Options

Definition (Option)

Is a contract written by a seller, that gives the right (but not the obligation) to the holder to trade in the future the underlying asset at a previously agreed price.

Option styles:

- European option- an option that may be only exercised on expiration;
- American option - an option that may be exercised on any trading day (also on the expiration);
- Bermudan option - an option that may be exercises only on specified dates;
- Barrier option - option which is exercised, for example, only if security's price reaches some trigger level;

Options cont.

Most popular options are Call and Put options: At a prescribed time in the future, (maturity: T):

- Call Option: The holder of the option may purchase a prescribed asset (shares, stocks : S) for a prescribed amount (strike: K) and the writer of the contract must sell the asset, if the holder decides to buy it.
- Put Option: The holder of the option may sell a prescribed asset (shares, stocks : S) for a prescribed amount (strike: K) and the writer of the contract must buy the asset, if the holder decides to sell it.

Options & Payoffs

The value of European call option at the expiry T is given by:

$$C(T, S_T) = \max\left(S_T - K, 0\right).$$

The value of European put option at the expiry T is given by:

$$P(T, S_T) = \max\left(K - S_T, 0\right).$$

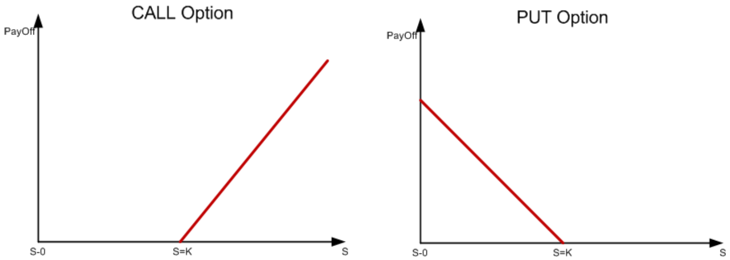


Figure: Payoff diagram for European Call (left), and European Put (right).

Terminology

During this course we will use the following terminology:

- Premium: The amount paid for the contract initially;
- Underlying (Asset): The financial instrument on which the option value depends. Stocks, commodities, currencies are all denoted by S_t;
- Strike (price), Exercise price: The amount for which the underlying can be bought (call) or sold (put), denoted by K;
- Maturity or Expiry date: Date on which the option can be exercised. This will be denoted by T;
- Intrinsic Value: The payoff that would be received if the underlying is at its current level when the option expires.

Terminology cont.

The option can be:

- In the money: An option with a positive intrinsic value. A call option when the asset price is above strike; a put option when the asset price is below the strike;
- Out of the money: An option with no intrinsic value;
- At the money: A call or a put that is close to the current asset level.

Owning/Selling Options:

- Long position: A positive amount of a quantity, or a positive exposure to a quantity. The holder, holding the option in the future, takes a long position by buying the derivative;
- Short position: A negative amount of a quantity, or a negative exposure to a quantity. The writer sells the option $\rightarrow$ He goes "short" in the option.

The value of an option

What determines the value of an option?

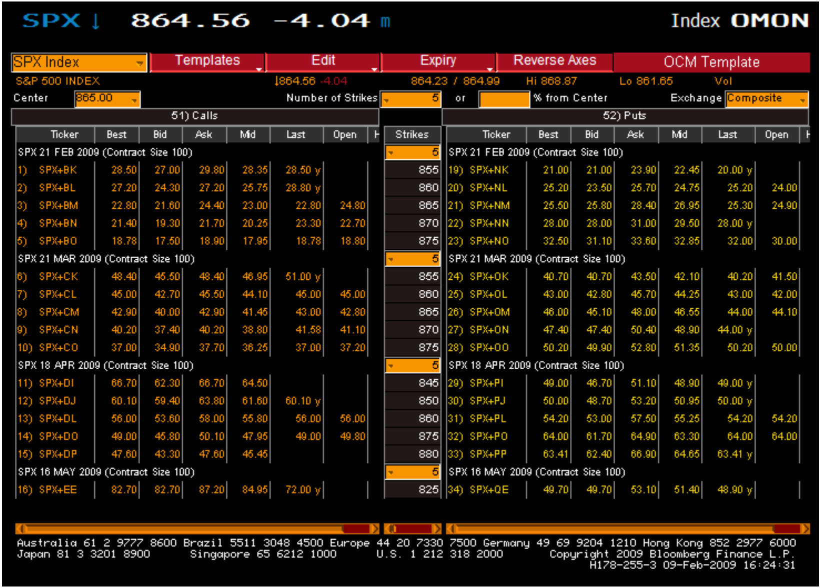
- what is the asset price today S_t?
- how long there is until expiry $T - t$?
- how volatile is the asset S_t?

