Delft University of Technology
Faculty of Electrical Engineering, Mathematics and Computer Science
Delft Institute of Applied Mathematics

Dynamic Portfolio Choice

A Simulation Approach with an Application to Multiple Assets

A thesis submitted to the
Delft Institute of Applied Mathematics
in partial fulfillment of the requirements

for the degree

MASTER OF SCIENCE
in
APPLIED MATHEMATICS

by

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Delft, the Netherlands
September 7, 2010

MSc THESIS APPLIED MATHEMATICS

"Dynamic Portfolio Choice"

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September 7, 2010

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Preface

This thesis is the final part of my Master of Science in Applied Mathematics at Delft University of Technology. From February until April 2010 I did an internship at ING Investment Management, where I mainly did my literature research. From April until September 2010 I did my actual research during an internship at the Pension Risk Management department of Ortec Finance in Rotterdam.

My thesis is rather on the 'Applied' side of Applied Mathematics. I always explained my specialization as being 'almost equal to econometrics', but of course it is not. However, I did try to combine the more practical view of the econometricians that mostly surrounded me during my internship with a more theoretical mathematical view. In my opinion it is this combination that is important to function as a mathematical engineer and solve practical problems by all kinds of mathematics. And measuring by the number of old study books I opened in the last months, I did use subjects from many fields of mathematics from my whole study period to complete this thesis.

It has to be said that the start was somewhat tough. After five years of clearly defined assignments, at the beginning my graduation project seemed to be an unbounded and unconstrained objective function. As the horizon until graduation became shorter, I started to appreciate doing research more and more and a story line appeared. Of course many people helped me along the way.

From Delft University I was supervised by dr. J.A.M. van der Weide, to whom I would like to express my gratitude. Thanks to ING Investment Management for a place to do my literature research and for helping me understanding the gap between the academic and the industry approach around asset allocation. At Ortec Finance I would like to thank Marnix Engels and Guus Boender for supervising, the ideas and feedback.

After nine months of work I would like to thank my roommates for their feedback, graphical assistance and for a fair amount of distraction by - amongst others - putting a swimming pool in front of my window. Finally many thanks to my parents, sister and Rick, for supporting me during my studies, during good and sometimes hard days, from my first, rather ugly design at Architecture to my final thesis in Applied Mathematics.

For questions or remarks, do not hesitate to contact me.

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September 7, 2010

Management Summary

In this thesis we found and extended a methodology to find an optimal, dynamic portfolio strategy for a long-term investor who can choose between multiple assets. Such a strategy is new in the sense that it combines the dynamic approach which is a hot topic in academic literature with the larger number of assets with rather abritrary return dynamics that is used in practice.

The methodology, developed by Brandt et al. (2005) is a fast, simulation-based dynamic approach towards portfolio choice. Originally it was only applied to a simple setting with two assets and one predicting state variable, with the idea that an extension to a more realistic setting would be possible. In this thesis we actually applied this algorithm to a more realistic investor's environment with a long investment horizon, six assets, five state variables, scenario-based asset returns and constraints on the portfolio weights.

In this situation the algorithm is still fast, taking less than an hour for 10,000 simulations. And more importantly, we found that the obtained dynamic strategy outperforms a static strategy in the most used performance measures of financial management: that is portfolio mean return and variance, probability of underperforming a risk-free strategy, Value-at-Risk and expected shortfall. Moreover, we have clearly visualized the outperformance by drawing the dynamic efficient frontier, which is a well-known concept from static portfolio optimization.

Also by using the efficient frontier, we gained a new insight about the modeling of an individual's attitude towards risk. Empirical evidence on individual's attitude towards risk and return and, one step further, on how to calibrate the parameter of risk aversion, is mixed. In this thesis we show that both commonly used assumptions lead to the *same* efficient frontier. This justifies the convenient choice to assume constant relative risk aversion.[1]

Also in the more realistic, multivariate complex setting the dynamic strategies outperform static strategies. Nevertheless, the performance of the methodology does detoriate as the investor's horizon or risk aversion increases. This decreasing effectiveness was already mentioned by the original authors, but in a simple setting we were able to succesfully eliminate these issues by improving the methodology with a technique called robust regression, which reduces the effect of outlying datapoints. Unfortunately this technique does not work sufficiently in the multivariate complex setting.

Even though the method remains suboptimal for high risk aversion and a long investment horizon, the dynamic strategy *does* outperform a static strategy convincingly for all parameter values. It is open to discussion whether the suboptimality is an issue inherent in application of the methodology to a complex setting, or if improvements are to be found. In our opinion there will always be a trade-off between the complexity (and thus volatility) of the investor's environment at one hand and the optimality of the dynamic strategy at the other hand. How this trade-off should be balanced is a management decision.

To Ortec Finance we recommend implementation. But, since this thesis made use of their old scenario generator, first the performance under the new, more complex scenario generator has to be investigated. Additionally, they have to develop an application to more complex investors, for example with volatile liabilities or with wealth corrected for inflation effects.

[1] This means that an investor always attaches an equal value to an equal relative increase of wealth.

Contents

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List of most important symbols and abbreviations

Symbols

n	=	number of assets
R_i	=	return on asset i
μ_i	=	expected return on asset i
σ_{ij}	=	covariance of asset i and j
Σ	=	$(n \times n)$-covariance matrix of R
x	=	$(n \times 1)$-vector of portfolio weights on risky assets
x_{f}	=	portfolio weight on risk-free asset
R^{f}	=	rate of return on the risk-free asset
s	=	Sharpe ratio
u	=	utility function
ξ	=	parameter of absolute risk aversion
γ	=	parameter of relative risk aversion
ζ	=	risk aversion parameter of quadratic utility function
T	=	length of investment horizon
C_t	=	consumption at time t
W_t	=	wealth at time t
Z_t	=	collection of state variables at time t
$J_t(\cdot,\cdot)$	=	value function at time t
$\mathbb{E}_t$	=	expectation conditional on W_t and Z_t
R_t^{e}	=	vector of excess returns on risky assets at time t
$\hat{x}_t$	=	approximate optimal portfolio weights on risky assets
ψ_{t+1}	=	$\prod_{s=t+1}^{T-1}(\hat{x}_s' R_{s+1}^{\mathrm{e}} + R^{\mathrm{f}})$, future portfolio return under optimal strategy
X_t	=	regression matrix at time t
ρ	=	objective function for M-estimators
w	=	weight function for M-estimators
k	=	tuning constant for Huber and bisquare estimator
P_{upf}	=	empirical probability of underperforming the risk-free strategy
VaR_α	=	Value-at-Risk at confidence level α
cVaR_α	=	expected shortfall at confidence level α
M	=	number of simulations

Abbreviations

HARA	=	hyperbolic absolute risk aversion
CARA	=	constant absolute risk aversion
CRRA	=	constant relative risk aversion
OLS	=	ordinary least-squares
LTS	=	least-trimmed squares
VAR	=	vector autoregressive

Chapter 1

Introduction

> *Every investor must decide the trade-off one is willing to make between eating well and sleeping well* - Burton G. Malkiel, [24].

If a person decides to invest his wealth, he has to choose which combination of assets meets his personal preferences best. This opens up a wide set of possible strategies. Clearly, he would prefer a high return on his money, but higher expected returns usually also increase the risk involved. Then, to speak in terms of Malkiel, the question is which portfolio will generate enough return to eat well without keeping the investor from sleeping at night because of the high risk.

The decision on the asset allocation[1] depends on the type of investor. An individual with a short-term horizon might buy a lottery ticket and will therefore behave differently than an investor with a long-term horizon, who decides to invest in real estate. Moreover, every individual has his own attitude towards the trade-off between risk and return, which is best illustrated by the following example.

Suppose a game where an individual is given the choice between immediately receiving 400 euro or gambling with 50 percent probability of winning 1000 euro, and 50 percent probability of receiving nothing. Taking the gamble has an expected pay-out of 500 euro, but the standard deviation of the pay-out is also 500, which is much higher than the standard devation of zero corresponding to receiving 400 euro for sure.

The actual choice will depend on an individual's attitude to risk and will therefore differ from person to person, but the decision in this example is easily made, because the number of possible outcomes is limited. But in the context of investors, the choice of an ideal portfolio is much more complicated, since in reality infinitely many portfolios of assets can be constructed, all with different combinations of risk and return.

Already during the literature phase of this project the gap between the academic approach and the practical application of portfolio choice became clear. At one hand, industry still uses a static and rather old-fashioned framework to come up with an optimal asset allocation among the many available assets in the market, while paying a lot of attention to the modeling, prediction and/or simulation of assets returns. The trade-off between risk and return is modeled in various

[1] Portfolio choice and asset allocation are equivalent terms.

ways and results are easily quantified and visualized. Unfortunately the static approach does not capture the problem well for a long-term investor, since during the investment period market conditions as well as personal preferences will change almost surely.

At the other hand, academic literature mainly places the portfolio choice in a setting which is above all mathematically convenient. The number of assets is mostly limited to only two, of which one has a deterministic return and of which the second has nicely distributed returns. The investor's attitude towards risk is almost always modeled as being independent of wealth, because this has nice analytical properties. Nevertheless, the latest literature *does* focus on a dynamical portfolio to adjust for changing beliefs.

