With $V(t_M, S(t_M)) = E(t_M, S(t_M))$ we find the option price via backward induction:

\[
\begin{align*}
V(t_M, S(t_M)) &= E(t_M, S(t_M)) \\
C(t_m, S(t_m)) &= e^{-r\Delta t}E_t\{V(t_{m+1}, S(t_{m+1}))\} \\
V(t_m, S(t_m)) &= \max\{C(t_m, S(t_m)), E(t_m, S(t_m))\}, \\
V(t_0, S(t_0)) &= C(t_0, S(t_0)),
\end{align*}
\]

\[m = M - 1, \ldots, 1,\]
Write, in the case of deterministic interest rates, as an integral:

\[ C(t_m, S(t_m)) = e^{-r\Delta t} \int_{-\infty}^{\infty} V(t_{m+1}, y)f(y|S(t_m))dy \]

- O’Sullivan(2005): Generalization to exponential Lévy processes, as the density can be recovered via Fourier inversion.
- With the midpoint rule, the density can be approximated and resolved by the FFT. Overall complexity of \( O(MN^2) \) for M-times exercisable Bermudan options.
The main premise of the CONV method is that $f(y|x)$ depends on $x$ and $y$ via

$$f(y|x) = f(y - x).$$

Assumption is clearly satisfied in exp. Lévy models, where $x$ and $y$ then represent log-asset prices. The assumption means that log-returns are independent.

$$C(t_m, x) = e^{-r\Delta t} \int_{-\infty}^{\infty} V(t_{m+1}, y)f(y|x)dy$$

$$= e^{-r\Delta t} \int_{-\infty}^{\infty} V(t_{m+1}, x + z)f(z)dz.$$  

The key insight is the notion that, apart from the discounting, the equation is a cross-correlation of $V$ with the density function $f$. 

Premultiplying by \( \exp(\alpha x) \) and taking its Fourier transform, gives:

\[
e^{r\Delta t} \mathcal{F}\{e^{\alpha x} C(t_m, x)\} = e^{r\Delta t} \int_{-\infty}^{\infty} e^{iux} e^{\alpha x} C(t_m, x) \, dx \\
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{iux} e^{\alpha x} V(t_{m+1}, x + z) f(z) \, dz \, dx \\
= \int_{-\infty}^{\infty} e^{i(u-i\alpha) y} V(t_{m+1}, y) \, dy \int_{-\infty}^{\infty} e^{-i(u-i\alpha) z} f(z) \, dz \\
= \tilde{V}(t_{m+1}, u - i\alpha) \phi(-(u - i\alpha)).
\]

- A computation for resolving the (conditional) density function is avoided, only the characteristic function \( \phi \) is involved.
- The option price is recovered by the inverse Fourier transform and undamping.
The extended characteristic function

\[ \phi(x + yi) = \int_{-\infty}^{\infty} e^{i(x+yi)z} f(z) dz, \]

is well-defined when \( \phi(yi) < \infty \), as \( |\phi(x + yi)| \leq |\phi(yi)| \).

\[ \Rightarrow \] This puts a restriction on the damping coefficient \( \alpha \), because \( \phi(\alpha i) \) must be finite.

The damping factor is necessary when considering e.g. a Bermudan put, as then \( V(t_{m+1}, x) \) tends to a constant when \( x \to -\infty \), and as such is not \( L^1 \)-integrable.

The difference with the Carr-Madan approach is that we take a transform with respect to the log-spot price instead of the log-strike price.

\[ \Rightarrow \] The idea for GBM is already present in a presentation by Eric Reiner (2000)
The algorithm may now be clear, with $E(t_0, x) = 0$:

- $V(t_M, x) = E(t_M, x)$ for all $x$
- For $m = M - 1$ to 0
  - Dampen $V(t_{m+1}, y)$ and take its Fourier transform
  - Multiply with $\phi(-u + i\alpha)$
  - Apply Fourier inversion and undamp
- $V(t_m, x) = \max\{E(t_m, x), C(t_m, x)\}$
- Next $m$
Expressions for hedge parameters

- The CONV formulae for two hedge parameters, $\Delta$ and $\Gamma$, defined as,

$$
\Delta = \frac{\partial V}{\partial S} = \frac{1}{S} \frac{\partial V}{\partial x}, \quad \Gamma = \frac{\partial^2 V}{\partial S^2} = \frac{1}{S^2} \left( -\frac{\partial V}{\partial x} + \frac{\partial^2 V}{\partial x^2} \right). \quad (1)
$$

- Define, $\mathcal{F}\{e^{\alpha x}V(t_0, x)\} = e^{-r\Delta t}A(u)$, where $A(u) = \mathcal{F}\{e^{\alpha y}V(t_1, y)\} \cdot \phi(-u + i\alpha)$.

