Pricing Options with Discrete Dividends by High Order Finite Differences and Grid Stretching

Kees Oosterlee
Numerical analysis group, Delft University of Technology
Joint work with Coen Leentvaar, Ariel Almendral Vázquez

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Outline

- Discretization for Black-Scholes equation
  - use only a few grid points
- Discrete dividend
- American-style options

⇒ “PDE on a grid” is straightforward, some modeling questions remain
Black-Scholes option pricing

Point of Departure (here)

- The asset price follows the lognormal random walk.
- Interest rate $r$ and volatility $\sigma_c$ are known functions of $t$.
- Transaction costs for hedging are not included in the model.
- There are no arbitrage possibilities.

$\Rightarrow$ Black-Scholes partial differential equation:
(for a European option)

$$\frac{\partial u}{\partial t} + \frac{1}{2} \sigma_c^2 S^2 \frac{\partial^2 u}{\partial S^2} + rS \frac{\partial u}{\partial S} - ru = 0$$

- The Black-Scholes equation is a parabolic partial differential equation
Options on dividend-paying equities

- At the time that a dividend is paid there will be a drop in the value of the stock.
- The price of an option on an dividend-paying asset is affected by these payments.
- Different structures are possible for the dividend payment (deterministic or stochastic with payments continuously or at discrete times).
- We consider discrete deterministic dividends, whose amount and timing are known.
- Arbitrage arguments require:

\[ u(S, t_d^-) = u(S - D, t_d^+) \]
Final/Boundary conditions

- **European Call option:** Right to buy assets at maturity $t = T$ for exercise price $K$.
- **Final condition:** $u(S, T) = \max(S - K, 0)$
- **Boundary conditions** $S = 0$: $u(0, t) = 0$,
  for $S \to \infty$: $u(S_{\text{max}}, t) = S_{\text{max}} - Ke^{-r(T-t)} - De^{-r(t_d-t)}$ or $u_{ss} = 0$.
- **The strategy to solve the Black-Scholes equation numerically** is as follows
  - Start solving from $t = T$ to $t = t_d$ with the usual pay-off.
  - Apply an interpolation to calculate the new asset and option price on the grid discounted with $D$.
  - Restart the numerical process with the PDE from the interpolated price as final condition from $t_d$ to $t = 0$. 
Discretization

$$\frac{\partial u}{\partial t} + \frac{1}{2} \sigma c S^2 \frac{\partial^2 u}{\partial S^2} + r S \frac{\partial u}{\partial S} - ru = 0$$

- Grid in space and time with $N$ and $M$ points; mesh width $h = 1/N$, $k = 1/M$
- Finite differences, based on Taylor’s expansion
- $O(h^2 + k^2)$ is easily achieved by central differencing and Crank-Nicolson discretization
- Our aim: High accuracy with only a few grid points

$\Rightarrow$ Grid stretching in space and 4th order discretizations in space and in time
Grid stretching

- A coordinate transformation that clusters points in the region of interest.
- Boundary at infinity: truncate the domain at a safe place (option value is not influenced) according to a well-known formula.
- An equidistant grid discretization can be used after the analytic transformation.
- Consider a general parabolic PDE with non-constant coefficients:

\[ \frac{\partial v}{\partial t} = \alpha(s) \frac{\partial^2 v}{\partial s^2} + \beta(s) \frac{\partial v}{\partial s} + \gamma(s)v(s, t) \]

\[ v(a, t) = L(t), \quad v(b, t) = R(t), \quad v(s, 0) = \phi(s). \]
Grid stretching

- Consider a coordinate transformation $y = \psi(s)$ (one-to-one), inverse $s = \varphi(y) = \psi^{-1}(y)$ and let $\hat{v}(y, t) := v(s, t)$.