In this thesis we consider a long-term investor and try to find an optimal dynamical portfolio strategy, that combines the best of both worlds: the dynamic character of the portfolio choice as used in literature with the advantages of the static approach, namely the flexibility to the number of assets having rather arbitrary return dynamics and the clear visualization of results.

More specifically, we actually apply the methodology published by Brandt, Goyal, Santa-Clara and Stroud in 2005 to an investor in a realistic setting, with multiple assets with non-standard return dynamics, constraints on his portfolio weights and with different types of risk averseness. Results are quantified and visualized wherever possible.

Overview of this thesis

In a paper that was published in 1952, Harry Markowitz developed a framework for static portfolio optimization which is considered as the starting point of Modern Portfolio Theory. In chapter 2 we explain the general set-up and describe this framework. We explain the concept of the efficient frontier and we discuss various portfolios, both in formulas as visually.

To make a ranking between all possible portfolios we use utility functions, which capture the investor's risk averseness and thus enable ranking of portfolios. Various types of utility functions are addressed in chapter 3.

As mentioned before, static portfolios do not take into account changing conditions and therefore only have limited value for a long-term investor. This brings us to dynamic portfolio choice, which is the subject of this thesis. In chapter 4 we start with the dynamic portfolio problem in continuous time and describe two solutions by Merton and Wachter. Because closed-form solutions are not known and might not even exist for realistic problems with less strict assumptions, it is common to discretize the problem using the Optimality Principle of Richard Bellman. At the end of the chapter we choose to continue with the simulation-based method developed by of Brandt, Goyal Santa-Clara and Stroud (hereafter referred to as BGSS), being the most promising to deal fast with multiple variables and various utilities.

In chapter 5 we extensively discuss the methodology of BGSS and explain amongst others how constraints on the portfolio weights, robust regression and different assumptions on the investor's utility function can be accommodated.

To understand the methodology and the impact of various extensions we used a very simple economic model with only one risky asset. Results are displayed in chapter 6. The chapter is concluded with a comparison between static and dynamic strategies. Next, in chapter 7 the

method is applied to a setting with multiple assets. The returns are simulated by economic scenario generator of Ortec Finance.

Chapter 8 contains the main conclusions of this thesis, recommendations for future research and additional notes for Ortec Finance. Finally, appendix A contains additional extensions of the methodology and results for a different set of scenarios. Appendix B shows the most used `MATLAB`-codes.

Chapter 2

Static Portfolio Choice

In this chapter we discuss the choice of a static portfolio following the paper of Markowitz ([25]), who is considered to be the founder of Modern Portfolio Theory. In 1952 he published the first paper on portfolio problems. He rightly stated that investors should not focus solely on expected return. Instead, for each investor there exists a trade-off between expected portfolio return and risk.

Before we describe the optimal portfolios of Markowitz, we describe the general mathematical and economical set-up in the first section. In section 2.2 we introduce the efficient frontier, which is a concept that consists of portfolios with optimal combinations of mean return and standard devation. Two important portfolios on the efficient frontier are the minimum-variance and the mean-variance portfolio, which are calculated in sections 2.3 and 2.4 respectively. Finally, in section 2.5, we add a risk-free asset to the model and show its impact on the efficient frontier and the optimal portfolio.

2.1 Mathematical and economical set-up

We assume the investor takes part in an economy where the standard assumptions of a perfect market hold. The most important assumptions are (Merton, [28])

- At any time, each asset can be sold and purchased in any amount at the market price.
- The bid-ask spread is zero, which means that the purchase price of each asset is the same as the selling price.
- The investor cannot influence asset prices: he is a price-taker.
- There are no transaction costs or taxes involved.

Now suppose there exist n risky assets in the market. At time 0 the investor chooses his portfolio. After one period at time 1, the value of the portfolio is evaluated. The investor thus has investment horizon 1. Moreover, he does not rebalance between time 0 and 1.

We denote the price of asset i at time 0 as $S_{i,0}$. Analogously the price of asset i at the end of one period is given by $S_{i,1}$. The rate of return (or just 'return') on the asset during this period

is the relative change in price during one period and defined as

$$R_i = \frac{S_{i,1} - S_{i,0}}{S_{i,0}}$$

The return R_i is modeled as a random variable with mean μ_i and variance σ_i^2 on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The σ-algebra of events resolved at the end of the period is called $\mathcal{F}$ and $\mathbb{P}$ is the probability measure on $\mathcal{F}$. Every $\omega \in \Omega$ corresponds to a particular scenario of market evolution at the end of the period. Scenario ω will occur with probability p and under this scenario the actual return on the ith asset is $R_i(\omega)$.

The investor divides his wealth over the n available assets by choosing the column vector of portfolio weights $x \in \mathbb{R}^n$. A portfolio weight is the asset value divided by the value of the complete portfolio. The weight of asset i in the portfolio is given by x_i. The weights have to sum to one, $\sum_{i=1}^n x_i = 1$. From now on we will write this as $e'x = 1$, where $e \in \mathbb{R}^n$ is the unit vector. The apostrophe is used for the transpose of a vector a matrix. Aditionally, one could impose a restriction on shorting assets, which implies $x \geq 0$, but this is not necessary.

Once the weight vector x has been chosen, we can easily calculate the expectation μ_{p} and variance σ_{p}^2 of the portfolio return:

$$\mu_{\text{p}} = \sum_{i=1}^n x_i \mu_i = \mu' x \tag{2.1}$$

$$\sigma_{\text{p}}^2 = \sum_{i=1}^n \sum_{j=1}^n x_i x_j \sigma_{ij} = x' \Sigma x \tag{2.2}$$

where μ is the mean vector of means of R and $\Sigma \in \mathbb{R}^{n \times n}$ is the covariance matrix, which contains the covariances between R_i and R_j, denoted by σ_{ij}. Because Σ is a covariance matrix, it is positive semidefinite and symmetric. Moreover, we assume Σ is invertible.[1]

In his set-up, Markowitz uses only the variance of portfolio return σ_{p}^2 as a measure for risk. Higher distributional moments, such as skewness and kurtosis, are not taken into account. Engels ([17]) showed that these moments can be neglected as long as the asset returns follow an elliptical distribution.

By varying the portfolio weight vector x, infinitely many portfolios can be constructed, but only so-called efficient portfolios will matter to the investor. An efficient portfolio is a portfolio which has minimum risk for a given expected return or which has maximum expected return for a given risk. All these portfolios together form the *efficient frontier*, which is the subject of the next section.

2.2 The efficient frontier

By changing the weight vector x the investor can achieve different combinations of expected return and risk. One could first try to find the portfolio where risk is minimized, given some

[1] Recall that a positive semidefinite matrix Σ has the property that $y'\Sigma y \geq 0$ for all $y \in \mathbb{R}^n$. The inverse Σ^{-1} is positive semidefinite as well.

expected return $\bar{\mu}$:

$$\begin{aligned} \min_x \quad & x'\Sigma x \\ \text{s.t.} \quad & e'x = 1 \\ & \mu' x = \bar{\mu} \end{aligned} \tag{2.3}$$

The two constraints make sure that the vector of portfolio weights sums up to one and that the portfolio return equals $\bar{\mu}$. They can be rewritten as $A'x = d$, where $A := (\mu\, e)$ and $d' := (\bar{\mu}\, 1)$.

Problem (2.3) is solved easily with the Lagrange method. Let $\theta \in \mathbb{R}^2$. Then the Lagrangian of the problem is:

$$\Lambda(x, \theta) = x'\Sigma x - \theta' A' x + \theta' d \tag{2.4}$$

We have to set the derivatives of (2.4) equal to zero:

$$\begin{cases} \Lambda_x = & 2\Sigma x + A\theta & = 0 \\ \Lambda_\theta = & A'x - d & = 0 \end{cases} \tag{2.5}$$

Rewriting the first equation of (2.5) gives:

$$x = -\frac{1}{2}\Sigma^{-1} A\theta \tag{2.6}$$

Left multiplying both sides of (2.6) with A' gives

$$d = A'x = -\frac{1}{2} A'\Sigma^{-1} A\theta \tag{2.7}$$

Define $H := A'\Sigma^{-1}A$. Clearly H is symmetric and positive definite. It is easy to see that

$$H := \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} \mu'\Sigma^{-1}\mu & \mu'\Sigma^{-1}e \\ \mu'\Sigma^{-1}e & e'\Sigma^{-1}e \end{pmatrix}$$

Using the definition of H, we can rewrite (2.6) and (2.7) as:

$$\begin{aligned} \theta &= -2H^{-1}d \\ x &= \Sigma^{-1}AH^{-1}d \end{aligned}$$

Now the variance σ_{ef} and the portfolio weights x_{ef} of the efficient portfolios follow immediately:

$$\begin{aligned} \sigma_{\text{ef}}^2 &= x'\Sigma x = (\Sigma^{-1}AH^{-1}d)'\Sigma\Sigma^{-1}AH^{-1}d \\ &= d'(H^{-1})'A'\Sigma^{-1}AH^{-1}d = d'H^{-1}d \\ x_{\text{ef}} &= \Sigma^{-1}AH^{-1}d \end{aligned}$$

If we note that

$$H^{-1} = \frac{1}{ac - b^2}\begin{pmatrix} c & -b \\ -b & a \end{pmatrix}$$

and that the expected portfolio return μ_{ef} was equal to $\bar{\mu}$, we find the equation for the efficient frontier, which is:

$$\sigma_{\text{ef}}^2 = \frac{1}{ac - b^2}(c\mu_{\text{ef}}^2 - 2b\mu_{\text{ef}} + a) \tag{2.8}$$

In $(\mu_{\text{ef}}, \sigma_{\text{ef}}^2)$-space equation (2.8) defines a parabola. The direction of the parabola is defined by the coefficient $\frac{c}{ac-b^2}$. Because Σ^{-1} is positive semidefinite, it follows that both $a = \mu'\Sigma^{-1}\mu \geq 0$ and $c = e'\Sigma^{-1}e \geq 0$. But also

$$(b\mu - ae)'\Sigma^{-1}(b\mu - ae) = a^2c - ab^2 + b^2a - ab^2 = a(ac - b^2) \geq 0$$

Because $a \geq 0$ it immediately follows that $ac - b^2 \geq 0$ as well. We get that $\frac{c}{ac-b^2} \geq 0$ and thus equation (2.8) defines an upward parabola.

