error of $\hat{\phi}_{Y_M}$). Application of (4.17) and (4.8) results in

$$|\epsilon(\hat{\phi}_{Y_M}(u))| = O\left((M-1)\left(|\epsilon_F| + \frac{2}{b-a}N|\epsilon_q|\right)\right) \quad \forall u.$$

When the number of monitoring dates, M, increases, larger values of N and n_q are necessary to reach a specified level of accuracy.

Moreover, when a large value of N is necessary for accuracy, we should also increase n_q to control the error. When N and n_q both increase, the expression $|N\epsilon_q|$ converges exponentially to zero[2] and we have that

$$|\epsilon(\hat{\phi}_{Y_M}(u))| = O((M-1)(|\epsilon_F| + |\epsilon_q|)) \quad \forall u.$$

4.2. Error in the option price. We now focus on the error in the arithmetic Asian option price. After application of (4.16) in (4.3) the error reads as

$$\epsilon = O(\epsilon_F) + O\left(\sum_{j=1}^{M-1} e_{m-j} \exp(-r\Delta t) {\sum_{k=0}^{N-1}}{}' \operatorname{Re}\left\{\phi_{Y_j}\left(\frac{k\pi}{b-a}\right) e^{-ik\pi\frac{a}{b-a}}\right\} V_k\right). \tag{4.18}$$

When replacing $e^{-r\Delta t}V_k$ (V_k defined in (3.17)) by the term

$$e^{-r\Delta t_j} W_k^j := e^{-r\Delta t_j} \begin{cases} \frac{2}{b-a}\left[\frac{S_0}{j+1}\chi_k(x^*,b) + \left(\frac{S_0}{j+1} - K\right)\psi_k(x^*,b)\right] & \text{for a call,} \\ \frac{2}{b-a}\left[\left(K - \frac{S_0}{j+1}\right)\psi(a,x^*) - \frac{S_0}{j+1}\chi(a,x^*)\right] & \text{for a put,} \end{cases} \tag{4.19}$$

with $\Delta t_j := j\Delta t/M$, the expression

$$\sum_{j=1}^{M-1} e_{m-j} \exp(-r\Delta t) {\sum_{k=0}^{N-1}}{}' \operatorname{Re}\left\{\phi_{Y_j}\left(\frac{k\pi}{b-a}\right) e^{-ik\pi\frac{a}{b-a}}\right\} V_k \quad \forall j,k$$

remains of the same order regarding N and n_q.

The error in (4.18) therefore satisfies

$$\epsilon = O(\epsilon_F) + O\left(\sum_{j=1}^{M-1} e_{m-j} e^{-r\Delta t_j} {\sum_{k=0}^{N-1}}{}' \operatorname{Re}\left\{\phi_{Y_j}\left(\frac{k\pi}{b-a}\right) e^{-ik\pi\frac{a}{b-a}}\right\} W_k^j\right).$$

[2] Note that N varies linearly but $|\epsilon_q|$ decays exponentially, so that $N|\epsilon_q|$ also decays exponentially.

We can now write, for the overall error,

$$\epsilon = O(\epsilon_F) + O\left(\sum_{j=1}^{M-1} e_{m-j} A(S_0, \Delta t_j)\right),$$

where $A(S_0, \tau)$ stands for the Asian option value with initial underlying price S_0 and time to maturity τ. Then,

$$|\epsilon| = O(|\epsilon_F|) + O\left(\sum_{j=1}^{M-1} |e_{m-j}| A(S_0, \Delta t_j)\right).$$

By Lemma 4.2 we find that

$$|\epsilon| = O(|\epsilon_F|) + O\left(\left(|\epsilon_F| + \frac{2}{b-a} N |\epsilon_q|\right) \sum_{j=1}^{M-1} A(S_0, \Delta t_j)\right). \tag{4.20}$$

Volatility inherent in an Asian option is smaller than that of an equivalent vanilla European option, due to the averaging feature. This makes Asian options cheaper than their plain vanilla equivalents. In other words, with the same maturity, the value of an Asian option, $A(S_0, \tau)$, is less than or equal to that of the corresponding vanilla European option, denoted by $E(S_0, \tau)$, written on the same underlying asset. The European option value will therefore be used as the upper bound for the corresponding arithmetic Asian option value in (4.20), and we have

$$|\epsilon| = O(|\epsilon_F|) + O\left(\left(|\epsilon_F| + \frac{2}{b-a} N |\epsilon_q|\right) \sum_{j=1}^{M-1} E(S_0, \Delta t_j)\right). \tag{4.21}$$