- Chain rule, the first and second derivative:

$$\frac{\partial v}{\partial s} = \frac{1}{\varphi'(y)} \frac{\partial \hat{v}}{\partial y},$$  \hspace{1cm} (1)
$$\frac{\partial^2 v}{\partial s^2} = \frac{1}{(\varphi'(y))^2} \frac{\partial^2 \hat{v}}{\partial y^2} - \frac{\varphi''(y)}{(\varphi'(y))^3} \frac{\partial \hat{v}}{\partial y}. \hspace{1cm} (2)$$

Application changes the factors $\alpha$, $\beta$ and $\gamma$ into:

$$\hat{\alpha}(y) = \frac{\alpha(\varphi(y))}{(\varphi'(y))^2}, \quad \hat{\beta}(y) = \frac{\beta(\varphi(y))}{\varphi'(y)} - \alpha(\varphi(y)) \frac{\varphi''(y)}{(\varphi'(y))^3}, \quad \hat{\gamma}(y) = \gamma(\varphi(y)). \hspace{1cm} (3)$$
Grid stretching

- Spatial transformation used for Black-Scholes [Clarke-Parrott, Tavella-Randall]:
  \[ y = \psi(s) = \sinh^{-1}(\mu (s - K)) + \sinh^{-1}(\mu K). \]  
  (4)

- The grid is refined around \( s = K \), i.e. the nondifferentiability in the final condition.
- Parameter \( \mu \) determines the rate of stretching; keep \( \mu K \) constant
- Stretching is possible at several places: grid is defined numerically
Discretization

• Fourth order in space (long stencils):

\[
\frac{\partial \hat{v}_i}{\partial t} = \frac{1}{12h^2} \hat{\alpha}_i (-\hat{v}_{i+2} + 16\hat{v}_{i+1} - 30\hat{v}_i + 16\hat{v}_{i-1} - \hat{v}_{i-2}) + \\
+ \frac{1}{12h} \hat{\beta}_i (-\hat{v}_{i+2} + 8\hat{v}_{i+1} - 8\hat{v}_{i-1} + \hat{v}_{i-2}) + \hat{\gamma}_i \hat{v}_i + O(h^4), \quad 2 \leq i \leq N - 2. 
\]

(5)

• Fourth order in time: BDF4 scheme (preceded by CN, BDF3). BDF4 reads

\[
\left( \frac{25}{12} I - kL \right) u^{j+1} = 4u^j - 3u^{j-1} + \frac{4}{3}u^{j-2} + \frac{1}{4}u^{j-3},
\]

(6)

• No stability complications observed

• Well-suited for linear complementarity problems (for American options)
Accuracy

European option pricing experiment, no dividend

- Error in $u_h$ and hedge parameters $\Delta_h, \Gamma_h$
- $K = 15, \ s_0 = K, \ \sigma_c = 0.3, \ r = 0.05, \ D = 0.03, \ T = 0.5.$

<table>
<thead>
<tr>
<th>Scheme</th>
<th>Grid</th>
<th>$|u - u_{ex}|_\infty$</th>
<th>$c_\infty$</th>
<th>$|\Delta - \Delta_{ex}|_\infty$</th>
<th>$c_\infty$</th>
<th>$|\Gamma - \Gamma_{ex}|_\infty$</th>
<th>$c_\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O(h^4 + k^4)$</td>
<td>10 × 10</td>
<td>$1.1 \times 10^{-2}$</td>
<td>2.4 × 10^{-2}</td>
<td>6.3 × 10^{-3}</td>
<td>1.3 × 10^{-3}</td>
<td>4.8</td>
<td></td>
</tr>
<tr>
<td></td>
<td>20 × 20</td>
<td>$1.1 \times 10^{-3}$</td>
<td>3.1 × 10^{-3}</td>
<td>7.6</td>
<td>1.3 × 10^{-3}</td>
<td>10.8</td>
<td></td>
</tr>
<tr>
<td></td>
<td>40 × 40</td>
<td>$9.4 \times 10^{-5}$</td>
<td>2.9 × 10^{-4}</td>
<td>9.7</td>
<td>9.7 × 10^{-5}</td>
<td>13.6</td>
<td></td>
</tr>
</tbody>
</table>