- CONV formula for $\Delta$ and $\Gamma$,

$$
\Delta = \frac{e^{-\alpha x}e^{-r\Delta t}}{S} \left[ \mathcal{F}^{-1}\{-iuA(u)\} - \alpha \mathcal{F}^{-1}\{A(u)\} \right],
$$

$$
\Gamma = \frac{e^{-\alpha x}e^{-r\Delta t}}{S^2} \left[ \mathcal{F}^{-1}\{(iu)^2A(u)\} - (1 + 2\alpha) \mathcal{F}^{-1}\{-iuA(u)\} \right.
$$

$$
+ \left. \alpha(\alpha + 1) \mathcal{F}^{-1}\{A(u)\} \right].
$$
Step 1 - The payoff transform

\[
\mathcal{F}\{ e^{\alpha y} V(t_{m+1}, y) \}(u) = \int_{-\infty}^{\infty} e^{iuy} e^{\alpha y} V(t_{m+1}, y) dy
\]

\[
\approx \Delta y \sum_{n=0}^{N-1} w_n e^{iu_j y_n} e^{\alpha y_n} V(t_{m+1}, y_n)
\]

Can be evaluated using the FFT, use the Trapezoidal rule, for example.

Need uniform grids for \( u, x \) (log-asset price at \( t_m \)) and \( y \) (log-asset price at \( t_{m+1} \)).

Further, the Nyquist relation must be satisfied: \( \Delta u \cdot \Delta x = 2\pi/N \).
Rederive discretized CONV formula by a Fourier series expansion of continuation value.

This reveals that

- Only moment restriction on \( \alpha \) is necessary (\( L^1 \) integrability is replaced by \( L^1 \)-summability);
- If \( \phi \) decays faster than a polynomial, the discretized CONV formula converges as \( O(1/N^2) \) for continuous payoff functions;
- If \( \phi \) decays as \( x^\beta \), the order is \( O(1/N^{\min\{1+\beta,2\}}) \) for continuous payoff functions.
Consider two discretizations:

- Discretization I: \( x = y \) throughout, and \( \ln S(0) \) lies on the grid;
- Discretization II: At each time, \( t_m \), we place \( d_m \) on the \( x \)-grid.

1. Estimate \( d_m \) in \( C(t_m, d_m) = E(t_m, d_m) \);
2. Place \( d_m \) on the \( x \)-grid and recalculate \( C(t_m) \);
3. Re-evaluate exercise decision and continue.
Pricing 10-times exercisable Bermudan put under GBM and VG

\[ S_0 = 100, \ K = 110, \ T = 1, \ r = 0.1, \ q = 0; \]

For GBM: \( \sigma = 0.25 \), reference = 11.1352431;

For VG: \( \sigma = 0.12, \ \theta = -0.14, \ \nu = 0.2 \), reference = 9.040646114;

<table>
<thead>
<tr>
<th>( n )</th>
<th>GBM</th>
<th></th>
<th>VG</th>
<th></th>
<th></th>
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<tr>
<td></td>
<td>time(msec)</td>
<td>abs. error</td>
<td>conv.</td>
<td>time(msec)</td>
<td>abs. error</td>
<td>conv.</td>
</tr>
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<td>–</td>
<td>0.28</td>
<td>-9.6e-02</td>
<td>–</td>
</tr>
<tr>
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<td>1.09</td>
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<tr>
<td>12</td>
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<td>4.0</td>
<td>9.29</td>
<td>-4.1e-05</td>
<td>3.9</td>
</tr>
</tbody>
</table>
The value of an American option can be approximated
- either by a Bermudan with many exercise dates,
- or, by Richardson extrapolation on a series of Bermudan options
  with an increasingly number of exercise dates

To this end assume that the Bermudan price $V(\Delta t)$, with $\Delta t$ the
time step between two consecutive exercise moments, can be written as:

$$V(\Delta t) = V(0) + \sum_{i=1}^{\infty} a_i(\Delta t)^{\gamma_i}$$
American option under GBM

- \( \lim_{M \to \infty} P(M) = \) American option value
  - Approximate the American option value by \( P(M) \) with a big \( M \).
  - Reconstruct a faster converging series \( P'(M) \) by
- \( S_0 = 100, K = 110, T = 1, \sigma = 0.25, r = 0.1, q = 0; \)
- Reference value: \( V_{\text{ref}}(0, S(0) = 12.169417 \) (Black-Scholes)
- Richardson extrapolation with 128, 64 and 32 exercise opportunities