We assume that

$$\max_{j=1,\ldots,M-1} E(S_0, j\Delta t_j) =: E(S_0, \Delta t_{j^*}),$$

so that the error in the Asian option price satisfies

$$|\epsilon| = O(|\epsilon_F|) + O\left(\left(|\epsilon_F| + \frac{2}{b-a} N |\epsilon_q|\right) (M-1) E(S_0, \Delta t_{j^*})\right). \tag{4.22}$$

What remains is an upper bound for the plain vanilla European option value, $E(S_0, (M-1)\Delta t_{j^*})$, which is given as follows.

Result 4.1. *The value of a plain vanilla European call option can be bounded by*

$$v_C(S_0, \tau) \le S_0 e^{-q\tau},$$

with S_0, τ, q the initial underlying price, the time to maturity, and the dividend rate, respectively.

The value of a vanilla European put option can be bounded by

$$v_P(S_0, \tau) \le K e^{-r\tau},$$

with K, r the strike price and the interest rate, respectively.

Summarizing, the error in the arithmetic Asian option with M monitoring dates can be approximated by

$$|\epsilon| \approx \begin{cases} O\left(\left(|\epsilon_F| + \frac{2}{b-a}N|\epsilon_q|\right)(M-1)S_0 e^{-q\Delta t_j}\right) & \text{for a call,} \\ O\left(\left(|\epsilon_F| + \frac{2}{b-a}N|\epsilon_q|\right)(M-1)K e^{-r\Delta t_j}\right) & \text{for a put.} \end{cases} \tag{4.23}$$

For $f(y|x) \in C^\infty[a,b]$, ϵ_F and ϵ_q converge exponentially with respect to N and n_q, respectively. Therefore, as N and n_q increase, the error in the Asian option price decreases exponentially:

$$|\epsilon| \sim \bar{P}(N, n_q)(\exp(-(N-1)\nu_F) + \exp(-(n_q - 1)\nu_q)),$$

where $\bar{P}(N, n_q)$ is a term which varies less than exponentially with respect to N and n_q, and $\nu_F > 0$, $\nu_q > 0$.

When the transitional probability density function has a discontinuous derivative, the error in the Asian option price converges algebraically.

5. Numerical results. In this section numerical results for Asian options under the Black–Scholes (BS), CGMY [6], and normal inverse Gaussian (NIG) [2] models are presented. We use the same parameter sets as in [15], based on three test cases:

- *BS case*: $r = 0.0367$, $\sigma = 0.17801$;
- *CGMY case*: $r = 0.0367$, $C = 0.0244$, $G = 0.0765$, $M = 7.5515$, $Y = 1.2945$;
- *NIG case*: $r = 0.0367$, $\alpha = 6.1882$, $\beta = -3.8941$, $\delta = 0.1622$.

These parameters have been obtained by calibration (see [15]). The characteristic functions for these processes are presented in Appendix B. In all numerical examples we set time to maturity $T - t_0 = 1$, and $S_0 = 100$. Strike price, K, and the number of monitoring dates, M, vary among the different experiments.

MATLAB 7.7.0 is used, and the CPU is an Intel(R) Core(TM)2 Duo CPU E6550 (@ 2.33GHz Cache size 4MB). CPU time is recorded in seconds.

The absolute error that we report below is defined as the absolute value of the difference between the approximate solution at t_0 and S_0 and a reference value which is computed by the ASCOS method with a large number of terms in the Fourier cosine expansions. The values have also been compared to reference values in the literature. With our own reference values, however, we can compare up to a higher accuracy.

5.1. Geometric Asian options. First, we confirm the exponential convergence of the ASCOS method for geometric Asian options under the BS model, for which an analytic result is available, in Figure 1. For increasing N-values the error decreases exponentially.

The performance of the ASCOS pricing method for the NIG and CGMY test cases is presented in Table 1. Geometric Asian call option prices with 12, 50, and 250 monitoring dates are shown. Reference values are taken from ASCOS computations with $N = 4096$. In all examples our method also gives the same option prices, up to a basis point, as those presented in [15].