General principles:

- The longer the time to expiry, the more time there is for the asset to rise or fall;
- The more the asset is volatile the higher the chance that it will rise or fall;

How are options traded?

The value of an option (Bloomberg L.P. 09.02.09)



SPX ↓ 864.56 -4.04 m Index OMON

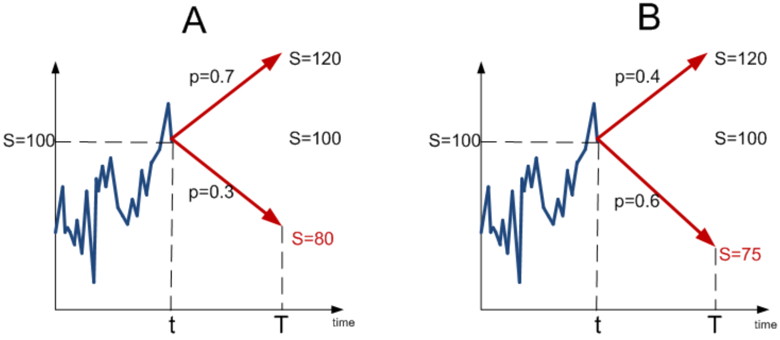
SPX Index | Templates | Edit | Expiry | Reverse Axes | OCM Template

S&P 500 INDEX ↓864.56 -4.04 864.23 / 864.99 Hi 868.87 Lo 861.65 Vol

Center 865.00 Number of Strikes 5 or % from Center Exchange Composite

51) Calls								52) Puts						
Ticker	Best	Bid	Ask	Mid	Last	Open	Strikes	Ticker	Best	Bid	Ask	Mid	Last	Open
SPX 21 FEB 2009 (Contract Size 100)							5	SPX 21 FEB 2009 (Contract Size 100)						
1) SPX+BK	28.50	27.00	29.80	28.35	28.50 y		855	19) SPX+NK	21.00	21.00	23.90	22.45	20.00 y	
2) SPX+BL	27.20	24.30	27.20	25.75	28.80 y		860	20) SPX+NL	25.20	23.50	25.70	24.75	25.20	24.00
3) SPX+BM	22.80	21.60	24.40	23.00	22.80	24.80	865	21) SPX+NM	25.50	25.80	28.40	26.95	25.30	24.90
4) SPX+BN	21.40	19.30	21.70	20.25	23.30	22.70	870	22) SPX+NN	28.00	28.00	31.00	29.50	28.00 y	
5) SPX+BO	18.78	17.50	18.90	17.95	18.78	18.80	875	23) SPX+NO	32.50	31.10	33.60	32.85	32.00	30.00
SPX 21 MAR 2009 (Contract Size 100)							5	SPX 21 MAR 2009 (Contract Size 100)						
6) SPX+CK	48.40	45.50	48.40	46.95	51.00 y		855	24) SPX+OK	40.70	40.70	43.50	42.10	40.20	41.50
7) SPX+CL	45.00	42.70	45.50	44.10	45.00	45.00	860	25) SPX+OL	43.00	42.80	45.70	44.25	43.00	42.00
8) SPX+CM	42.90	40.00	42.90	41.45	43.00	42.80	865	26) SPX+OM	46.00	45.10	48.00	46.55	44.00	44.10
9) SPX+CN	40.20	37.40	40.20	38.80	41.58	41.10	870	27) SPX+ON	47.40	47.40	50.40	48.90	44.00 y	
10) SPX+CO	37.00	34.90	37.70	36.25	37.00	37.20	875	28) SPX+OO	50.20	49.90	52.80	51.35	50.20	50.00
SPX 18 APR 2009 (Contract Size 100)							5	SPX 18 APR 2009 (Contract Size 100)						
11) SPX+DI	66.70	62.30	66.70	64.50			845	29) SPX+PI	49.00	46.70	51.10	48.90	49.00 y	
12) SPX+DJ	60.10	59.40	63.80	61.60	60.10 y		850	30) SPX+PJ	50.00	48.70	53.20	50.95	50.00 y	
13) SPX+DL	56.00	53.60	58.00	55.80	56.00	56.00	860	31) SPX+PL	54.20	53.00	57.50	55.25	54.20	54.20
14) SPX+DO	49.00	45.80	50.10	47.95	49.00	49.80	875	32) SPX+PO	64.00	61.70	64.90	63.30	64.00	64.00
15) SPX+DP	47.60	43.30	47.60	45.45			880	33) SPX+PP	63.41	62.40	66.90	64.65	63.41 y	
SPX 16 MAY 2009 (Contract Size 100)							5	SPX 16 MAY 2009 (Contract Size 100)						
16) SPX+EE	82.70	82.70	87.20	84.95	72.00 y		825	34) SPX+QE	49.70	49.70	53.10	51.40	48.90 y	

The value of an option Example:



For $K = 100$ which call option is more expensive $C_A < C_B$, $C_A > C_B$, $C_A = C_B$?