<table>
<thead>
<tr>
<th>(( N = 2^n ))</th>
<th>( P(N/2) )</th>
<th>Richardson</th>
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<tr>
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<td>time(msec)</td>
<td>error</td>
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<tr>
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<td>-2.2e-03</td>
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<tr>
<td>9</td>
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</table>
Yet Another Method: Fourier-Cosine Expansion

- The COS method:
  - Exponential convergence;
  - Greeks are obtained at no additional cost.
  - For discretely-monitored barrier and Bermudan options as well;
- The basic idea:
  - Replace the density by its Fourier-cosine series expansion;
  - Series coefficients have simple relation with characteristic function.
Fourier-Cosine expansion of density function on interval \([a, b]\):

\[
f(x) = \sum_{n=0}^{\infty} F_n \cos \left( n\pi \frac{x - a}{b - a} \right),
\]

with \(x \in [a, b] \subset \mathbb{R}\) and the coefficients defined as

\[
F_n := \frac{2}{b - a} \int_{a}^{b} f(x) \cos \left( n\pi \frac{x - a}{b - a} \right) dx.
\]

\(F_n\) has direct relation to ch.f., \(\phi(u) := \int_{\mathbb{R}} f(x)e^{iux} dx\)

\((\int_{\mathbb{R}\setminus[a,b]} f(x) \approx 0),\)

\[
F_n \approx A_n := \frac{2}{b - a} \int_{\mathbb{R}} f(x) \cos \left( n\pi \frac{x - a}{b - a} \right) dx
\]

\[
= \frac{2}{b - a} \text{Re} \left\{ \phi \left( \frac{n\pi}{b - a} \right) \exp \left( -i \frac{ka\pi}{b - a} \right) \right\}.
\]
Replace $F_n$ by $A_n$, and truncate the summation:

$$f(x) \approx \frac{2}{b-a} \sum_{n=0}^{N-1} \text{Re} \left\{ \phi \left( \frac{n\pi}{b-a}; x \right) \exp \left( in\pi \frac{-a}{b-a} \right) \right\} \cos \left( n\pi \frac{x-a}{b-a} \right)$$

Example: 

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}, \quad [a, b] = [-10, 10] \text{ and } x = \{-5, -4, \cdots , 4, 5\}.$$  

<table>
<thead>
<tr>
<th>$N$</th>
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<th>16</th>
<th>32</th>
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<td>0.0028</td>
<td>0.0025</td>
<td>0.0031</td>
<td>0.0032</td>
</tr>
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</table>

Exponential error convergence in $N$. 
Pricing European Options

- Start from the risk-neutral valuation formula:

\[ V(x, t_0) = e^{-r\Delta t} \mathbb{E}^Q [V(y, T)|x] = e^{-r\Delta t} \int_{\mathbb{R}} V(y, T) f(y|x) dy. \]

- Truncate the integration range:

\[ V(x, t_0) = e^{-r\Delta t} \int_{[a,b]} V(y, T) f(y|x) dy + \varepsilon. \]

- Replace the density by the COS approximation, and interchange summation and integration:

\[ \hat{V}(x, t_0) = e^{-r\Delta t} \sum_{n=0}^{N-1} \text{Re} \left\{ \phi \left( \frac{n\pi}{b-a}; x \right) e^{-in\pi \frac{a}{b-a}} \right\} V_n, \]

where the series coefficients of the payoff, \( V_n \), are analytic.
Log-asset prices: \( x := \ln(S_0/K) \) and \( y := \ln(S_T/K) \),

The payoff for European options reads

\[
V(y, T) \equiv [\alpha \cdot K(e^y - 1)]^+.
\]

For a call option, we obtain

\[
V_k^{\text{call}} = \frac{2}{b - a} \int_0^b K(e^y - 1) \cos \left( k\pi \frac{y - a}{b - a} \right) \, dy
\]

\[
= \frac{2}{b - a} K \left( \chi_k(0, b) - \psi_k(0, b) \right),
\]

For a vanilla put, we find

\[
V_k^{\text{put}} = \frac{2}{b - a} K \left( -\chi_k(a, 0) + \psi_k(a, 0) \right).
\]
The characteristic function of the log-asset price for Heston’s model:

\[ \varphi_{hes}(u; \sigma_0) = \exp \left( iur \Delta t + \frac{\sigma_0}{\gamma^2} \left( \frac{1 - e^{-D\Delta t}}{1 - Ge^{-D\Delta t}} \right) (\kappa - i \rho \gamma u - D) \right) \cdot \exp \left( \frac{\kappa \sigma}{\gamma^2} \left( \Delta t(\kappa - i \rho \gamma u - D) - 2 \log \left( \frac{1 - Ge^{-D\Delta t}}{1 - G} \right) \right) \right), \]

with \( D = \sqrt{(\kappa - i \rho \gamma u)^2 + (u^2 + iu)\gamma^2} \) and \( G = \frac{\kappa - i \rho \gamma u - D}{\kappa - i \rho \gamma u + D} \).