From Table 1 we see that the option prices have converged up to basis point precision with $N = 128$ and $N = 512$, respectively, for the NIG and CGMY test cases. Exponential

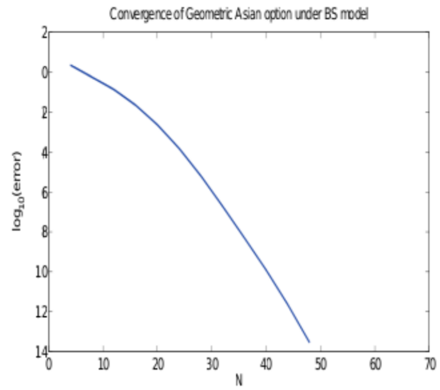

Figure 1. Convergence of geometric Asian options under the BS model with $M = 250$, $S_0 = 100$, $K = 90$.

Table 1
Convergence of geometric Asian options for the NIG and CGMY test cases with $S_0 = 100$, $K = 110$.

NIG model				
M		$N = 64$	$N = 128$	$N = 192$
12	Abs. error	1.42e-04	2.81e-05	1.33e-08
	CPU time	4.9e-04	7.7e-04	8.3e-04
50	Abs. error	1.23e-04	3.07e-05	1.24e-08
	CPU time	9.3e-04	1.4e-03	2.1e-03
250	Abs. error	1.13e-04	3.13e-05	2.11e-08
	CPU time	3.1e-03	5.8e-03	8.2e-03
CGMY model				
M		$N = 256$	$N = 512$	$N = 1024$
12	Abs. error	2.1e-03	9.87e-06	6.27e-11
	CPU time	2.7e-03	4.1e-03	9.9e-03
50	Abs. error	1.20e-02	1.24e-05	6.71e-11
	CPU time	1.2e-02	1.7e-02	4.3e-02
250	Abs. error	1.16e-02	3.65e-05	3.84e-11
	CPU time	0.050	0.10	0.22

convergence is observed for these exponential Lévy asset price processes, and, as a result, the geometric Asian options can be priced within milliseconds by the ASCOS method. In a comparison with the results in [15], ASCOS is approximately 100 times faster in the NIG test case and 20 times faster in the CGMY case.

Table 2 presents the convergence behavior when we approximate continuously monitored geometric Asian options ($M = 1$) by discretely monitored geometric Asian options combined with the four-point Richardson extrapolation (3.26). Here d is as defined in (3.26); that is, discretely monitored Asian options with $2^d, 2^{d+1}, 2^{d+2}, 2^{d+3}$ monitoring dates are used to approximate the continuously monitored Asian options. The reference values have been obtained by employing the ASCOS method with $N = 4096$, $M = 512$.

The discretely monitored Asian prices with 4, 8, 16, and 32 monitoring dates, i.e., $d = 2$, have converged to the reference Asian price in Table 2. Note that one may also develop a Richardson extrapolation scheme to approximate discrete Asian options with many monitoring

Table 2

Convergence of geometric Asian options for the NIG and CGMY cases with $S_0 = 100$, $K = 110$. For the NIG model we use $N = 128$ and for the CGMY model $N = 512$.

	NIG		CGMY	
d	Abs. error	CPU time	Abs. error	CPU time
1	3.78e-04	0.0018	2.06e-04	0.0120
2	5.92e-05	0.0023	1.21e-04	0.0247
3	3.31e-05	0.0052	5.71e-05	0.0499

dates by a Richardson extrapolation based on fewer dates.

We need approximately 2 and 25 milliseconds to get the continuously monitored Asian option prices within basis point precision for the NIG and CGMY test cases, respectively, which is competitive with the existing methods in [15, 14, 8].

5.2. Arithmetic Asian options. In all numerical experiments in this subsection, the reference values are obtained by the ASCOS method with $N = 4096$, $n_q = 6400$.

Figure 2 presents the logarithm (basis 10) of the absolute error in the value of an arithmetic Asian option under the BS model with 50 monitoring dates, against the index d with $N = 64d$ and $n_q = 100d$, where exponential convergence in the option price with respect to N and n_q, increasing simultaneously, is observed. Our method is an efficient alternative for [14], which has algebraic convergence as the error decays linearly on a log-log scale.