To find the answer we follow the reasoning of an option writer...

The value of an option Example cont.:

Writers, in order to hedge their short call option position buy some (Δ) stocks, so their position equals:

$$\Pi(t) = C_t - \Delta \cdot S_t,$$

Now, we consider two scenarios:

- Stock goes UP:

$$\begin{aligned} \Pi_{up} &= C_t - \Delta S_t + \Delta S_{up} - S_{up} + K && (1) \\ &= C_t - \Delta S_t + \Delta S_{up} - 20 && (2) \end{aligned}$$

- Stock goes DOWN:

$$\Pi_d = C_t - \Delta S_t + \Delta S_d; \qquad (3)$$

Since writers don't want to take any risk:

$$\left\{ \begin{array}{rcl} C_t - \Delta S_t + \Delta S_{up} - 20 &=& 0 \\ C_t - \Delta S_t + \Delta S_d &=& 0; \end{array} \right. \qquad (4)$$

The value of an option Example cont.:

Simple algebra gives:

$$\Delta = \frac{20}{S_{up} - S_d}.$$

So now we have:

- Case A:

$$\Delta = \frac{1}{2},$$

  which gives $C_0 = 10$

- Case B:

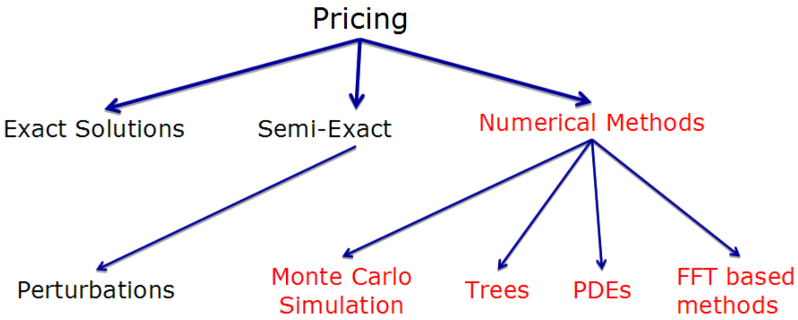
$$\Delta = \frac{4}{9},$$

  which gives $C_0 = 11.1$.

How does a writer make money???

Pricing Techniques

There are several different approaches for obtaining information on the prices of options:



Put-Call Parity

We seek for a relation between value $V(t, S_t)$ of European call and put options with the same strike price K and expiry T. Suppose we have given two portfolios:

- Π_A : one call option plus Ke^{-rT} cash (invested in the bank);
- Π_B : one put option plus one unit of the asset.

At expiry portfolio Π_A is worth:

$$\max(S_T - K, 0) + K = \max(S_T, K),$$

The portfolio Π_B is worth on expiry:

$$\max(K - S_T, 0) + S_T = \max(K, S_T).$$

So, we conclude:

$$C(t, S_t) + Ke^{-r(T-t)} = P(t, S_t) + S_t,$$

This relationship is the so-called Put-Call Parity.
Note that we did not make any assumptions about stock S_t!

Upper-Lower bounds on option values

Similar arguments can be used to obtain simple upper/lower bounds on the values of call & put options. Suppose we have given two portfolios:

- Π_A : one call option plus Ke^{-rT} cash (invested in the bank);
- Π_B : one unit of the asset S_t.

We see that portfolio Π_B has a value S_T, which is never greater than the payoff Π_A. So we can write:

$$C \geq S_T - Ke^{-r(T-t)},$$

since the call option cannot have a negative value, we may write:

$$C \geq \max\left(S_T - Ke^{-r(T-t)}, 0\right),$$

on the other hand, the call option can never be worth more than the underlying asset, so:

$$C \leq S.$$

Using the same arguments for the put option we have:

$$P \geq \max\left(Ke^{-r(T-t)} - S, 0\right) \text{ and } P \leq Ke^{-r(T-t)}.$$

Arbitrage

- There is never an opportunity to make a risk-free profit that gives a greater return than that provided by a bank.

Example: Suppose we have given a portfolio:

$$\Pi = P(t, S_t) - C(t, S_t) + S_t,$$

with $P(t, S_t)$ and $C(t, S_t)$ having expiry date T,and strike K.

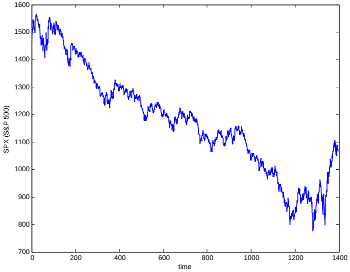
Question: What is the "fair" price of this portfolio ?