For Lévy and Heston models, the ChF can be represented by

\[ \phi(u; x) = \varphi_{levy}(u) \cdot e^{iux} \quad \text{with} \quad \varphi_{levy}(u) := \phi(u; 0), \]

\[ \phi(u; x, \sigma_0) = \varphi_{hes}(u; \sigma_0) \cdot e^{iux}, \]
For the CGMY/KoBol model:

\[ \varphi_{\text{levy}}(u) = \exp (iu(r - q)\Delta t - \frac{1}{2}u^2\sigma^2\Delta t) \cdot \exp (\Delta t C \Gamma(-Y)[(M - iu)^Y - M^Y + (G + iu)^Y - G^Y]), \]

where \( \Gamma(\cdot) \) represents the gamma function.

- The parameters should satisfy \( C \geq 0, G \geq 0, M \geq 0 \) and \( Y < 2 \).
- The characteristic function of the log-asset price for NIG:

\[ \varphi_{\text{NIG}}(u) = \exp \left( iu\mu + \delta(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + iu)^2}) \right) \]

with \( \alpha, \delta > 0, \beta \in (-\alpha, \alpha - 1) \)
We can present the $V_k$ as $V_k = U_k K$, where

$$U_k = \begin{cases} \frac{2}{b-a} (\chi_k(0, b) - \psi_k(0, b)) & \text{for a call} \\ \frac{2}{b-a} (-\chi_k(a, 0) + \psi_k(a, 0)) & \text{for a put.} \end{cases}$$

The pricing formula simplifies for Heston and Lévy processes:

$$v(x, t_0) \approx Ke^{-r\Delta t} \cdot \Re \left\{ \sum_{n=0}^{N-1} \varphi \left( \frac{n\pi}{b-a} \right) U_n \cdot e^{in\pi \frac{x-a}{b-a}} \right\},$$

where $\varphi(u) := \phi(u; 0)$.
Pricing for 21 strikes $K = 50, 55, 60, \cdots, 150$ under Heston’s model. Other parameters: $S_0 = 100, r = 0, q = 0, T = 1, \kappa = 1.5768, \gamma = 0.5751, \bar{\sigma} = 0.0398, \sigma_0 = 0.0175, \rho = -0.5711$.

<table>
<thead>
<tr>
<th></th>
<th>$N$</th>
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<th>128</th>
<th>160</th>
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<td>Carr-Madan</td>
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<td>(msec.)</td>
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<tr>
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<td>2.15e-03</td>
<td>2.08e-07</td>
<td></td>
</tr>
</tbody>
</table>

Error analysis for the COS method is provided in the COS paper.
The pricing formulae

\[
\begin{cases}
  C(x, t_m) = e^{-r\Delta t} \int_{\mathbb{R}} V(y, t_{m+1}) f(y|x) \, dy \\
  V(x, t_m) = \max(E(x, t_m), C(x, t_{m}))
\end{cases}
\]

and \( V(x, t_0) = e^{-r\Delta t} \int_{\mathbb{R}} V(y, t_1) f(y|x) \, dy \).

- Use Newton’s method to locate the early exercise point \( x_m^* \), which is the root of \( E(x, t_m) - C(x, t_m) = 0 \).
- Recover \( V_n(t_1) \) recursively from \( V_n(t_M), V_n(t_{M-1}), \ldots, V_n(t_2) \).
- Use the COS formula for \( V(x, t_0) \).
\( \mathcal{V}_k \)-Coefficients

Once we have \( x_m^* \), we split the integral, which defines \( \mathcal{V}_k(t_m) \):

\[
\mathcal{V}_k(t_m) = \begin{cases} 
C_k(a, x_m^*, t_m) + G_k(x_m^*, b), & \text{for a call,} \\
G_k(a, x_m^*) + C_k(x_m^*, b, t_m), & \text{for a put,}
\end{cases}
\]

for \( m = M - 1, M - 2, \cdots, 1 \). whereby

\[
G_k(x_1, x_2) := \frac{2}{b - a} \int_{x_1}^{x_2} E(x, t_m) \cos \left( k\pi \frac{x - a}{b - a} \right) dx.
\]

and

\[
C_k(x_1, x_2, t_m) := \frac{2}{b - a} \int_{x_1}^{x_2} \hat{C}(x, t_m) \cos \left( k\pi \frac{x - a}{b - a} \right) dx.
\]