However, it is common to evaluate the efficient frontier in $(\sigma_{\text{ef}}, \mu_{\text{ef}})$-space. Engels ([17]) showed that in this space (2.8) defines the right side of a hyperbola with center $(0, \frac{b}{c})$ and asympotes

$$\mu_{\text{ef}} = \frac{b}{c} \pm \sqrt{\frac{ac - b^2}{c}}\sigma_{\text{ef}}$$

The efficient frontier with center asymptotes is shown in figure 2.1.[2]

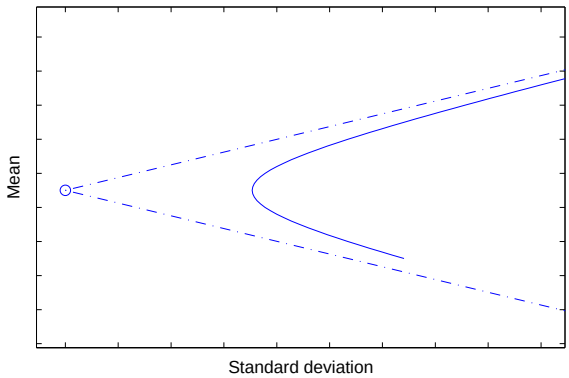

Figure 2.1: *Efficient frontier (solid line) with asymptotes and center*

[2]The efficient frontier can also be calculated with other risk measures than variance. The use of risk measures that only take into account the downside risk, such as semi-variance, the Gini-coefficient or the Value-at-Risk has become increasingly popular. The portfolios on the efficient frontier obtained by such measures can, but are not necessarily different from the portfolios obtained with variance as a risk measure.

2.3 Minimum-variance portfolio

If the investor does not care about expected return and solely wants to minimize his risk, we have to minimize σ_{ef} (or equivalently σ^2_{ef}) with respect to μ_{ef}. We differentiate (2.8):

$$\begin{aligned} \frac{d\sigma^2_{\text{ef}}}{d\mu_{\text{ef}}} &= \frac{2c\mu_{\text{ef}} - 2b}{ac - b^2} = 0 \\ 2c\mu_{\text{ef}} &= 2b \\ \mu_{\min} &= \frac{b}{c} \end{aligned} \tag{2.9}$$

We see that the efficient portfolio with overall minimum variance has expected return $\mu_{\min} := \frac{b}{c}$. Substituting (2.9) in formula (2.8) of the efficient frontier gives $\sigma^2_{\min} = \frac{1}{c}$. The corresponding weights can be derived by using $d' = (\frac{b}{c}\ 1)$, which gives $x_{\min} = \frac{1}{c}\Sigma^{-1}e$. In figure 2.2 we can see that the minimum variance portfolio is the point on the efficient frontier where the tangency line is vertical.

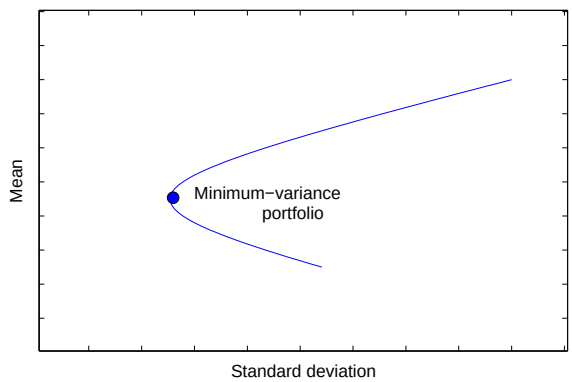

Figure 2.2: *Efficient frontier with the minimum-variance portfolio*

2.4 Mean-variance optimization

As the name already says, the minimum-variance portfolio only focuses on minimizing the risk involved. However, risk might be acceptable for the investor if this corresponds to higher expected returns. As first stated by Markowitz, every investor has his own opinion about the trade-off between risk and return, which is captured by parameter γ, the parameter of relative risk aversion. Its value is dependent on the risk averseness of the investor. The greater γ, the more important the variance becomes in the objective function and thus the more risk averse an investor is. We assume that every investor is risk averse, so γ has to be positive. A negative value of γ would imply that an investor prefers a higher risk for the same expected return. The

optimal portfolio for an investor with risk aversion γ is the solution of:

$$\begin{aligned} \max \quad & \mu' x - \frac{\gamma}{2} x' \Sigma x \\ \text{s.t.} \quad & e' x = 1 \end{aligned} \tag{2.10}$$

Equation (2.10) is known as the *mean-variance* problem. We can find the solution of (2.10) with the Lagrange method. Let $\theta \in \mathbb{R}$. The Lagrangian is given by:

$$\Lambda(x, \theta) = \mu' x - \frac{\gamma}{2} x' \Sigma x - \theta e' x + \theta \tag{2.11}$$

Taking the derivatives of (2.11) with respect to x and θ gives:

$$\begin{cases} \Lambda_x &= \mu - \gamma \Sigma x - \theta e = 0 \\ \Lambda_\theta &= -e' x + 1 = 0 \end{cases} \tag{2.12}$$

Rewriting the first equation of (2.12) gives $x = \frac{1}{\gamma} \Sigma^{-1} (\mu - \theta e)$. Substituting this into the second equation, we find:

$$\begin{aligned} 1 &= e' \frac{1}{\gamma} \Sigma^{-1} (\mu - \theta e) = \frac{1}{\gamma} (e' \Sigma^{-1} \mu - \theta e' \Sigma^{-1} e) = \frac{1}{\gamma} (b - \theta c) \\ \theta &= \frac{b - \gamma}{c} \end{aligned}$$

The optimal solution of the mean-variance problem is:

$$\begin{aligned} x_{\text{mv}} &= \frac{1}{\gamma} \Sigma^{-1} (\mu - \frac{b - \gamma}{c} e) \\ &= \frac{1}{\gamma} \Sigma^{-1} (\mu - \frac{b}{c} e) + \Sigma^{-1} \frac{e}{c} \end{aligned}$$

Now we can derive the portfolio's expected return and variance (Engels, [17]):

$$\begin{aligned} \mu_{\text{mv}} &= \mu' x_{\text{mv}} = \frac{1}{\gamma} \mu' \Sigma^{-1} (\mu + \frac{\gamma - b}{c} e) \\ &= \frac{a}{\gamma} + \frac{b}{c} - \frac{b^2}{\gamma c} \\ \sigma^2_{\text{mv}} &= x'_{\text{mv}} \Sigma x_{\text{mv}} \\ &= \frac{1}{\gamma^2} (\mu + \frac{\gamma - b}{c} e)' \Sigma^{-1} \Sigma \Sigma^{-1} (\mu + \frac{\gamma - b}{c} e) \\ &= \frac{ac - b^2 + \gamma^2}{c \gamma^2} \end{aligned}$$

The solution is visualized in figure 2.3. Each orange line represents the combinations of mean and standard deviation between which an investor with a fixed risk aversion is indifferent.[3] Graphically we see that the mean-variance portfolio is at the point of tangency between a so-called utility curve and the efficient frontier.

[3] Or as we will define in the next chapter, each orange line represents portfolios with the same utility.

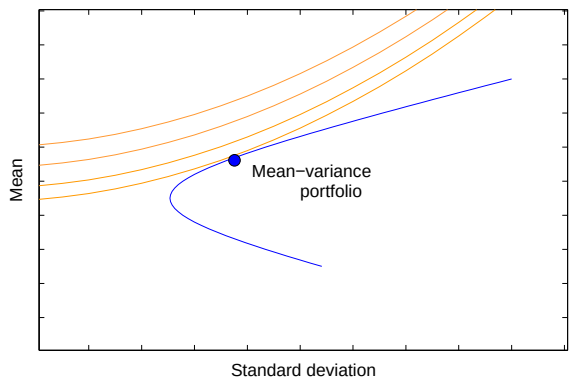

Figure 2.3: *Efficient frontier, utility curves and the optimal mean-variance portfolio*

2.5 Including a risk-free asset

Usually portfolio theory assumes the existence of a risk-free asset next to the n risky assets. Even though a truly risk-free asset exists only in theory, short-dated government bonds or cash are used in practice. For a short horizon, the interest on cash on the bank account is known and fixed and in normal economic conditions it is not likely that a bank will default. Analogously, it is not likely that a government will default. The return on a bond is constant and the short maturity of the bonds protects the investor from small changes in interest rate. Note that the assumption of a risk-free asset is acceptable on short-term problems, but for investors with a long horizon the return is not known.