Modeling Asset Prices

When investing, the main concern is that the return on the investment is satisfactory. Suppose we have given asset S_t, then

$$\text{Return} = \frac{\text{Stock tomorrow} - \text{Stock today}}{\text{Stock today}} = \frac{S_{t+\delta_t} - S_t}{S_t}$$

Lets see it in practice! We take S&P index from 30-04-2002 to 09-11-2007 daily monitored.



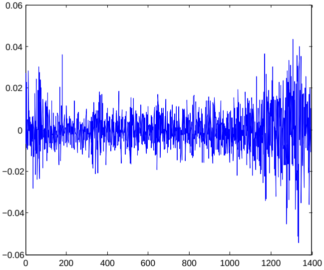
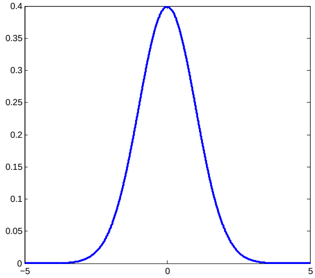
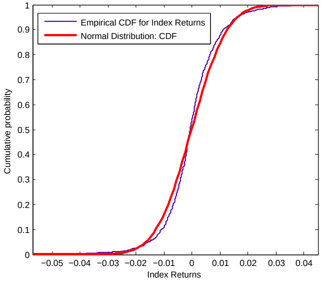


Figure: Left: S&P Index, Right: Index Return

Modeling Asset Prices

- From the data in this example we find that the mean is -0.000172 and the standard deviation is 0.0121;
- The distribution of the returns has been scaled and translated to give it a mean of zero, a standard deviation of one and an area under the curve of one.





Randomness of the stock prices

Daily returns for a certain asset look like noise!

- What can be then done?
- We can model the noise!

Wiener Process

Definition (Wiener Process)

A stochastic process W_t for $t \in [0, \infty)$ is called a Wiener Process (or Brownian motion) if the following conditions are satisfied:

- *It starts at zero: $W_0 = 0$,*
- *It has stationary, independent increments,*
- *For every $t > 0$, W_t has a normal distribution with mean 0 and variance t,*
- *It has a.s. continuous paths with NO JUMPS.*

Process trajectories

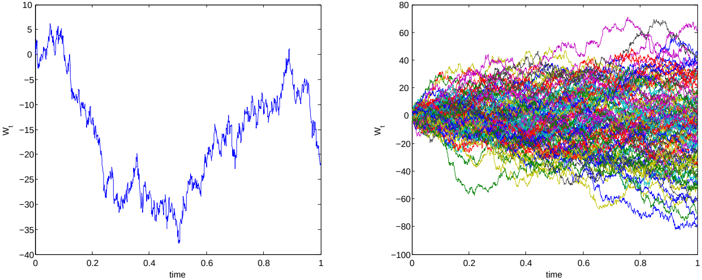


Figure: Sample paths of Brownian motion on $[0, 1]$. Left: 1 path, Right: 100 paths.

Stochastic Processes

Suppose we observe the stock price of Company Y at every fixed instance t since some initial time t_0 and known T.

- We can interpret the observed stock values as a realization $X_t(\omega)$ of the random variable X_t.
- We need a model which takes into account almost continuous realizations of the stock prices.

Definition (Stochastic Process)

A stochastic process X_t is a collection of random variables

$$(X_t, t \in T) = (X_t(\omega), t \in T, \omega \in \Omega)$$

We note that a stochastic process X_t is a function of two variables:

- for a fixed instant t it's a variable:

$$X_t = X_t(\omega) \text{ for } \omega \in \Omega,$$

- for a fixed random outcome $\omega \in \Omega$, it's a function of time

$$X_t = X_t(\omega), \text{ for } t \in T.$$

Modeling the returns:

The most popular Stochastic Process for generating prices is the Geometric Brownian Motion process (GBM):

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

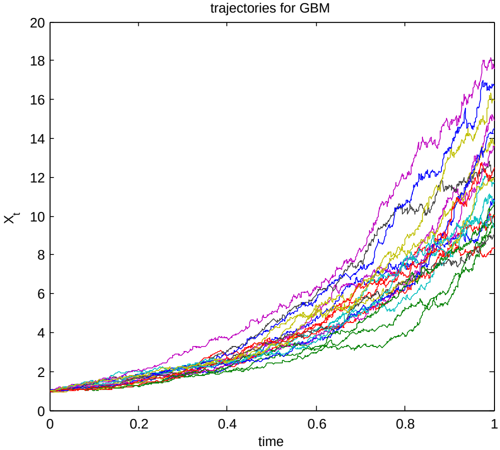
which can be translated to:

$$\frac{S_{t+\delta_t} - S_t}{S_t} = \mu\delta_t + \sigma(W_{t+\delta_t} - W_t),$$

where:

- μdt is a deterministic return,
- σdW_t is the random change with dW_t a sample from a normal distribution with mean 0 and variance δt.