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<tr>
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<th>$c_\infty$</th>
<th>$|\Delta - \Delta_{ex}|_\infty$</th>
<th>$c_\infty$</th>
<th>$|\Gamma - \Gamma_{ex}|_\infty$</th>
<th>$c_\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu = 12$</td>
<td>10 × 10</td>
<td>$2.7 \times 10^{-1}$</td>
<td>1.7 × 10^{-1}</td>
<td>4.2</td>
<td>4.2 × 10^{-2}</td>
<td>9.9</td>
<td></td>
</tr>
<tr>
<td>stretching</td>
<td>20 × 20</td>
<td>$1.5 \times 10^{-2}$</td>
<td>1.5 × 10^{-2}</td>
<td>4.2</td>
<td>4.2 × 10^{-3}</td>
<td>8.9</td>
<td></td>
</tr>
<tr>
<td>$O(h^4 + k^4)$</td>
<td>40 × 40</td>
<td>$9.1 \times 10^{-4}$</td>
<td>1.7 × 10^{-3}</td>
<td>5.3</td>
<td>9.9 × 10^{-4}</td>
<td>8.0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>80 × 80</td>
<td>$5.7 \times 10^{-5}$</td>
<td>1.5 × 10^{-4}</td>
<td>4.2</td>
<td>4.2 × 10^{-5}</td>
<td>12.7</td>
<td></td>
</tr>
</tbody>
</table>
Accuracy

European option pricing experiment
Example European option

Multiple discrete dividends

- Multiple discrete dividends: analytic solution not available
- Parameters: $s_0 = K = 100$, $r = 0.06$, $\sigma_c = 0.25$, multiple dividends of 4 (ex-dividend date is each half year), $T = 1, 2, 3, 4, 5, 6$. Grid: $s_{max} = RK(3 \leq R \leq 7)$, $\mu = 0.15$

<table>
<thead>
<tr>
<th>Grid</th>
<th>$T = 1$</th>
<th>Grid</th>
<th>$T = 2$</th>
<th>Grid</th>
<th>$T = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$20 \times 20$</td>
<td>10.660</td>
<td>$20 \times 40$</td>
<td>15.202</td>
<td>$20 \times 80$</td>
<td>18.607</td>
</tr>
<tr>
<td>$40 \times 40$</td>
<td>10.661</td>
<td>$40 \times 80$</td>
<td>15.201</td>
<td>$40 \times 160$</td>
<td>18.600</td>
</tr>
<tr>
<td>Lewis (Wilmott Mag. 2003)</td>
<td>10.661</td>
<td>15.199</td>
<td>18.598</td>
<td></td>
<td></td>
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</table>

<table>
<thead>
<tr>
<th>Grid</th>
<th>$T = 4$</th>
<th>Grid</th>
<th>$T = 5$</th>
<th>Grid</th>
<th>$T = 6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$20 \times 80$</td>
<td>21.370</td>
<td>$20 \times 100$</td>
<td>23.697</td>
<td>$20 \times 120$</td>
<td>25.710</td>
</tr>
<tr>
<td>$40 \times 160$</td>
<td>21.362</td>
<td>$40 \times 200$</td>
<td>23.691</td>
<td>$40 \times 240$</td>
<td>25.698</td>
</tr>
<tr>
<td>Lewis</td>
<td>21.364</td>
<td>23.697</td>
<td>25.710</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Example European option

Zero interest rate

- Case $r = 0$, the ex-dividend date $t_d$ should not matter for the option price.
- Black-Scholes does not satisfy this market principle.
- Correction of volatility in Black-Scholes:

$$dS = \mu Sdt + (S - De^{-rt_d})dW \quad t \in [0, t_d]$$

$$\sigma(S, t, D) = \begin{cases} 
\sigma_c \frac{S - D \exp(-rt_d)}{S} & t \in [0, t_d] \\
\sigma_c & t \in [t_d, T]
\end{cases}$$