The risk-free asset has a low, but certain return R^{f} and involves no risk: $\sigma_{\mathrm{f}} = 0$.[4] Furthermore, the risk-free asset is uncorrelated with the risk assets. We denote the weight of the risk-free asset in the portfolio with x_{f}. It is common to assume that one can lend ($x_{\mathrm{f}} > 0$) and borrow ($x_{\mathrm{f}} < 0$) at the risk-free rate.

Just as before we want to derive the efficient frontier by minimizing the variance, given some expected return $\bar{\mu}$:

$$\begin{aligned} \min_{(x,x_{\mathrm{f}})} \quad & x'\Sigma x \\ \text{s.t.} \quad & \mu' x + x_{\mathrm{f}} R^{\mathrm{f}} = \bar{\mu} \\ & e' x + x_{\mathrm{f}} = 1 \end{aligned}$$

Again using a Lagrangian, Engels ([17]) showed that the efficient frontier is defined by all pairs $(\mu_{\mathrm{ef}}, \sigma_{\mathrm{ef}})$ that satisfy:

$$\mu_{\mathrm{ef}} = (\sqrt{c(R^{\mathrm{f}})^2 - 2bR^{\mathrm{f}} + a})\sigma_{\mathrm{ef}} + R^{\mathrm{f}} \tag{2.13}$$

The old efficient frontier (2.8) was a curve, but now we have obtained a linear equation. In other words: the efficient frontier changes to a straight line when a risk-free asset is added. The

[4] For consistency with later chapters we write R^{f} with a superscript rather than a subscript.

new efficient frontier (2.13) is called *Capital Market Line*. The Capital Market Line starts at the risk-free rate and is tangent to the old efficient frontier. The point of tangency is called the *market portfolio*, which contains all risky assets proportional to their availability in the market. The market portfolio has expectation μ_{m} and variance σ_{m}^2. It is shown in figure 2.4.

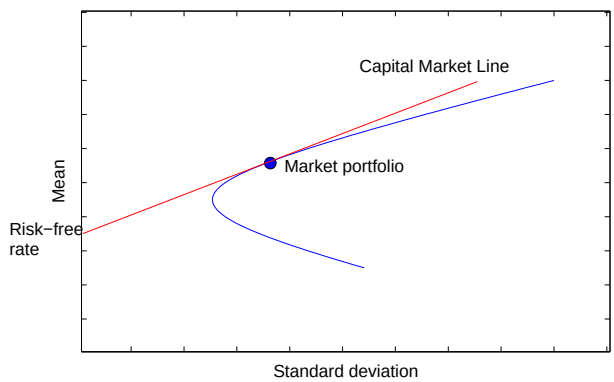

Figure 2.4: *Efficient frontier with Capital Market Line and market portfolio*

An investor now has to make a choice which proportion of his wealth he wants to invest in the market portfolio, x_{m}, and how much he wants to borrow or lend at the risk-free rate, x_{f}. His coefficient of relative risk aversion is γ. Then the mean-variance problem with a risk-free asset is:

$$\begin{aligned} \max_{(x_{\mathrm{m}},x_{\mathrm{f}})} \quad & x_{\mathrm{f}}R^{\mathrm{f}} + x_{\mathrm{m}}\mu_{\mathrm{m}} - \frac{\gamma}{2}x_{\mathrm{m}}^2\sigma_{\mathrm{m}}^2 \\ \text{s.t.} \quad & x_{\mathrm{m}} + x_{\mathrm{f}} = 1 \end{aligned}$$

which is solved by (Engels, [17]):

$$x_{\mathrm{m}} = \frac{\mu_{\mathrm{m}} - R^{\mathrm{f}}}{\gamma\sigma_{\mathrm{m}}^2} \quad \text{and} \quad x_{\mathrm{f}} = 1 - \frac{\mu_{\mathrm{m}} - R^{\mathrm{f}}}{\gamma\sigma_{\mathrm{m}}^2} \tag{2.14}$$

The mean return and variance of the corresponding portfolio are:

$$\begin{aligned} \mu_{\mathrm{mv}*} &= \frac{(R^{\mathrm{f}})^2 - 2\mu_{\mathrm{m}}R^{\mathrm{f}} + \mu_{\mathrm{m}}^2}{\gamma\sigma_{\mathrm{m}}^2} + R^{\mathrm{f}} = \frac{1}{\gamma}(\frac{\mu_{\mathrm{m}} - R^{\mathrm{f}}}{\sigma_{\mathrm{m}}})^2 + R^{\mathrm{f}} =: \frac{s^2}{\gamma} + R^{\mathrm{f}} \\ \sigma_{\mathrm{mv}*}^2 &= \frac{1}{\gamma^2}(\frac{\mu_{\mathrm{m}} - R^{\mathrm{f}}}{\sigma_{\mathrm{m}}})^2 = \frac{s^2}{\gamma^2} \end{aligned}$$

The quantity s is known as the Sharpe ratio. It measures the ratio of expected excess return and the risk taken in terms of standard deviation. (Sharpe, [33])

Assuming the existence of a risk-free asset, we see that every investor who cares only about the mean and variance should invest a part of his wealth in a market portfolio of risky assets, while investing the remainder at the risk-free interest rate. The market portfolio contains the

unique best mix of risky assets and no investor should alter the relative proportions of risky assets within this portfolio.[5] The only choice of the investor is how much he wants to invest in this market portfolio and in the risk-free asset. [6]

2.6 Summary of chapter 2

In this chapter we explained how we can construct the efficient frontier for an investor who can choose between a certain number of risky assets and who uses variance of returns as a measure for risk. The efficient frontier is a curve that contains all the portfolios with an optimal combination of mean and variance of the portfolio return. To calculate this curve, we only need to estimate the first two moments of the joint asset return distribution.

Several points on this curve deserved special attention. First we derived analytically how the portfolio with overall minimum variance should be composed. Next, we explained how the optimal portfolio can be found for an investor who has a trade-off between expected mean return and variance. This trade-off can be modeled by a risk aversion parameter γ.

Finally we added a risk-free asset to the model. This changed the efficient frontier from a hyperbola to a straight line. We observed that, depending on his risk averison, each investor only has to choose between the risk-free asset and the market portfolio, which contains constant proportions of all the risky assets.

[5]This is under the assumption that all investors work with the same estimate of the mean μ and covariance Σ of returns. In economics this is be called *homogeneous expectations*.

[6]This is the *mutual fund theorem* of James Tobin ([37]). The mutual fund theorem implies that every portfolio problem can be generalized towards a problem with only two assets: a risk-free asset and a risky asset, which is the market portfolio, or equivalently, a mutual fund.

Chapter 3

Utility functions

Asset allocation is about decision making under uncertainty. In some way the investor has to come up with a ranking between different asset allocations. For these kind of choice problems, utility functions have been developed, which is the subject of this chapter. In chapter 2 we already implicitly used utility function (2.10) to find the optimal mean-variance portfolio, when we described the investor's attitude towards the trade-off between risk and return by $\mu' x - \frac{\gamma}{2} x' \Sigma x$.

The foregoing formula is a function of mean portfolio return and variance. However, utility functions are not necessarily explicitly dependent on return and risk. It is more common to define utility as a function of realized wealth W or consumption C, where the curvature of the utility function takes into account the investor's risk aversion.[1] As mentioned before, risk aversion is the degree in which one prefers a lower, more certain payoff over a possibly higher but less certain payoff.

An investor will aim at maximizing his expected utility of wealth, instead of maximizing his expected wealth. Now let us define the concept of utility functions formally.

Definition A function $u : \mathcal{S} \to \mathbb{R}$ is called a *utility function* if it is strictly concave, strictly increasing and continuous on $\mathcal{S}$.

If u either defines the utility of wealth W or consumption C, we have that $\mathcal{S} = [0, \infty)$ since we assume that negative wealth or consumption does not exist. Concavity is assumed in order to capture the trade-off between risk and return (Shreve, [35]): if u is not concave, this would imply that an investor is risk-loving and prefers a higher risk for the same return, which is not realistic. Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is concave if:

$$f(\varrho x + (1 - \varrho)y) \geq \varrho f(x) + (1 - \varrho) f(y) \quad \forall x, y \in \mathbb{R},\ \varrho \in (0, 1)$$

The curvature of the utility function determines the intensity of the investor's risk aversion. To measure this, we take the second derivative and scale it by the first derivative of u. Therefore we suppose that the utility function u is twice continuously differentiable on $\mathcal{S}$. The *coefficient*

[1] Next to this, a utility function can be dependent on time, for example by a discount factor or by a certain deadline, after which the utility is decreased.

of absolute risk aversion is defined as:

$$ARA(W) := -\frac{u''(W)}{u'(W)} \tag{3.1}$$

and the *coefficient of relative risk aversion* is defined as:

$$RRA(W) := -W\frac{u''(W)}{u'(W)} \tag{3.2}$$

By the concavity of u both ARA and RRA will be positive.