Modeling the returns: Cont



Types of processes

We compare the behaviour of the following SDEs:

- Arithmetic Brownian motion (ABM)

$$dX_t = \mu dt + \sigma dW_t,$$

- Geometric Brownian Motion (GBM)

$$dX_t = \mu X_t dt + \sigma X_t dW_t,$$

- Ornstein-Uhlenbeck mean reverting processes (OU)

$$dX_t = \kappa(\theta - X_t)dt + \sigma dW_t,$$

  where: μ, σ, κ and θ are known constants, and W_t is a Wiener process.

We also cover multidimensional extensions. What is the difference between these processes?

Process trajectories

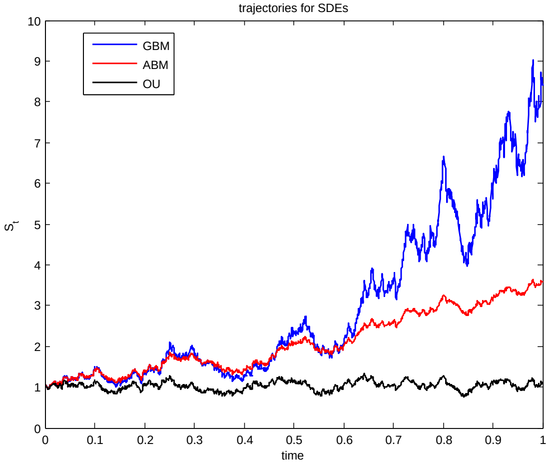


Figure: Process trajectories generated from the same random path. The parameters are: $\mu = 0.05$, $\sigma = 0.7$, $\kappa = 1.5$ and $\theta = 1$.

ABM: example

Consider the ABM process:

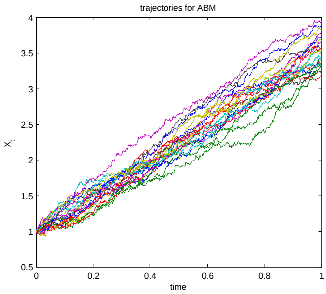
$$dS_t = \mu dt + \sigma dW_t,$$

which is a Gaussian process with expectation and covariance functions:

$$\mathbb{E}(S_t) = \mu t,$$

and covariance function

$$\text{cov}(S_t, S_s) = \sigma^2 \min(t, s), \text{ with } s, t > 0.$$



Ito's Lemma

Ito's Lemma is most fundamental for stochastic processes. It helps in deriving solutions to stochastic differential equations (SDE).

Lemma (Ito's Lemma)

Suppose X_t follows an Ito process:

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t, \text{ with } X_0 = x_0,$$

is the short form for:

$$X_t = X_0 + \int_{t_0}^{t} a(s, X_s)ds + \int_{s_0}^{t} b(s, X_s)dW_s,$$

with $a(t, X_t)$ the drift term and $b(t, X_t)$ the diffusion term. Let now $g(t, X_t)$ be a function with continuous derivatives (up to second order). Then $Y_t := g(t, X_t)$ follows an Ito process with the same Wiener process W_t:

$$dY_t = \left(\frac{\partial g}{\partial x} a + \frac{\partial g}{\partial t} + \frac{1}{2} \frac{\partial^2 g}{\partial x^2} b^2 \right) dt + \frac{\partial g}{\partial x} b dW_t.$$

Ito's Lemma: Example

Example: Suppose we want find the dynamics for

$$g(t, S_t) = \log S_t,$$

where S_t is given by:

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

By the Ito formula we have:

$$dg(t,s) \quad = \quad \frac{\partial g(t,S_t)}{\partial t}dt + \frac{\partial g(t,S_t)}{\partial S_t}dS_t + \frac{1}{2}\frac{\partial^2 g(t,S_t)}{\partial S_t^2}(dS_t)^2, \quad (5)$$

we have:

$$\frac{\partial g(t,S_t)}{\partial t} = 0,\ \frac{\partial g(t,S_t)}{\partial S_t} = \frac{1}{S_t} \text{ and } \frac{\partial^2 g(t,S_t)}{\partial S_t^2} = -\frac{1}{S_t^2},$$

so:

$$dg(t,s) \quad = \quad 0 \cdot dt + \frac{1}{S_t}dS_t - \frac{1}{2}\frac{1}{S_t^2}(dS_t)^2, \quad (6)$$

Ito's Lemma: Example cont.