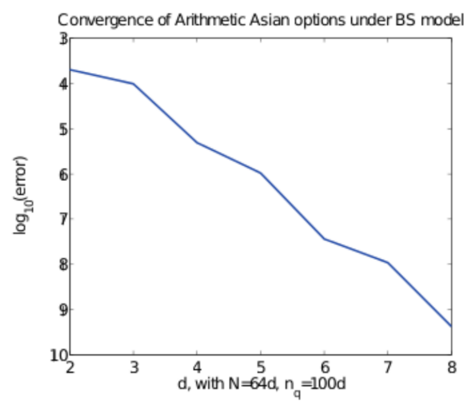

Figure 2. Convergence of arithmetic Asian options for the BS test case with $M = 50$, $S_0 = 100$, $K = 90$.

Table 3 then presents the convergence and the CPU time of an arithmetic Asian option for the NIG test case with $M = 12$, $M = 50$, and $M = 250$ (monthly, weekly, and daily monitored, respectively). Exponential convergence is *not* influenced significantly by an increase in the number of monitoring dates, M, and neither is the CPU time. This is because the quadrature rule, which dominates the CPU time, is used only once. This feature is especially beneficial for pricing Asian options with many monitoring dates and continuously monitored Asian options. However, with a larger number of monitoring dates, based on our error analysis, a larger number of Fourier cosine terms may be required to reach the same level of accuracy, thus resulting in a higher CPU time which grows as $n_q N^2$. With $N = 256$, $n_q = 400$, we find converged option prices (up to basis point precision) for the NIG case with all monitoring

Table 3
Convergence of arithmetic Asian options for the NIG test case with $S_0 = 100$, $K = 110$.

M	Time and error	$N = 128$ $n_q = 200$	$N = 256$ $n_q = 400$	$N = 384$ $n_q = 600$
12	Abs. error	2.0e-3	1.71e-4	5.16e-6
	CPU time	2.41	15.13	46.09
50	Abs. error	2.26e-4	6.94e-5	2.17e-6
	CPU time	2.43	15.16	46.22
250	Abs. error	7.8e-3	9.33e-5	8.49e-6
	CPU time	2.42	15.23	46.68

dates.

Similar convergence behavior has been observed for other Lévy processes. For instance, in the case of the CGMY model, when $M = 12, 50$, the option prices converge to basis point precision with $N = 256$, $n_q = 400$, and the computation time is within 15 seconds. With $M = 250$, the ASCOS method reaches basis point accuracy for the CGMY model when $N = 320$, $n_q = 500$ in approximately 27 seconds, which is much less than the approximately 210 seconds it takes with the method in [15] to reach an accuracy of $O(10^{-3})$ for the same CGMY test case with $M = 250$. Note that due to the exponential convergence rate of the Clenshaw–Curtis quadrature and the Fourier cosine expansion, the number of terms needed to reach a certain accuracy level remains limited, which reduces the computational cost and the CPU time of our pricing method.

In Table 4 we finally compute continuously monitored arithmetic Asian call options under the NIG model, with $S_0 = 100$ and different strikes, by the repeated Richardson extrapolation based on discretely monitored arithmetic Asian call options (3.26). The option prices converge somewhat slower with respect to parameter d when compared to the geometric Asian case. However, the CPU time of the ASCOS method does not increase when d increases, so that we can use a larger value for d, for instance $d = 6$ ($M = 64, 128, 256, 512$), and obtain accurate results.

Table 4
Convergence of arithmetic Asian options under the NIG model with $S_0 = 100$, $N = 256$, $n_q = 400$.

d	$K = 90$		$K = 100$	
	Option value	CPU time	Option value	CPU time
4	12.6748	60.05	5.1191	60.01
5	12.6744	60.13	5.1186	59.94
6	12.6743	60.35	5.1185	60.17

6. Conclusions.

In this article, we proposed an efficient pricing method for European-style Asian options, the ASCOS method, based on Fourier cosine expansions and Clenshaw–Curtis quadrature. The method performs well for different exponential Lévy processes, different parameter values, and different numbers of Asian option monitoring dates. The method is accompanied by a detailed error analysis, giving evidence for an exponential convergence rate for geometric and arithmetic Asian options. Due to the exponential convergence, our pricing method is highly efficient and significant speedup has been achieved compared to competitor pricing methods.

The ASCOS method performs in a robust manner when the number of monitoring dates increases, and, interestingly, the CPU time does not increase significantly. This makes the pricing method especially advantageous for weekly and even daily monitored arithmetic Asian options, as well as for continuously monitored Asian options whose value can be approximated by discretely monitored Asian options in combination with Richardson extrapolation.

