Exercise 1 We write the central difference discretization for a second derivative in the form

\[ (u_i'')_h = \frac{1}{h^2} (a_1 u_{i-1} + a_2 u_i + a_3 u_{i+1}) \]

where \(a_1, a_2\) and \(a_3\) represent parameters, that must be calculated so that the scheme has a high accuracy. By Taylor’s expansion of \(u_{i-1}\) and \(u_{i+1}\) around \(u_i\) we find

\[ (u_i'')_h = \frac{1}{h^2} \left( u_i (a_1 + a_2 + a_3) + h u_i' (-a_1 + a_3) + \frac{1}{2!} h^2 u_i'' (a_1 + a_3) \right. \]

\[ \left. + \frac{1}{3!} h^3 u_i''' (-a_1 + a_3) + \frac{1}{4!} h^4 u_i'''' (a_1 + a_3) + \cdots \right) \]

Use this equation to compute the best possible approximation for the second derivative of \(u\) under the assumption that \(h\) is very small.

Exercise 2 One can obtain a higher order central difference scheme for the approximation of a first derivative \(u'\), if one uses, for example, \(u\)-values at four grid points, \(x = -2h, x = -h, x = h\) and \(x = 2h\) (see Fig. 1.1).

![Figure 1.1: Grid points for a higher order central difference scheme.](image_url)

The approximation for the first derivative at \(x = 0\) is written in the form

\[ (u_0')_h = a_1 u_1 + a_2 u_2 + a_3 u_3 + a_4 u_4, \]

where \(a_1, a_2, a_3\) and \(a_4\) are parameters that must be calculated so that the scheme has the required accuracy.

Calculate the parameters \(a_1, a_2, a_3\) and \(a_4\) so, that the approximation of \(u'\) is exact for the functions \(u(x) = 1, u(x) = x, u(x) = x^3\) and \(u(x) = x^5\).

Exercise 3 Let the following Sturm-Liouville boundary value problem be given:

\[-u'' + p(x)u' + q(x)u = r(x), \quad u(a) = \alpha, \quad u(b) = \beta\]
with \( q(x) \geq q_0 > 0 \) for \( x \in [a,b] \).
We search for approximations \( \tilde{u}_i \) of the exact values \( u(x_i) \), \( x_i = a + ih \),
i = 1,..,n and \( h = \frac{b-a}{n+1} \).
If one replaces \( u'(x_i) \) by \( \frac{\tilde{u}_{i+1} - \tilde{u}_{i-1}}{2h} \) and \( u''(x_i) \) by \( \frac{\tilde{u}_{i-1} - 2\tilde{u}_i + \tilde{u}_{i+1}}{h^2} \) for
\( i = 1,...,n \), \( \tilde{u}_0 = \alpha \), \( \tilde{u}_{n+1} = \beta \), a system of equations is obtained for the
vector \( \tilde{u} := (\tilde{u}_1, ..., \tilde{u}_n)^T \)

\[
A\tilde{u} = c \quad \text{with} \quad A \in \mathbb{R}^{n \times n}, \quad c \in \mathbb{R}^n.
\]
a) Determine \( A \) and \( c \).
b) For which \( h > 0 \) does \( A \) satisfy the requirement \( a_{i,j} \leq 0 \) for \( i \neq j \)?

**Exercise 4**

a) Set up three second-order accurate discretizations for the mixed derivative \( u_{xy} \), that can be represented in stencil notation by,

\[
A_h = \frac{1}{h^2} \begin{bmatrix}
  a_2 & a_1 & 0 \\
  a_1 & a_3 & a_1 \\
  0 & a_1 & a_2
\end{bmatrix}_h,
B_h = \frac{1}{h^2} \begin{bmatrix}
  0 & b_1 & b_2 \\
  b_1 & b_3 & b_1 \\
  b_2 & b_1 & 0
\end{bmatrix}_h,
C_h = \frac{1}{h^2} \begin{bmatrix}
  c_1 & 0 & c_2 \\
  0 & 0 & 0 \\
  c_3 & 0 & c_4
\end{bmatrix}_h;
\]

b) Consider the equation

\[
-\Delta u - \tau u_{xy} = 0 \quad (\Omega = (0,1)^2)
\]

\[
u = g \quad (\partial \Omega).
\]

Write down a discretization in stencil notation for an interior grid point.

c) Matrix \( A \) is called Z-matrix, if \( a_{ij} \leq 0 \) for \( i \neq j \). The Z-matrix
property is a basis for convergence proofs of certain iterative solution
methods. For which \( \tau \)-values and which discretization approaches discussed under (a), does the boundary value problem (1.0.1) result in a
Z-matrix? Which discretization approach for (1.0.1) would give the
Z-matrix property for general \(-2 < \tau < 2\)?

**Exercise 5** Consider the 1D problem \( Lu(x) = -u''(x) = f(x) \) on the interval \( \Omega = (0,1) \) with boundary conditions \( u(0) = u_0, u(1) = u_1 \). Set up the
discrete problem (matrix and right-hand side) for the discretization

\[
L_h = \frac{1}{h^2} \begin{bmatrix}
  -1 & 2 & -1
\end{bmatrix}
\]

and \( h = 1/4 \)

a) without elimination of boundary conditions,
b) with elimination of boundary conditions.
Exercise 6 Consider \( Lu(x) = -\Delta u(x, y) = f(x, y) \) on the domain \( \Omega = (0, 1)^2 \) with boundary conditions \( u(x, y) = g(x, y) \). Set up the discrete problem (matrix and right-hand side) for the standard five-point discretization of \( L \) and \( h = 1/3 \)

a) without elimination of boundary conditions,

b) with elimination of boundary conditions,

using a lexicographic ordering of grid points.

Exercise 7 Consider \( Lu(x) = -\Delta u(x, y) = f(x, y) \) on the domain \( \Omega = (0, 1)^2 \) with boundary conditions \( u(x, y) = g(x, y) \). Set-up the discrete problem (matrix and right-hand side) for the standard five-point discretization of \( L \) and \( h = 1/3 \) using a red–black ordering of grid points. I.e.,

\[
\begin{array}{cccc}
15 & 7 & 16 & 8 \\
5 & 13 & 6 & 14 \\
11 & 3 & 12 & 4 \\
1 & 9 & 2 & 10 \\
\end{array}
\]

or

\[
\begin{align*}
u_1 &= u_h(0, 0) & u_2 &= u_h(2h, 0) & u_3 &= u_h(h, h) & u_4 &= u_h(1, h) \\
u_5 &= u_h(0, 2h) & u_6 &= u_h(2h, 2h) & u_7 &= u_h(h, 1) & u_8 &= u_h(1, 1) \\
u_9 &= u_h(h, 0) & u_{10} &= u_h(1, 0) & u_{11} &= u_h(0, h) & u_{12} &= u_h(2h, h) \\
u_{13} &= u_h(h, 2h) & u_{14} &= u_h(1, 2h) & u_{15} &= u_h(0, 1) & u_{16} &= u_h(2h, 1)
\end{align*}
\]

a) without elimination of boundary conditions,

b) with elimination of boundary conditions,

Exercise 8 Determine the sparsity structure of the five-point stencil if the grid points are ordered in alternating lines

\[
\begin{array}{cccc}
13 & 14 & 15 & 16 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
1 & 2 & 3 & 4 \\
\end{array}
\]