Since

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

$$\begin{aligned} dg(t,s) &= \frac{1}{S_t} dS_t - \frac{1}{2}\frac{1}{S_t^2}(dS_t)^2 \\ &= \frac{1}{S_t}\left(\mu S_t dt + \sigma S_t dW_t\right) - \frac{1}{2}\frac{1}{S_t^2}\left(\mu S_t dt + \sigma S_t dW_t\right)^2 \\ &= \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t. \end{aligned}$$

We have found that

$$dg(t, S_t) = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t.$$

Black-Scholes model (1973)

We start with the assumption about the price stochastic process:

$$\frac{\mathrm{d}S_t}{S_t} = \mu \mathrm{d}t + \sigma \mathrm{d}W_t^{\mathbb{P}},$$

and Bank account:

$$\frac{\mathrm{d}B_t}{B_t} = r\mathrm{d}t,$$

where $\mathbb{P}$ represents real-world measure.

- We define $V(t, S_t)$ which represents the value of the option at time time t.
- Further we consider a trading strategy under which one holds one option and continuously trades in the stock in order to hold some Δ shares.

We see that at time t, the value of the holdings will be:

$$\Pi(t, S_t) = V(t, S_t) - \Delta S_t.$$

Black-Scholes cont.

Since S_t is stochastic, our portfolio is as well, the dynamics of our portfolio Π_t we get from Ito's lemma:

$$\mathrm{d}\Pi = \mathrm{d}V - \Delta \mathrm{d}S. \tag{7}$$

We need to find $\mathrm{d}V$ so once more time by applying Ito's we have:

$$\begin{aligned}
\mathrm{d}V &= V_S \mathrm{d}S + V_t \mathrm{d}t + \frac{1}{2} F_{S,S} (\mathrm{d}S)^2 + \frac{1}{2} F_{t,t} (\mathrm{d}t)^2 + F_{S,t} \mathrm{d}t \mathrm{d}S \\
\mathrm{d}V &= \frac{\partial V}{\partial S} \mathrm{d}S + \frac{\partial V}{\partial t} \mathrm{d}t + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \mathrm{d}S^2 \\
\mathrm{d}V &= \frac{\partial V}{\partial S} \mathrm{d}S + \frac{\partial V}{\partial t} \mathrm{d}t + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \left(\sigma^2 S_t^2 \mathrm{d}t \right) \\
\mathrm{d}V &= \left(\mu S_t \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} \right) \mathrm{d}t + \sigma S_t \frac{\partial V}{\partial S} \mathrm{d}W_t
\end{aligned}$$

so now by using Equation (7) we have

Black-Scholes cont.

$$\mathrm{d}\Pi \quad = \quad \mathrm{d}V - \Delta \mathrm{d}S,$$

$$\mathrm{d}\Pi = \left(\mu S_t \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2}\right) \mathrm{d}t + \sigma S_t \frac{\partial V}{\partial S} \mathrm{d}W_t - \Delta \left(\mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t\right).$$

Since we want the risk to be hedged (no noise involved), and to grow with risk free rate, we firstly have

$$\Delta = \frac{\partial V}{\partial S},$$

and further:

$$r\Pi dt \quad = \quad \left(\mu S_t \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2}\right) dt - \frac{\partial V}{\partial S} \left(\mu S_t dt\right).$$

Black-Scholes cont.

So finally:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} - r\Pi \quad = \quad 0.$$

Moreover we also know that:

$$\Pi = V - \frac{\partial V}{\partial S} S,$$

so:

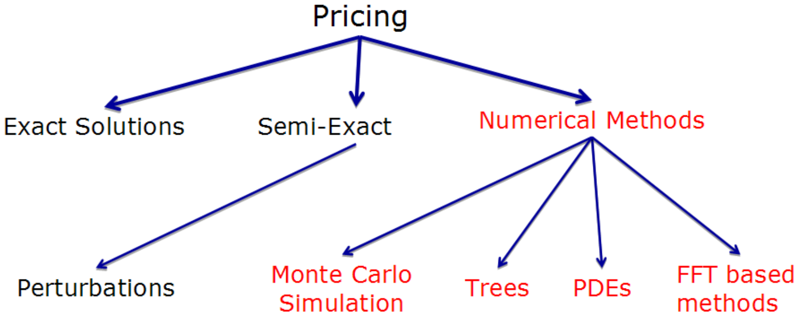
$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} - r\left(V - \frac{\partial V}{\partial S}\right) \quad = \quad 0.$$

And finally:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

Alternative Approaches to Option Pricing

There are several different approaches for obtaining information on the prices of options:



Feynman-Kac formula

Feynman-Kac have established a link between partial differential equations (PDEs) and stochastic processes. It offers a method for solving certain PDEs by simulating random paths of a stochastic process. Suppose we are given the PDE:

$$\frac{\partial V}{\partial t} + \mu(x,t)\frac{\partial V}{\partial x} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 V}{\partial x^2} = 0,$$

subject to the boundary condition $V(x,T) = \eta(x)$, then the Feynman-Kac formula reads:

$$V(x,t) = \mathbb{E}\left(\eta(X_T)|\mathcal{F}_t\right)$$

where: X is an Ito process driven by the equation:

$$\mathrm{d}X = \mu(X,t)\mathrm{d}t + \sigma(X,t)\mathrm{d}W(t),$$

with $W(t)$ is a Wiener process, with initial for $X(t)$, $X(0) = x$.

proof of the Feynman-Kac formula.

We know the PDE for $V(x,t)$, so the Ito's dynamics for V are given by:

$$
\begin{aligned}
\mathrm{d}V &= \frac{\partial V}{\partial x}\mathrm{d}x + \frac{\partial V}{\partial t}\mathrm{d}t + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(\mathrm{d}x)^2 \\
\mathrm{d}V &= \left(\frac{\partial V}{\partial t} + \mu(x,t)\frac{\partial V}{\partial x} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 V}{\partial x^2}\right)\mathrm{d}t + \sigma(x,t)\frac{\partial V}{\partial x}\mathrm{d}W(t) \\
\mathrm{d}V &= \sigma(x,t)\frac{\partial V}{\partial x}\mathrm{d}W(t)
\end{aligned}
$$

Integrating both sides one gets

$$
\int_t^T dV = V(X_T,T) - V(x,t) = \int_t^T \sigma(x,t)\frac{\partial V}{\partial x}dW(t)
$$

now by taking expectation we have

$$
V(x,t) = \mathbb{E}\left(V(X_T,T)\right) = \mathbb{E}\left(\eta(X_T)\right)
$$

□

Feynman-Kac in practice

Exercise: Solve the following PDE

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial x^2} &= 0 \\ V(x,T) &= x^2 \end{aligned}$$

where σ is a constant.
Answer: From Feynman-Kac we have:

$$V(x,t) = \mathbb{E}\left(\eta(X_t)|\mathcal{F}_t\right) = \mathbb{E}\left(X_T^2|\mathcal{F}_t\right)$$

where:

$$\begin{aligned} dX_s &= 0 \cdot dt + \sigma dW_s \\ X_t &= x \end{aligned}$$

So we have: $X_T = x + \sigma\left(W_T - W_t\right)$, and X_T has the distribution $N(x, \sigma\sqrt{T-t})$. Finally $V(t,x) = \sigma^2(T-t) + x^2$

What about the drift? Which one to choose?

Risk Neutral probability and Option Pricing

- There are two major approaches to pricing an option. The first one is the PDE approach, the second one is the risk-neutral probability approach.
- The basic idea of the risk-neutral probability approach is the change the probability measure from the true (statistical) probability to risk-neutral probability.
- The major difference of the two measures is the expected return of the stock:
  - In the true probability measure, the expected return is μ
  - In the risk-neutral probability measure, the expected stock return is risk-free rate r

Which one to choose μ or r ?

Definition of Risk-neutral measure

How Risk-neutral probability is used in asset pricing theory.

- A risk neutral probability is the probability of an future event or state that both trading parties in the market agree upon.
  A simple example:
  - For a future event, two parties A and B enter into a contract, in which A pays B 1€ if it happens and 0€ if it doesn't.
  - For such an agreement, there is a price for B to pay A. If they agree that B pays 0.4€ to A, this means the two parties think that the probability of the event that happens is 40%. Otherwise, they won't reach that agreement and sign a contract.
  - This price reflects the common beliefs towards the probability that the event happens. 40% is the risk neutral probability of the event that happens.

Definition of Risk-neutral measure

- It is not any historical statistic or prediction of any kind. It is not the true probability, either.
- One should ask what kind of information is offered from Risk-neutral probability and where can we find this measure in the real world.
- For the simple example mentioned above, once the price is established, the risk-neutral measure is also determined.
- Whenever you have a pricing problem in which the event is measurable under this measure, you have to use this measure to avoid arbitrage. If you don't, it's like you are simply giving out another price for the same event at the same time, which is an obvious arbitrage opportunity.