Appendix A. Beta function formulation. After some manipulations with symbolic software, we find that integral (3.22) can be written in a form with incomplete Beta functions as follows:

$$
\begin{aligned}
&\int_a^b (e^x+1)^{i\frac{k-l}{b-a}} \cos\left((x-a)\frac{l\pi}{b-a}\right) dx \\
&= \frac{1}{2} e^{-\frac{l(ia+\pi)}{d}} \left(e^{\frac{2ial}{d}} \left(-\beta\left(-e^a, -\frac{il}{d}, 1+\frac{ik}{d}\right) + \beta\left(-e^b, -\frac{il}{d}, 1+\frac{ik}{d}\right)\right) \right. \\
&\qquad \left. + e^{\frac{2l\pi}{d}} \left(-\beta\left(-e^a, \frac{il}{d}, 1+\frac{ik}{d}\right) + \beta\left(-e^b, \frac{il}{d}, 1+\frac{ik}{d}\right)\right)\right), \qquad \text{(A.1)}
\end{aligned}
$$

where $i = \sqrt{-1}$, $d = \frac{b-a}{\pi}$, and $\beta(x, y, z)$ is the incomplete Beta function

$$\beta(x, y, z) = \int_0^x t^{y-1}(1-t)^{z-1} dt.$$

The computation of the incomplete Beta functions in (A.1) is, however, involved with these complex-valued arguments.

Appendix B. Exponential Lévy processes and characteristic functions. With exponential Lévy models, the underlying asset is written as an exponential function of a Lévy process and the characteristic function of the log-asset price can be found in closed form as

$$\phi(u; x_0) = \exp(iux_0)\varphi(u, t), \qquad \text{(B.1)}$$

where $x_0 = \log(S_0)$ and $\varphi(u, t)$, the characteristic function of an increment in the log-asset, is defined as in (2.5).

The simplest and most widely used exponential Lévy process is the geometric Brownian motion (GBM) model, where the logarithm of the asset price follows a Brownian motion. Under the GBM model, the characteristic function of the Lévy increment, $\varphi(u, t)$ in (B.1), has the following form:

$$\varphi_{\mathrm{GBM}}(u, t) = \exp\left(iu\mu t - \frac{1}{2}u^2\sigma^2 t\right),$$

where μ and σ are the percentage drift and percentage volatility, respectively, of the underlying process.

One problem with the GBM model is that it is not able to reproduce the volatility skew or smile present in most financial markets. Over the past few years it has been shown that several other exponential Lévy models are, at least to some extent, able to reproduce the skew or smile.

One particular model we consider is the *CGMY model* [6]. The underlying Lévy process is characterized by four parameters C, G, M, and Y. Parameter $Y : Y < 2$ controls whether the CGMY process has finite or infinite activity. Parameter $C : C > 0$ controls the kurtosis of the distribution, and nonnegative parameters G, M give control over the rate of exponential decay on the right and left tails of the density, respectively.

For a CGMY model, the characteristic function of increment reads as

$$\varphi_{\mathrm{CGMY}}(u,t) = \exp\left(iu\mu t\right) \cdot \exp\left(tC\Gamma(-Y)\left((M-iu)^Y - M^Y + (G+iu)^Y - G^Y\right)\right),$$

where $\Gamma(x)$ is the gamma function.

The NIG process [2] is a variance-mean mixture of a Gaussian distribution with an inverse Gaussian. The pure jump characteristic function of increment under the NIG model reads as

$$\varphi_{\mathrm{NIG}}(u,t) = \exp\left(iu\mu t - \frac{1}{2}u^2\sigma^2 t\right) \cdot \exp\left(t\delta\left(\sqrt{\alpha^2-\beta^2} - \sqrt{\alpha^2-(\beta+iu)^2}\right)\right),$$

with $\alpha, \delta > 0$ and $\beta \in (-\alpha, \alpha - 1)$. The α-parameter controls the steepness of the density; β is a skewness parameter: $\beta > 0$ implies a density skew to the right, $\beta < 0$ implies a density skew to the left, and $\beta = 0$ implies the density is symmetric around 0. δ is a scale parameter in the sense that the rescaled parameters $\alpha \to \alpha\delta$ and $\beta \to \beta\delta$ are invariant under location-scale changes of x.

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