**Theorem**

The \( G_k(x_1, x_2) \) are known analytically and the \( C_k(x_1, x_2, t_m) \) can be computed in \( O(N \log_2(N)) \) operations with the Fast Fourier Transform.
Bermudan Details

- Formula for the coefficients \( C_k(x_1, x_2, t_m) \):

\[
C_k(x_1, x_2, t_m) = e^{-r \Delta t} \text{Re} \left\{ \sum_{j=0}^{N-1} \varphi_{\text{levy}} \left( \frac{j \pi}{b-a} \right) V_j(t_{m+1}) \cdot M_{k,j}(x_1, x_2) \right\},
\]

where the coefficients \( M_{k,j}(x_1, x_2) \) are given by

\[
M_{k,j}(x_1, x_2) := \frac{2}{b-a} \int_{x_1}^{x_2} e^{ij \pi \frac{x-a}{b-a}} \cos \left( k \pi \frac{x-a}{b-a} \right) dx,
\]

- With fundamental calculus, we can rewrite \( M_{k,j} \) as

\[
M_{k,j}(x_1, x_2) = -\frac{i}{\pi} \left( M_{k,j}^c(x_1, x_2) + M_{k,j}^s(x_1, x_2) \right),
\]
Hankel and Toeplitz

Matrices $\mathcal{M}_c = \{M_{c,k,j}(x_1, x_2)\}_{k,j=0}^{N-1}$ and $\mathcal{M}_s = \{M_{s,k,j}(x_1, x_2)\}_{k,j=0}^{N-1}$ have special structure for which the FFT can be employed: $\mathcal{M}_c$ is a Hankel matrix,

$$\mathcal{M}_c = \begin{bmatrix} m_0 & m_1 & m_2 & \cdots & m_{N-1} \\ m_1 & m_2 & \cdots & \cdots & m_N \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ m_{N-2} & m_{N-1} & \cdots & \cdots & m_{2N-3} \\ m_{N-1} & \cdots & m_{2N-3} & m_{2N-2} \end{bmatrix}_{N \times N}$$

and $\mathcal{M}_s$ is a Toeplitz matrix,

$$\mathcal{M}_s = \begin{bmatrix} m_0 & m_1 & \cdots & m_{N-2} & m_{N-1} \\ m_{-1} & m_0 & m_1 & \cdots & m_{N-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ m_{2-N} & \cdots & m_{-1} & m_0 & m_1 \\ m_{1-N} & m_{2-N} & \cdots & m_{-1} & m_0 \end{bmatrix}_{N \times N}$$
Bermudan puts with 10 early-exercise dates

Table: Test parameters for pricing Bermudan options

<table>
<thead>
<tr>
<th>Test No.</th>
<th>Model</th>
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<th>$K$</th>
<th>$T$</th>
<th>$r$</th>
<th>$\sigma$</th>
<th>Other Parameters</th>
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<td>0.1</td>
<td>0</td>
<td>$C = 1, G = 5, M = 5, Y = 1.5$</td>
</tr>
</tbody>
</table>

\[ \text{log}_{10}|error| \]

\[ \text{milliseconds} \]

\[ \text{log}_{10}|error| \]

\[ \text{milliseconds} \]
The price of an $M$-times monitored up-and-out option satisfies

$$C(x, t_{m-1}) = e^{-r(t_m-t_{m-1})} \int_{\mathbb{R}} V(x, t_m) f(y|x) dy$$

$$V(x, t_{m-1}) = \begin{cases} e^{-r(T-t_{m-1})} Rb, & x \geq h \\ C(x, t_{m-1}), & x < h \end{cases}$$

where $h = \ln(H/K)$, and

$$V(x, t_0) = e^{-r(t_m-t_{m-1})} \int_{\mathbb{R}} V(x, t_1) f(y|x) dy.$$

The technique:

- Recover $\mathcal{V}_n(t_1)$ recursively, from $\mathcal{V}_n(t_M), \mathcal{V}_n(t_{M-1}), \cdots, \mathcal{V}_n(t_2)$ in $O((M-1)N \log_2(N))$ operations.
- Split the integration range at the barrier level (no Newton required)
- Insert $\mathcal{V}_n(t_1)$ in the COS formula to get $V(x, t_0)$, in $O(N)$ operations.
### Table: Test parameters for pricing barrier options