Hyperbolic absolute risk aversion (HARA) is the most general class of utility functions that is used in practice. It is a rich family, in the sense that by suitable adjustment of the parameters one can create a utility function with absolute or relative risk aversion increasing, decreasing or constant. A function u exhibits HARA if its absolute risk aversion is of the form:

$$ARA(W) = (\frac{W}{\gamma} + \frac{\eta}{\xi})^{-1} \tag{3.3}$$

subject to restrictions $\gamma \neq 0$, $\xi > 0$, $(\frac{\xi W}{\gamma} + \eta) > 0$ and $\eta = 1$ if $\gamma = \infty$ (Merton, [27]). We add that $W \in [0, \infty)$.

The general solution to differential equation (3.3), open to additive and multiplicative constant terms, can be expressed as:

$$u(W) = \frac{\gamma}{1-\gamma}(\frac{\xi W}{\gamma} + \eta)^{1-\gamma} \tag{3.4}$$

Campbell and Viceira ([12]) learn us that three subclasses of HARA utility exist which produce results consistent with mean-variance analysis. The two most used subclasses are the utility functions for which either the absolute risk aversion (3.1) or the relative risk aversion (3.2) is constant. The third type is quadratic utility, for which absolute risk aversion is increasing in wealth.

If RRA is constant and thus $\eta = 0$, we speak of constant relative risk aversion (CRRA).[2] For $\gamma = 1$, the solution to the differential equation (3.3) is log utility, (3.5):

$$u(W) \quad = \log(W) \quad \text{for} \quad \gamma = 1 \tag{3.5}$$

This also follows from (3.4) as γ tends to 1. For $\gamma \neq 1$, we get the power utility function (3.6):

$$u(W) \quad = \tfrac{W^{1-\gamma}}{1-\gamma} \quad \text{for} \quad \gamma \neq 1 \tag{3.6}$$

In this thesis we assume that every individual is risk-averse, which implies $\gamma > 0$.

Secondly, if ARA is constant, we speak of constant absolute risk aversion (CARA). We note that ARA becomes a constant as γ tends to infinity. From the restrictions we have that $\eta = 1$

[2]Note that CRRA implies decreasing absolute risk aversion.

for $\gamma = \infty$. Then the solution to (3.3) is straightforward and is called the exponential utility function:[3]

$$u(W) = 1 - e^{-\xi W} \tag{3.7}$$

where we assume that $\xi > 0$.

Finally, if $\eta \neq 0$ and $\gamma = -1$, we have quadratic utility:

$$u(W) = -\frac{1}{2}\xi^2 W^2 + \eta\xi W - \frac{1}{2}\eta^2$$

Again the solution is open to the addition and multiplication of constants. Usually the quadratic utility is written as

$$u(W) = W - \frac{\zeta}{2}W^2 \tag{3.8}$$

where $\zeta > 0$ is the parameter of increasing risk aversion.

Figure 3.1 below shows the exponential utility function for $\xi = 2$, the power utility function for $\gamma = 2$ and the quadratic utility function for $\zeta = 0.5$. The curvature is obviously different. We see that as long as wealth is low, a small increase of wealth is rewarded by a higher increase in utility than as wealth is high. This effect is most present for the power utility function and less present for the quadratic utility function.

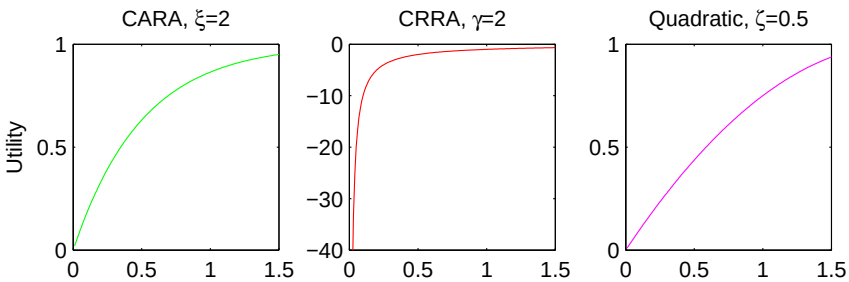

Figure 3.1: *Exponential utility with $\xi = 2$, power utility with $\gamma = 2$ and quadratic utility with $\zeta = 0.5$.*

The difference between the three types of utility is best explained by an example. Suppose we have two investors, A and B. Investor A has a wealth of 10 euros while B owns 1 millon euros. Under CARA, an absolute increase by, for example, 1 euro will result in the same increase in utility for both investors. Under CRRA a relative increase of wealth by, for example, 10% will lead to the same increase in utility for both A and B. Finally, for quadratic utility, an increase of wealth of both 10% or 10 euros will lead to a lower increase in utility for the wealthier investor B than for A.

[3] One can also evaluate the limit of (3.4), using the calculus identity $\lim_{n\to\infty}(1+\frac{x}{n})^{-n} = e^x$. We get that

$$\lim_{\gamma\to\infty} \frac{\gamma}{1-\gamma}(\frac{\xi W}{\gamma}+1)^{1-\gamma} = \lim_{\gamma\to\infty} \frac{\gamma}{1-\gamma}(\frac{\xi W}{\gamma}+1)(\frac{\xi W}{\gamma}+1)^{-\gamma} = -1\cdot 1\cdot e^{-\xi w}$$

This solution is open to addition of constants. It is common to add 1 which gives (3.7).

In the environment of investors, CRRA seems the most realistic assumption. Except Scrooge McDuck, who appears to become more greedy as he becomes richer and thus is the 'living' example of an investor with quadratic utility, introspection suggests that a millonaire will be relatively unconcerned with a risk that might worry a poor person and will pay less to avoid such a risk. Next to this, power and log utility have attractive analytical properties, which we will see later in this thesis. Therefore most authors assume an investor with constant relative risk aversion. In any case, it is commonly thought that absolute risk aversion should not increase in wealth, which makes quadratic utility not a very plausible assumption.

Despite these intuitive reasonings, empirical results on the type of risk aversion and realistic values of parameters are mixed. For example, Schooley and Worden, ([32]), report that an individual's relative risk aversion is empirically constant or decreasing in wealth, depending on the samples and measure of wealth. Friend and Blume, ([19]), found with a high level of significance that in the context of static portfolios households typically have constant relative risk aversion with parameter γ being at least 1, and more likely to exceed 2. Institutional investors probably have a higher γ than households. Bali, ([2]), used multiple time series to conclude with high statistical significance that γ is between 1 and 5. Finally Chou et al., ([13]), found that the coefficient of relative risk aversion might be increasing in wealth which would suggest the use of quadratic utility.

Because empirical results are contradictory, we will investigate the influence of different types of utility functions on the optimal portfolio strategy and the efficient frontier in which this results. As shown by Yu et al., ([39]), optimal portfolio weights differ if different utility functions are used, but they do not show which utility function is optimal. We hope to answer this question.

Chapter 4

From static to dynamic portfolios

In chapter 2 we have seen the framework developed by Markowitz and the extension with a risk-free asset. It is an intuitive and useful framework which is very easy to implement. Therefore classical single-period portfolio choice continues to dominate most academic and practical applications and extensions of portfolio theory. Nevertheless, an obvious shortcoming of mean-variance analysis is the assumption that mean and covariance of asset returns remain constant until the end of the investment horizon. It is not capable of adapting to possible changes in the financial markets.[1] A dynamic asset allocation would enable adaption to these changing conditions and is therefore much more realistic than the static Markowitz-type models, but its implementation involves many complications. This is well described by the following quote:

> *Classical mean-variance brilliantly declares victory and goes home just before the hard part begins* - John. H. Cochrane ([14]).

In section 4.1 we describe two set-ups, under which the dynamic portfolio choice problem can be solved analytically in continuous time. In section 4.2 thereafter discretization of the continuous problem is discussed.

4.1 Dynamic portfolio theory in continuous time

4.1.1 An explicit continuous time solution by Merton

The first contribution to dynamic portfolio choice was by Merton (1969, 1971). Based on the observations that means and variances do not stay constant over time but change in response to economic conditions, investment strategies should be designed to protect against these fluctuations and if possible take advantage of these fluctuations. He considered the portfolio choice problem in continuous time with intertemporal consumption. He concluded that the optimal portfolio should combine a standard, mean-variance component with a intertemporal hedging component to provide insurance against shocks in return moments. (Detemple, [16])

[1] For this reason static, single-period portfolio choice is often referred to as *myopic* portfolio choice, from the Greek word $\mu\omega\upsilon\pi\varsigma$, meaning short-sighted.