<table>
<thead>
<tr>
<th></th>
<th>$t_d = 0$</th>
<th>$t_d = 3$ months</th>
<th>$t_d = 6$ months</th>
<th>$t_d = 9$ months</th>
<th>$t_d = 12$ months</th>
</tr>
</thead>
<tbody>
<tr>
<td>Black-Scholes</td>
<td>8.3386</td>
<td>8.5522</td>
<td>8.7590</td>
<td>8.9587</td>
<td>9.1511</td>
</tr>
<tr>
<td>Vol. correction</td>
<td>8.3386</td>
<td>8.3386</td>
<td>8.3386</td>
<td>8.3386</td>
<td>8.3386</td>
</tr>
</tbody>
</table>

European Call, $K = 100$, $D = 7$, $r = 0$, $T = 1$, $\sigma = 0.3$
American Options

Linear Complementarity

- American options are contracts that may be exercised early. This right to exercise is valuable: The American option cannot be worth less than the equivalent European.

- The problem we need to solve for an American call option contract reads:

\[
Au := \frac{\partial u}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 u}{\partial S^2} + r S \frac{\partial u}{\partial S} - ru \leq 0
\]

\[
u(S, T) = \max (S - K, 0), \quad u(S, t_d) = \max \{S - Ke^{r(T-t_d)} - D, S - K\},
\]

\[
u(S, t) \geq \text{final condition}
\]

\[
\frac{\partial u}{\partial S} \quad \text{continuous}
\]

\[
u(S_{max}, t) = \max \{S_{max} - Ke^{r(T-t)} - De^{r(t_d-t)}, S_{max} - Ke^{r(t_d-t)}\}, \quad t < t_d
\]

- Early exercise valuable only if \( D > K(1 - e^{-r(T-t_d)}) \), just before the asset goes ex-dividend [Kwok].

- Reformulation of the obstacle problem into a linear complementarity problem:
American Put with one Discrete Dividend

- $K = 100, T = 0.5, d = 2.0, t_d = 0.3, \sigma_c = 0.4, r = 0.08, \mu = 0.15, s_{\text{max}} = 3K$

<table>
<thead>
<tr>
<th>Grid</th>
<th>$u_h(80, t = 0)$</th>
<th>$u_h(100, t = 0)$</th>
<th>$u_h(120, t = 0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20 x 20</td>
<td>0.223</td>
<td>0.105</td>
<td>0.043</td>
</tr>
<tr>
<td>40 x 40</td>
<td>0.223</td>
<td>0.105</td>
<td>0.043</td>
</tr>
<tr>
<td>Meyer (J. C. Fin. 2001):</td>
<td>0.223</td>
<td>0.105</td>
<td>0.043</td>
</tr>
</tbody>
</table>

- $d = 0$ (black line), $d = 2$ (blue line) vs. $d = 0.98S$ (red line)
American Call with one Dividends

- Problem parameters: \( K = 100, \sigma_c = 0.3, r = 0.05, t_d = 51 \text{ weeks}, D_1 = 4, T = 50 \text{ weeks versus } T=1 \text{ year} \)

<table>
<thead>
<tr>
<th>( t_d = 51 \text{ weeks, } T=50 \text{ weeks} )</th>
<th>Vorst</th>
<th>Haug</th>
<th>Black Scholes</th>
<th>Vol. correction</th>
<th>Eur. corr.</th>
</tr>
</thead>
<tbody>
<tr>
<td>13.63</td>
<td>13.64</td>
<td>14.08</td>
<td>13.65</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- American price is lower than European
- One should maybe (but this will not happen in practice !) adapt the European price to avoid this contradiction with the volatility correction
Conclusions

- Accurate option values with grid stretching in space and 4th order discretization in space and time
- Option price and hedge parameters are accurate with 20 - 40 points
- Multiple discrete dividend payment can be included in a straightforward way
- American style early exercise does not pose any problems
⇒ Discrete dividends lead to interesting modeling issues.