<table>
<thead>
<tr>
<th>Test No.</th>
<th>Model</th>
<th>$S_0$</th>
<th>$K$</th>
<th>$T$</th>
<th>$r$</th>
<th>$q$</th>
<th>Other Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>NIG</td>
<td>100</td>
<td>100</td>
<td>1</td>
<td>0.05</td>
<td>0.02</td>
<td>$\alpha = 15, \beta = -5, \delta = 0.5$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Option Type</th>
<th>Ref. Val.</th>
<th>$N$</th>
<th>time (milli-sec.)</th>
<th>error</th>
</tr>
</thead>
<tbody>
<tr>
<td>DOP</td>
<td>2.139931117</td>
<td>$2^7$</td>
<td>3.7</td>
<td>1.28e-3</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$2^8$</td>
<td>5.4</td>
<td>4.65e-5</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$2^9$</td>
<td>8.4</td>
<td>1.39e-7</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$2^{10}$</td>
<td>14.7</td>
<td>1.38e-12</td>
</tr>
<tr>
<td>DOC</td>
<td>8.983106036</td>
<td>$2^7$</td>
<td>3.7</td>
<td>1.09e-3</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$2^8$</td>
<td>5.3</td>
<td>3.99e-5</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$2^9$</td>
<td>8.3</td>
<td>9.47e-8</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$2^{10}$</td>
<td>14.8</td>
<td>5.61e-13</td>
</tr>
</tbody>
</table>
Credit default swaps (CDSs), the basic building block of the credit risk market, offer investors the opportunity to either buy or sell default protection on a reference entity.

The protection buyer pays a premium periodically for the possibility to get compensation if there is a credit event on the reference entity until maturity or the default time, which ever is first.

If there is a credit event the protection seller covers the losses by returning the par value. The premium payments are based on the CDS spread.
CDS and COS

- CDS spreads are based on a series of default/survival probabilities, that can be efficiently recovered using the COS method. It is also very flexible w.r.t. the underlying process as long as it is Lévy.
- The flexibility and the efficiency of the method are demonstrated via a calibration study of the iTraxx Series 7 and Series 8 quotes.
Definition of default: For a given recovery rate, \( R \), default occurs the first time the firm’s value is below the “reference value” \( R V_0 \).

As a result, the survival probability in the time period \((0, t]\) is nothing but the price of a digital down-and-out barrier option without discounting.

\[
P_{surv}(t) = P_Q(X_s > \ln R, \text{ for all } 0 \leq s \leq t) \\
= P_Q\left(\min_{0 \leq s \leq t} X_s > \ln R\right) \\
= E_Q\left[1\left(\min_{0 \leq s \leq t} X_s > \ln R\right)\right]
\]
Assume there are only a finite number of observing dates.

\[ P_{\text{surv}}(\tau) = \mathbb{E}_Q \left[ 1(X_{\tau_1} \in [\ln R, \infty)) \cdot 1(X_{\tau_2} \in [\ln R, \infty)) \cdot \cdots \cdot 1(X_{\tau_M} \in [\ln R, \infty)) \right] \]

where \( \tau_k = k\Delta \tau \) and \( \Delta \tau := \tau / M \).

The survival probability then has the following recursive expression:

\[
\begin{align*}
P_{\text{surv}}(\tau) & := p(x = 0, \tau_0) \\
p(x, \tau_m) & := \int_{\ln R}^{\infty} f_{X_{\tau_{m+1}} | X_{\tau_m}}(y | x) p(y, \tau_{m+1}) \, dy, \quad m = M - 1, \cdots, 2, 1 \\
p(x, \tau_M) & := 1(x > \ln R) \text{ and equals 0 otherwise}
\end{align*}
\]

\( f_{X_{\tau_{m+1}} | X_{\tau_m}}(\cdot | \cdot) \) denotes the conditional probability density of \( X_{\tau_{m+1}} \) given \( X_{\tau_m} \).
The Fair Spread of a Credit Default Swap

- The *fair spread*, $C$, of a CDS at the initialization date is the spread that equalizes the present value of the premium leg and the present value of the protection leg, i.e.

$$C = \frac{(1 - R) \left( \int_0^T \exp(-r(s)s) dP_{def}(s) \right)}{\int_0^T \exp(-r(s)s) P_{surv}(s) ds},$$