Now let us describe the approach of Merton. Suppose there is one risk-free asset with return R^{f} and n risky assets, with prices following geometric Brownian motions and thus are log-normally distributed. Then Merton ([27]) proves a mutual fund theorem very similar to Tobin ([37]):

Theorem 4.1.1. *Given n assets whose changes in prices are log-normally distributed, there exists a unique pair of mutual funds constructed from linear combinations of these assets such that, independent of the form of the utility function, wealth distribution or time horizon, individuals will be indifferent between choosing from a linear combination of these two funds or a linear combination of the original n assets.*

In other words, we can work with just two assets without loss of generality. One asset is risk-free with return R^{f}. The second asset is risky and is in fact a composite of the n risky assets. Merton showed that its price follows a Geometric Brownian Motion:

$$dS_t = \mu S_t dt + \sigma S_t dB_t \tag{4.1}$$

where B_t is a single Wiener process.[2] An investor with horizon T has to decide at time t on the optimal portfolio weights x_t and the optimal amount to consume at each period. At time t, his wealth is denoted by W_t. The dynamics of his wealth are the proportions invested in each asset multiplied with the return on the risky asset plus the proportion invested in the riskfree asset multiplied with the return R^{f}, minus the amount C_t consumed at time t:

$$\begin{aligned} dW_t &= W_t x_t \frac{dS_t}{S_t} + W_t(1-x_t)R^{\mathrm{f}}dt - C_t dt \\ &= ((x_t(\mu - R^{\mathrm{f}}) + R^{\mathrm{f}})W_t - C_t)dt + x_t \sigma W_t dB_t \end{aligned}$$

The investor has two utility functions: one function u which captures his attitude towards consumption throughout the investment period and another function $\tilde{u}$ for his attitude towards final wealth.[3] Both consumption and wealth are discounted by a factor ρ.[4] The objective of the investor can be written as:

$$\mathcal{J}_t(W_t) = \sup_{\{x_s, C_s : s \in (t,T)\}} \mathbb{E}[\int_t^T e^{-\rho s} u(C_s) ds + e^{-\rho T} \tilde{u}(W_T) | \mathcal{F}_t] \tag{4.2}$$

subject to boundary condition $\mathcal{J}_T(W_T) = e^{-\rho T}\tilde{u}(W_T)$. At each time, x_s is also one of the parameters to optimize, because wealth and thus consumption are directly related to the portfolio weights chosen.

Mertons ([26]) shows that for all $t_0 < t < T$ (4.2) can be written as:

$$\mathcal{J}_{t_0}(W_{t_0}) = \sup_{\{x_t, C_t\}} \mathbb{E}[\int_{t_0}^t e^{-\rho s} u(C_s) ds + \mathcal{J}_t(W_t) | \mathcal{F}_0] \tag{4.3}$$

[2] A Wiener process is a stochastic process with three characteristics: (i) $B_0 = 0$, (ii) B_t is almost surely continuous and (iii) B_t has independent increments with distribution $B_t - B_s \sim N(0, t-s)$ for $0 \leq s \leq t$. (Shreve, [34])

[3] For example, an investor might attach a different value to his consumption pattern during his life than to his heritage, which can be seen as his final wealth.

[4] Having 1 euro today is worth more than being promised to get 1 euro one year from today: today's euro can be invested against interest rate r, being worth $1+r$ one year from today. Thus, today's value of a 1 euro one year from now is $\frac{1}{1+r}$. This is at the basis of discounting. As the frequency at which interest is paid out tends to infinity, it can be shown that an amount A invested for n years for rate R grows to Ae^{Rn}. (Hull, [21])

This rewriting is based on the Principle of Optimality of Richard Bellman. The discrete version of this principle is discussed in section 4.2. For the proof of the continuous version we refer to the book of Stokey, Lucas & Prescott ([36]) or Pham ([30]).

To derive an explicit expression for portfolio weights and consumption rules, Merton assumed the bequest function $\tilde{u}(W_T)$ to be equal to 0. Furthermore we assume that u is a member of the HARA class of utility functions (3.4) as described in the previous chapter. Merton shows that the explicit solution of (4.3) is given by:

$$\mathcal{J}_t(W) = \gamma\xi^{\gamma-1}e^{-\rho t}\Big[\frac{\gamma(1-e^{-(\frac{\rho-(1-\gamma)v}{\gamma})(T-t)})}{\rho-\delta v}\Big]^{\gamma}\Big[\frac{W_t}{\gamma}+\frac{\eta}{\xi R^{\mathrm{f}}}(1-e^{-R^{\mathrm{f}}(T-t)})\Big]^{1-\gamma}$$

where $v = R^f + \frac{(\mu-R^{\mathrm{f}})^2}{2\gamma\sigma^2}$. Then the optimal consumption rule and portfolio weights are represented by:

$$C_t^* = \frac{[\rho-(1-\gamma)v][W_t+\frac{\gamma\eta}{\xi R^{\mathrm{f}}}(1-e^{R^{\mathrm{f}}(t-T)})]}{\gamma(1-\exp[\frac{\rho-(1-\gamma)v}{\gamma}(t-T)])} - \frac{\gamma\eta}{\xi} \tag{4.4}$$

$$x_t^* W_t = \frac{\mu-R^{\mathrm{f}}}{\gamma\sigma^2}W_t + \frac{\eta(\mu-R^{\mathrm{f}})}{\xi R^{\mathrm{f}}\sigma^2}(1-e^{R^{\mathrm{f}}(t-T)}) \tag{4.5}$$

Explicit expressions for the portfolio weights and consumption have been found. If we let $\eta = 0$ in (3.4), then the HARA utility reduces to constant relative risk aversion. Then the portfolio weights (4.5) are independent of C_t and W_t and constant in time:

$$x^* = \frac{\mu-R^{\mathrm{f}}}{\gamma\sigma^2} \tag{4.6}$$

Note that these weights are exactly equal to the weight attached to the market portfolio in chapter 2, (2.14). This shows us that the risk aversion γ used in the mean-variance optimalization is exactly the same as the coefficient of relative risk aversion used here.

If we assume CARA utility, thus $\gamma = \infty$ and $\eta = 1$, then the optimal weights are

$$x_t^* W_t = \frac{(\mu-R^{\mathrm{f}})}{\xi\sigma^2}\frac{1-e^{R^{\mathrm{f}}(t-T)}}{R^{\mathrm{f}}}$$

This solution is different from the myopic case. The money value of the risky part of the portfolio, x^*W_t is held constant, only corrected for a kind of discount factor.

We have seen that for the dynamic portfolio choice problem in continuous time a closed-form solution exists if the investor has HARA utility and returns on the risky asset are IID. Moreover, in the special case of constant relative risk aversion, the long-term investor will act the same as in the static formulation of chapter 1.

An investor with CRRA utility and IID returns is not the only type of investor who will act myopically. Samuelson ([31]) showed that an investor with log utility will behave myopically as well, even if asset returns are not IID. Unfortunately these assumptions on asset returns rarely hold in reality and portfolio weights will be different from the myopic case. This is shown exactly in the set-up of Wachter (2002).

4.1.2 Another explicit solution by Wachter

Wachter combines the myopic portfolio strategy with a hedging component. In the model, the investor again has power utility with risk aversion γ. He wants to maximize expected utility of terminal wealth. There is one risky asset, which follows Brownian motion (4.1) with a time-varying mean μ_t, and a risk-free asset with constant return R^{f}. The market price of risk $\theta_t = \frac{\mu_t - R^{\mathrm{f}}}{\sigma}$ follows a mean-reverting Ornstein-Uhlenbeck process:

$$d\theta_t = k(\bar{\theta} - \theta_t)dt - \sigma_\theta dB_t$$

where k, $\bar{\theta}$ and σ_θ are positive constants and θ_0 is given. So Wachter assumes a perfect negative correlation between the market price of risk θ_t and the return on the risky stock.

Provided that $\sigma_\theta^{-2}k^2 + \rho(1 - 2\sigma_\theta^{-1}k) \geq 0$ holds, where $\rho = 1 - \frac{1}{\gamma}$, we can derive the exact portfolio weights. Define the constants:

$$\begin{aligned} G &:= \sigma_\theta^{-1}k - \sqrt{\sigma_\theta^{-2}k^2 + \rho(1 - 2\sigma_\theta^{-1}k)} \\ \alpha &:= 2(k - \sigma_\theta G) \end{aligned}$$

Let $\tau = T - t$ and define the variables:

$$\begin{aligned} B(\tau) &:= \frac{2(1 - e^{-\frac{1}{2}\alpha\tau})^2}{\alpha(\alpha - (\rho - G)\sigma_\theta(1 - e^{-a\tau}))}k\bar{\theta} \\ C(\tau) &:= \frac{1 - e^{-\alpha\tau}}{\alpha - (\rho - G)\sigma_\theta(1 - e^{-a\tau})} \end{aligned}$$

The optimal demand for the risky asset is $x_t^* = x_{1t}^* + x_{2t}^*$, where

$$\begin{aligned} x_{1t}^* &= \frac{1}{\gamma\sigma}\theta_t = \frac{\mu_t - R^{\mathrm{f}}}{\gamma\sigma^2} \\ x_{2t}^* &= -\frac{\rho\sigma_\theta}{\gamma\sigma}[B(\tau) + C(\tau)\theta_t] \end{aligned}$$

The first component is just the mean-variance demand (with a time-varying mean μ_t) which also follows from the myopic problem and Merton's solution. The second component is the intertemporal hedging demand. (Wachter, [38], Oberuc, [29])

4.2 Dynamic portfolio theory in discrete time

Unfortunately closed-form solutions to the continous dynamic problem only exist under very strict assumptions of the return dynamics. Once the number of assets is increased or once the return dynamics become more complicated, currently there are no closed-form analytic solutions to the continuous models, which is why the dynamic portfolio choice problem is often discretized to be solved numerically.