Martingales

Definition (Martingale)

A stoch. process $X = \{X_t; t \geq 0\}$ *is a martingale wrt the* $\{\mathcal{F}_t\}$ *if*

- X *is adapted to the filtration* $\mathcal{F}_{t\geq 0}$,
- *for all* t, $\mathbb{E}\left(|X_t|\right) < \infty$,
- *for all* s *and* t *with* $s < t$ *we have* $\mathbb{E}\left(X_t|\mathcal{F}_s\right) = X_s$.

Exercise: Show that W_t is a martingale:

- W_s is $\mathcal{F}_s^W$-measurable
- $\mathbb{E}\left(|W_s|\right) = \sqrt{\frac{2s}{\pi}}$
- for any $s \leq t$ we have

$$\mathbb{E}\left(W_t|\mathcal{F}_s^W\right) = \mathbb{E}\left(W_t - W_s + W_s|\mathcal{F}_s^W\right) = W_s$$

Martingales

Exercise: Suppose we have given a process

$$X_t = \exp\left(\alpha W_t - \alpha^2 \frac{t}{2}\right).$$

Simply first two conditions are satisfied, and the final condition reads:

$$\begin{aligned}
\mathbb{E}\left(X_t|\mathcal{F}_s\right) &= \exp\left(-\alpha^2 \frac{t}{2}\right)\mathbb{E}\left(e^{\alpha W_t}|\mathcal{F}_s\right) \\
&= \exp\left(-\alpha^2 \frac{t}{2} + \alpha W_s\right)\mathbb{E}\left(e^{\alpha(W_{t-s})}|\mathcal{F}_s\right) \\
&= \exp\left(-\alpha^2 \frac{t}{2} + \alpha W_s\right)\mathbb{E}\left(e^{\alpha(W_{t-s})}\right).
\end{aligned}$$

Martingales cont.

$$\mathbb{E}\left(X_t|\mathcal{F}_s\right) \quad = \quad \exp\left(-\alpha^2\frac{t}{2}+\alpha W_s\right)\mathbb{E}\left(\exp\left(\alpha\left(W_{t-s}\right)\right)\right)$$

We recall that if Y has a normal distribution with mean μ and variance σ^2 so $\exp(Y)$ has lognormal distribution with expectation

$$\mathbb{E}(X) = \exp\left(\mu+\sigma^2/2\right).$$

Finally we have

$$\begin{aligned}\mathbb{E}\left(X_t|\mathcal{F}_s\right) &= \exp\left(-\alpha^2\frac{t}{2}+\alpha W_s\right)\exp\left(\alpha^2\frac{(t-s)}{2}\right)\\ &= \exp\left(\alpha W_s-\alpha^2\frac{s}{2}\right)=X_s.\end{aligned}$$

So we have shown that process X_t is a martingale with respect to filtration $\mathcal{F}_s$

Fundamental Theorem for Pricing

A stochastic integral process is a martingale

Theorem

Let $g \in \mathcal{L}^2$. For any $t \geq 0$

$$X(t) = \int_0^t g(s)dW(s). \tag{8}$$

Then, the process $X = X(t); t \geq 0$ is a $\mathcal{F}_t^W$-martingale.

Finding a risk-free drift

Suppose we have an asset process given by

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t^{\mathbb{P}},$$

and a Bank account:

$$\frac{dB_t}{B_t} = rdt.$$

We know that we would like to find discounted payoff under riskneutral measure $\mathbb{Q}$, so the process $\frac{S_t}{B_t}$ has to be a martingale, i.e.

$$\mathbb{E}^{\mathbb{Q}}\left(\frac{S_T}{B_T}|\mathcal{F}_t\right) = \frac{S_t}{B_t}.$$

Let us find a dynamics of $\frac{S_t}{B_t}$

Finding a risk-free drift cont.

Dynamics of $F = \frac{S_t}{B_t}$ we find from Ito lemma:

$$\begin{aligned}
\mathrm{d}\left(\frac{S_t}{B_t}\right) &= F_S \mathrm{d}S + F_B \mathrm{d}B + \frac{1}{2}F_{S,S}(\mathrm{d}S)^2 + \frac{1}{2}F_{B,B}(\mathrm{d}B)^2 + F_{S,B}\mathrm{d}B\mathrm{d}S \\
&= \frac{1}{B_t}\mathrm{d}S_t - \frac{S_t}{B_t^2}\mathrm{d}B_t + 0 + 0 + 0 \\
&= \frac{1}{B_t}\left(\mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t\right) - \frac{S_t}{B_t^2} r B_t \mathrm{d}t \\
&= \frac{S_t}{B_t}\left(\mu - r\right)\mathrm{d}t + \frac{S_t}{B_t}\sigma S_t \mathrm{d}W_t
\end{aligned}$$

In order to make the dynamics of $\frac{S_t}{B_t}$ driftless we need to have: $\mu = r$.