- It is actually based on a series of survival probabilities on different time intervals:

$$C = \frac{(1 - R) \sum_{j=0}^{J} \frac{1}{2} [\exp(-r_j t_j) + \exp(-r_{j+1} t_{j+1})] [P_{surv}(t_j) - P_{surv}(t_{j+1})]}{\sum_{j=0}^{J} \frac{1}{2} [\exp(-r_j t_j) P_{surv}(t_j) + \exp(-r_{j+1} t_{j+1}) P_{surv}(t_{j+1})] \Delta t}$$
Replace the conditional density by the COS (semi-analytical) expression, the survival probability then satisfies

\[
\begin{aligned}
    P_{\text{surv}}(\tau) &= p(x = 0, \tau_0), \\
p(x, \tau_0) &= \sum'_{n=0} \phi_n(x) \cdot P_n(\tau_1),
\end{aligned}
\]

The only thing one needs is \( \{P_n(\tau_1)\}_{n=0}^{N-1} \), which can be recovered from \( \{P_n(\tau_M)\}_{n=0}^{N-1} \) via backwards induction.
• Starting from the definition of $P_n(\tau_m)$, we apply the COS reconstruction of $p(y, \tau_m)$ to get

\[ P(\tau_m) = \text{Re} \{ \Omega \Lambda \} P(\tau_{m+1}), \]

• Applying this recursively backwards in time, we get

\[ P(\tau_1) = (\text{Re} \{ \Omega \Lambda \})^{M-1} P(\tau_M) \]

• For this recursive matrix-vector-product, there exists a fast algorithm, e.g.

\[ P(\tau_1) = \text{Re} \{ \Omega \ [ \text{Re} \{ \Omega \ [ \text{Re} \{ \Omega \ [ \Lambda P(t_3)] ] ] ] ] } \}

• The FFT algorithm can be applied because $\Omega = H + T$, where $H$ is a Hankel matrix and $T$ is a Toeplitz matrix.
Convergence of Survival Probabilities

- Ideally, the survival probabilities should be monitored daily, i.e. \( \Delta \tau = 1/252 \). That is, \( M = 252T \), which is a bit too much for \( T = 5, 7, 10 \) years.

- For Black-Scholes’ model, there exist rigorous proof of the convergence of discrete barrier options to otherwise identical continuous options [Kou,2003].

- We observe similar convergence under NIG, CGMY:

![Survival Probabilities under NIG and CGMY](image)
The error convergence of the COS method is usually exponential in $N$.

**Figure**: Convergence of $P_{\text{surv}}(\Delta \tau = 1/48)$ w.r.t. $N$ for NIG and CGMY.
Calibration Setting

- The data sets: weekly quotes from iTraxx Series 7 (S7) and 8 (S8). After cleaning the data we were left with 119 firms from Series 7 and 123 firms from Series 8. Out of these firms 106 are common to both Series.
- The interest rates: EURIBOR swap rates.
- We have chosen to calibrate the models to CDSs spreads with maturities 1, 3, 5, 7, and 10 years.
To avoid the ill-posedness of the inverse problem we defined here, the objective function is set to

$$F_{obj} = \text{rmse} + \gamma \cdot \|X_2 - X_1\|_2,$$

where

$$\text{rmse} = \sqrt{\sum_{\text{CDS}} \left(\frac{\text{market CDS spread} - \text{model CDS spread}}{\text{number of CDSs on each day}}\right)^2},$$

$$\| \cdot \|_2$$ denotes the $L_2$-norm operator, and $X_2$ and $X_1$ denote the parameter vectors of two neighbor data sets.
Good Fit to Market Data

Table: Summary of calibration results of all 106 firms in both S7 and S8 of iTraxx quotes

<table>
<thead>
<tr>
<th>RMSEs</th>
<th>NIG in S7</th>
<th>CGMY in S7</th>
<th>NIG in S8</th>
<th>CGMY in S8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average (bp.)</td>
<td>0.89</td>
<td>0.79</td>
<td>1.65</td>
<td>1.54</td>
</tr>
<tr>
<td>Min. (bp.)</td>
<td>0.22</td>
<td>0.29</td>
<td>0.27</td>
<td>0.46</td>
</tr>
<tr>
<td>Max. (bp.)</td>
<td>2.29</td>
<td>1.97</td>
<td>4.27</td>
<td>3.52</td>
</tr>
</tbody>
</table>
Evolution of CDSs of ABN Amro Bank NV with maturity $T = 1$ year

Evolution of CDSs of ABN Amro Bank NV with maturity $T = 5$ year

Evolution of CDSs of ABN Amro Bank NV with maturity $T = 10$ year
An Extreme Case

Evolution of CDSs of DSG International PLC with maturity $T = 1$ year

Evolution of CDSs of DSG International PLC with maturity $T = 5$ year

Evolution of CDSs of DSG International PLC with maturity $T = 10$ year

Market CDSs
CGMY calibration results
NIG calibration results
Figure: Evolution of the NIG parameters and densities of “ABN AMRO Bank”
Figure: Evolution of the NIG parameters and densities of “DSG International PLC”
Both Lévy processes gave good fits, but