Next to this, an investor with a long-term horizon will presumably not trade continuously. It is more likely that he only rebalances his portfolio at certain discrete timepoints, which is another reason why discretization is a natural step.

4.2.1 Discretization of the dynamic portfolio choice problem

As in the myopic case, we still work with a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. $\mathcal{F}$ is the σ-algebra of events resolved at time T and $\mathbb{P}$ is the probability measure on $\mathcal{F}$. Let $\{\mathcal{F}_t\}_{t\in[0,T]}$ be a filtration on this probability space, where $\mathcal{F}_T = \mathcal{F}$.

At time t, the investor aims at maximizing the expected utility of his wealth[5] at some terminal date T by trading periodically at times $t, t+1, \ldots, T-1$ in n risky assets and one risk-free asset. His wealth at each time is directly related to his wealth one timestep earlier. The investor's problem is given by

$$\begin{aligned} J_t(W_t, Z_t) \quad &= \quad \max_{\{x_s\}_{s=t}^{T-1}} \mathbb{E}[u(W_T)|W_t, Z_t] \\ &\text{s.t.} \quad W_{s+1} = W_s \cdot (x_s' R_{s+1}^{\mathrm{e}} + R^{\mathrm{f}})\, \forall s \geq t \end{aligned} \tag{4.7}$$

where x_s is a vector of portfolio weights on the risky assets chosen at time s and kept until $s+1$. R^{f} is the gross return on the risk-free asset[6] and R_{s+1}^{e} is the vector of excess returns of the risky assets over the risk-free rate. As before, W_t is the investor's wealth at time t. Since the value of W_t is influenced by the decisions on the portfolio, it is an endogenous variable. Finally Z_t is the collection of state variables at time t. State variables are exogenous variables and in this setting mainly economic factors such as interest rate, inflation rate or value of a stock index.[7] Both Z_t and W_t are $\mathcal{F}_t$-measurable.

$J_t(W_t, Z_t)$ is called the *value function.* It is short for the conditional expectation at time t of the utility of terminal wealth W_T, generated by the current wealth W_t, exogenous state variables Z_t and the subsequent optimal portfolio weights $\{x_s\}_{s=t}^{T-1}$. The value function is the discretized version of (4.3) in Merton's portfolio problem.

Clearly, future decisions on the portfolio will depend on the decisions made earlier: one can imagine that an investor with a high realized portfolio return might behave differently from an investor with a very low realized past portfolio return. At each time between time t and T, an optimal decision has to be made, having in mind that future optimal decisions will relate to this decision.

This idea is described by Richard Bellman. In our problem definition a policy is every set $\{x_t\}_{t=1}^{T-1}$ that satisfies the constraints on the portfolio weights.[8] An optimal policy is a policy that maximizes the expected value of $u(W_T)$. Bellman ([5]) formulated the Principle of Optimality:

> *An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.*

[5] Unlike the continuous models, we assume there is no intertemporal consumption. Only the utility of terminal wealth is of importance.

[6] This means that for a risk-free interest rate of 6%, R^{f}=1.06 rather than 0.06.

[7] Technically Z_t also contains the investor's wealth W_t. However, because the investor's wealth is endogenous to the portfolio choice, it is written separately from the exogenous information Z_t

[8] So far we did not mention constraints on x_t, apart from the portfolio weights summing up to one. Natural additional constraints are that all individual weights should in $[0, 1]$, which implies that borrowing and going short is forbidden.

The analytic translation of this statement is discussed in Bellman's book ([4]), but in his article 'Dynamic Programming and Stochastic Control Processes' ([5]) he states that we can rewrite the investor's problem (4.7) as a recursive problem:

$$J_t(W_t, Z_t) = \max_{x_t} \mathbb{E}_t[J_{t+1}(W_{t+1}, Z_{t+1})] \tag{4.8}$$

$$\text{s.t.} \quad W_{s+1} = W_s \cdot (x_s' R_{s+1}^{\mathrm{e}} + R^{\mathrm{f}}) \, \forall s \geq t \tag{4.9}$$

where for notational convenience the conditional expectation is written with a subscript:

$$\mathbb{E}_t[X] = \mathbb{E}[X|W_t, Z_t]$$

The first step of the proof of (4.8) is easy and follows directly from the law of iterated expectations.

$$\begin{aligned} J_t(W_t, Z_t) &= \max_{\{x_s\}_{s=t}^{T-1}} \mathbb{E}_t[u(W_T)] \\ &= \max_{x_t} \max_{\{x_s\}_{s=t+1}^{T-1}} \mathbb{E}_t[\mathbb{E}_{t+1}[u(W_T)]] \end{aligned}$$

The second step is to prove that the interchange of expectation and maximum is allowed:

$$\max_{x_t} \max_{\{x_s\}_{s=t+1}^{T-1}} \mathbb{E}_t[\mathbb{E}_{t+1}[u(W_T)]] = \max_{x_t} \mathbb{E}_t[\max_{\{x_s\}_{s=t+1}^{T-1}} \mathbb{E}_{t+1}[u(W_T)]]$$

Despite the intuitive correctness of this step, the proof of this step is rigorous and too technical to show here.

We see that there exists a recursive relation between the value function at time t and the conditional expectation of the value function at time $t+1$. Equation (4.8) is known as the *Bellman equation* and is at the basis of the discretized portfolio choice problem. The relation between current wealth and wealth one step before, (4.9), is called the *budget constraint*.

4.2.2 Numerical approaches

Since computing power and numerical methods have advanced to the point at which realistic multiperiod portfolio choice problems can be solved numerically, numerical approaches towards dynamic portfolio optimization have become a popular topic. In the last thirteen years many papers have been published with different numerical approaches, amongst others by Brennan, Schwartz and Lagnado (1997, [9]), Campbell and Viceira (1999, [12]) and Barberis (2000, [3]). Unfortunately almost all approaches suffer from inflexibility regarding more complex (and thus more realistic) dynamics of asset returns, the inability to handle more than a few assets or state variables or the absence of constraints on the portfolio weights.

A very elegant numerical approach which is worth mentioning is by Detemple, Garcia and Rindisbacher (2003, [15]). They applied a special kind of stochastic calculus, so-called Malliavin derivatives, to solve the dynamic portfolio choice problem. Nevertheless, also their approach suffers from strict assumptions on the modelling of the market.

In 2005 Brandt, Goyal, Santa-Clara and Stroud published a simulation-based method, which uses a Taylor expansion of the value function and regression over all simulated sample paths after an idea of Longstaff and Schwartz. The method of BGSS is promising, both in terms of speed as flexibility to asset return and state variable dynamics. Therefore we choose to develop this method further. In their article they only apply the method to a two asset problem. We first will test the model in a simple environment and then apply the model on the multiple asset case. As remarked already by Detemple, ([16]) and Garlappi and Skoulakis, ([20]), addition of constraints on the portfolio weights might be necessary to bound the error of the method.

Chapter 5

Methodology

We start this chapter by describing the methodology as published by Brandt, Goyal, Santa-Clara and Stroud in 2005. It involves an expansion of the value function, section 5.1 and a backward recursion process, section 5.3. We give details for an investor with constant relative risk aversion. The conditional expectations are evaluated using across-path regression, which is explained in section 5.4. To adjust the regression for outliers, we discuss various robust regression techniques.

In section 5.5 we discuss the optimal portfolio strategy for higher order Taylor expansions. How to impose constraints is the subject of section 5.6. Finally we show more details on the method and its implementation for an investor with constant absolute risk aversion and quadratic utility in section 5.7.

5.1 Expanding the value function

From the previous chapter, recall the Bellman equation (4.8) with its budget constraint (4.9):

$$\begin{aligned} J_t(W_t, Z_t) &= \max_{x_t} \mathbb{E}_t[J_{t+1}(W_{t+1}, Z_{t+1})] \\ &\text{s.t.} \quad W_{s+1} = W_s \cdot (x_s' R^{\mathrm{e}}_{s+1} + R^{\mathrm{f}}) \, \forall s \geq t \end{aligned}$$

The BGSS method starts with substitution of the budget constraint (4.9) in $J_{t+1}(W_{t+1}, Z_{t+1})$:

$$J_{t+1}(W_{t+1}, Z_{t+1}) = J_{t+1}(W_t(x_t' R^{\mathrm{e}}_{t+1} + R^{\mathrm{f}}), Z_{t+1}) \tag{5.1}$$

Next, we employ a second-order Taylor-expansion of (5.1) around $W_t R^{\mathrm{f}}$, which will lead to an explicit solution for the portfolio weight x_t:[1]

$$\begin{aligned} J_{t+1}(W_t(x_t' R^{\mathrm{e}}_{t+1} + R^{\mathrm{f}}), Z_{t+1}) \approx\ & J_{t+1}(W_t R^{\mathrm{f}}, Z_{t+1}) \\ & + \partial_1 J_{t+1}(W_t R^{\mathrm{f}}, Z_{t+1})(W_t x_t' R^{\mathrm{e}}_{t+1}) \\ & + \frac{1}{2} \partial_1^2 J_{t+1}(W_t R^{\mathrm{f}}, Z_{t+1})(W_t x_t' R^{\mathrm{e}}_{t+1})^2 \end{aligned} \tag{5.2}$$

[1] In section 5.5 we employ higher-order Taylor-expansions to take the skewness and kurtosis of the return distributions into account.