- The NIG model returns more consistent measures from time to time and from one company to another.
- From a numerical point of view, the NIG model is also more preferable.
  - Small $N$ (e.g. $N = 2^{10}$) can be applied.
  - The NIG model is much less sensitive to the initial guess of the optimum-searching procedure.
  - Fast convergence to the optimal parameters are observed (usually within 200 function evaluations). However, averagely 500 to 600 evaluations for the CGMY model are needed.
\[ [a, b] := \left[ (c_1 + x_0) - L\sqrt{c_2 + \sqrt{c_4}}, \quad (c_1 + x_0) + L\sqrt{c_2 + \sqrt{c_4}} \right], \]
Table: Cumulants of $\ln(S_t/K)$ for various models.

<table>
<thead>
<tr>
<th>Model</th>
<th>$c_1$</th>
<th>$c_2$</th>
<th>$c_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>BS</strong></td>
<td>$(\mu - \frac{1}{2} \sigma^2) t$,</td>
<td>$\sigma^2 t$,</td>
<td>$0$</td>
</tr>
<tr>
<td><strong>NIG</strong></td>
<td>$(\mu - \frac{1}{2} \sigma^2 + w) t + \delta t \beta / \sqrt{\alpha^2 - \beta^2}$</td>
<td>$\delta t \alpha^2 (\alpha^2 - \beta^2)^{-3/2}$</td>
<td>$3 \delta t \alpha^2 (\alpha^2 + 4 \beta^2) (\alpha^2 - \beta^2)^{-7/2}$</td>
</tr>
<tr>
<td></td>
<td>$w = -\delta (\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + 1)^2})$</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Kou</strong></td>
<td>$c_1 = t \left( \mu + \frac{\lambda p}{\eta_1} + \frac{\lambda (1-p)}{\eta_2} \right)$</td>
<td>$c_2 = t \left( \sigma^2 + 2 \frac{\lambda p}{\eta_1} + 2 \frac{\lambda (1-p)}{\eta_2} \right)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$c_4 = 24 t \lambda \left( \frac{p}{\eta_1^4} + \frac{1-p}{\eta_2^4} \right)$</td>
<td>$w = \lambda \left( \frac{p}{\eta_1 + 1} - \frac{1-p}{\eta_2 - 1} \right)$</td>
<td></td>
</tr>
<tr>
<td><strong>Merton</strong></td>
<td>$c_1 = t (\mu + \lambda \bar{\mu})$</td>
<td>$c_2 = t (\sigma^2 + \lambda \bar{\mu}^2 + \bar{\sigma}^2 \lambda)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$c_4 = t \lambda (\bar{\mu}^4 + 6 \bar{\sigma}^2 \bar{\mu}^2 + 3 \bar{\sigma}^4 \lambda)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>VG</strong></td>
<td>$c_1 = (\mu + \theta) t$</td>
<td>$c_2 = (\sigma^2 + \nu \theta^2) t$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$c_4 = 3 (\sigma^4 \nu + 2 \theta^4 \nu^3 + 4 \sigma^2 \theta^2 \nu^2) t$</td>
<td>$w = \frac{1}{\nu} \ln(1 - \theta \nu - \sigma^2 \nu / 2)$</td>
<td></td>
</tr>
<tr>
<td><strong>CGMY</strong></td>
<td>$c_1 = \mu t + Ct \Gamma(1 - Y) \left( M^{Y-1} - G^{Y-1} \right)$</td>
<td>$c_2 = \sigma^2 t + Ct \Gamma(2 - Y) \left( M^{Y-2} + G^{Y-2} \right)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$c_4 = Ct \Gamma(4 - Y) \left( M^{Y-4} + G^{Y-4} \right)$</td>
<td>$w = -C \Gamma(-Y) [(M - 1)^Y - M^Y + (G + 1)^Y - G^Y]$</td>
<td></td>
</tr>
</tbody>
</table>

where $w$ is the drift correction term that satisfies $\exp(-wt) = \varphi(-i, t)$. 
American Options and Extrapolation

Let $v(M)$ denote the value of a Bermudan option with $M$ early exercise dates, then we can rewrite the 3-times repeated Richardson extrapolation scheme as

$$v_{AM}(d) = \frac{1}{12} \left( 64v(2^{d+3}) - 56v(2^{d+2}) + 14v(2^{d+1}) - v(2^{d}) \right),$$

where $v_{AM}(d)$ denotes the approximated value of the American option.
Further Reading: Fourier Pricing