where $\partial_1 J_{t+1}$ denotes the partial derivative with respect to the first variable of the value function. The Taylor expansion (5.2) is substituted in (4.8) and we obtain approximation $\tilde{J}_t(W_t, Z_t)$ for $J_t(W_t, Z_t)$:

$$\begin{aligned}\tilde{J}_t(W_t, Z_t) &= \max_{x_t} \mathbb{E}_t[J_{t+1}(W_t R^{\mathrm{f}}, Z_{t+1}) + \partial_1 J_{t+1}(W_t R^{\mathrm{f}}, Z_{t+1})(W_t x_t' R^{\mathrm{e}}_{t+1}) \\ &\quad + \frac{1}{2}\partial_1^2 J_{t+1}(W_t R^{\mathrm{f}}, Z_{t+1})(W_t x_t' R^{\mathrm{e}}_{t+1})^2] \end{aligned} \tag{5.3}$$

At each time t we can find x_t that maximizes the right-hand side of (5.3) by taking the gradient towards x_t and setting it equal to zero:

$$\begin{aligned}\vec{0} &= \nabla\{\mathbb{E}_t[J_{t+1}(W_t R^{\mathrm{f}}, Z_{t+1}) + \partial_1 J_{t+1}(W_t R^{\mathrm{f}}, Z_{t+1})(W_t x_t' R^{\mathrm{e}}_{t+1}) \\ &\quad + \frac{1}{2}\partial_1^2 J_{t+1}(W_t R^{\mathrm{f}}, Z_{t+1})(W_t x_t' R^{\mathrm{e}}_{t+1})^2]\} \end{aligned} \tag{5.4}$$

Let us assume that the use of the dominated convergence theorem is allowed. Then interchange of expectation and derivative is allowed and we can rewrite (5.4) as:

$$\begin{aligned}\vec{0} &= W_t \mathbb{E}_t[\partial_1 J_{t+1}(W_t R^{\mathrm{f}}, Z_{t+1}) R^{\mathrm{e}}_{t+1}] \\ &\quad + W_t^2 \mathbb{E}_t[\partial_1^2 J_{t+1}(W_t R^{\mathrm{f}}, Z_{t+1})(x_t' R^{\mathrm{e}}_{t+1}) R^{\mathrm{e}}_{t+1}] \end{aligned} \tag{5.5}$$

By the commutativity of the inner product and the associativity of matrix operations we have that $(x_t' R^{\mathrm{e}}_{t+1}) R^{\mathrm{e}}_{t+1} = (R^{\mathrm{e}}_{t+1} R^{\mathrm{e}'}_{t+1}) x_t$. Rewriting (5.5) leads to an explicit expression for x_t, which depends on two conditional expectations. Because this will also be an approximation, we denote the portfolio weight by $\tilde{x}_t$:[2]

$$\begin{aligned}\tilde{x}_t &= -\{W_t \mathbb{E}_t[\partial_1^2 J_{t+1}(W_t R^{\mathrm{f}}, Z_{t+1})(R^{\mathrm{e}}_{t+1} R^{\mathrm{e}'}_{t+1})]\}^{-1} \\ &\quad \times \mathbb{E}_t[\partial_1 J_{t+1}(W_t R^{\mathrm{f}}, Z_{t+1})(R^{\mathrm{e}}_{t+1})] \end{aligned} \tag{5.6}$$

Let us define

$$A_{t+1} := \partial_1 J_{t+1}(W_t R^{\mathrm{f}}, Z_{t+1}) R^{\mathrm{e}}_{t+1} \tag{5.7}$$

$$B_{t+1} := \partial_1^2 J_{t+1}(W_t R^{\mathrm{f}}, Z_{t+1}) R^{\mathrm{e}}_{t+1} R^{\mathrm{e}'}_{t+1} \tag{5.8}$$

so that we can write (5.6) as:[3]

$$\tilde{x}_t = -\{W_t \mathbb{E}_t[B_{t+1}]\}^{-1} \times \mathbb{E}_t[A_{t+1}] \tag{5.9}$$

[2]The rewriting relies on the assumption that matrix $\mathbb{E}_t[\partial_1^2 J_{t+1}(W_t R^{\mathrm{f}}, Z_{t+1})(R^{\mathrm{e}}_{t+1} R^{\mathrm{e}'}_{t+1})]$ is non-singular and thus invertible. Analytically this is not necessarily true. For example, for $n = 1$, neglecting $\partial_1^2 J_{t+1}(W_t R^{\mathrm{f}}, Z_{t+1})$, the matrix is singular if $\mathbb{E}_t[(R^{\mathrm{e}}_{1,t+1})^2] = 0$. For $n = 2$, the inverse does not exist if $\mathrm{Cov}(R^{\mathrm{e}}_{1,t+1}, R^{\mathrm{e}}_{2,t+1} | \mathcal{F}_t) = 0$. But as we will see in section 5.4, the conditional expectations are approximated for each element of the matrix separately, based on simulations and regression analysis. As the number of simulations grows, the probability that the approximated matrix has a determinant equal to zero becomes extremely small.

[3]The two expectations in (5.9) can be seen as the second moment matrix of returns scaled by the second derivative of the value function and the risk premia of the assets scaled by the first derivative of the value function (Brandt et al., [7]).

Now let us take a closer look at the derivative ∂_1 of $J_{t+1}(W_{t+1}, Z_{t+1})$ that is used in (5.7) and (5.8). Recall that by its original definition (4.7), we had:

$$J_{t+1}(W_{t+1}, Z_{t+1}) \quad = \quad \max_{\{x_s\}_{s=t+1}^{T-1}} \mathbb{E}_{t+1}[u(W_T)] \tag{5.10}$$

Using the budget constraints, we can write terminal wealth in terms of current wealth:

$$W_T = W_t \prod_{s=t}^{T-1} (x_s' R_{s+1}^{\mathrm{e}} + R^{\mathrm{f}}) \tag{5.11}$$

Now suppose that for $s = t+1, \ldots, T-1$ the optimal portfolio weights have already been determined. They are denoted by the set $\{\hat{x}_s\}_{s=t+1}^{T-1}$. Then we can write terminal wealth W_T in terms of W_{t+1} and the optimal future weights:

$$W_T = W_{t+1} \prod_{s=t+1}^{T-1} (\hat{x}_s' R_{s+1}^{\mathrm{e}} + R^{\mathrm{f}}) \tag{5.12}$$

For notational convenience, we define the portfolio return from $t+1$ until T under the estimated optimal strategy by:

$$\psi_{t+1} = \prod_{s=t+1}^{T-1} (\hat{x}_s' R_{s+1}^{\mathrm{e}} + R^{\mathrm{f}}) \tag{5.13}$$

so that we can write (5.12) as $W_T = W_{t+1}\psi_{t+1}$. We subsitute this in (5.10). Because the optimal future portfolio weights are assumed to be known already, the maximizer in (5.10) disappears. We get:

$$J_{t+1}(W_{t+1}, Z_{t+1}) \quad = \quad \mathbb{E}_{t+1}[u(W_{t+1}\psi_{t+1})] \tag{5.14}$$

Now let us take the partial derivative of (5.14) towards the first variable, which is W_{t+1} in the above:

$$\partial_1 J_{t+1}(W_{t+1}, Z_{t+1}) = \mathbb{E}_{t+1}[\partial u(W_{t+1}\psi_{t+1})\psi_{t+1}] \tag{5.15}$$

The extra term in the expectation follows from the chain rule. Because u is a function of one variable, we write ∂ instead of ∂_1.

Analogously, for the second derivative of J_{t+1} we get

$$\partial_1^2 J_{t+1}(W_{t+1}, Z_{t+1}) = \mathbb{E}_{t+1}[\partial^2 u(W_{t+1}\psi_{t+1})\psi_{t+1}^2] \tag{5.16}$$

5.2 Simulating sample paths

In the second step Monte Carlo simulation is used to generate M independent sample paths of the vector $\{Y_s\}_{s=1}^T = \{(R_s^{\mathrm{e}}, Z_s)\}_{s=1}^T$:

$$Y_{t+1} = f(Y_t, Y_{t-1}, \ldots; \epsilon_{t+1})$$

where ϵ_{t+1} is a random innovation.

Each sample path m describes one realized evolution of the asset returns and state variables from time 1 to T. The Y_t's can be simulated in several ways, but in econometrics it is common to use vector autoregressive modelling. This is discussed in section 6.1.

5.3 Backward recursion by approximating terminal wealth

For each date t and sample path m we solve recursively backwards to obtain the optimal portfolio weights. Assume that for $s = t+1, \ldots, T-1$ the optimal portfolio weights have already been computed. They are denoted by $\hat{x}_s$.

In the Taylor expansion we used $W_t R^{\mathrm{f}}$, the current wealth growing at the risk-free rate as an expansion point. We follow this and approximate terminal wealth by the current wealth growing at the risk-free rate for one period and subsequently growing at the return generated by the portfolio weights $\hat{x}_s$:[4]

$$\hat{W}_T = W_t R^{\mathrm{f}} \prod_{s=t+1}^{T-1} (\hat{x}_s' R_{s+1}^{\mathrm{e}} + R^{\mathrm{f}}) = W_t R^{\mathrm{f}} \psi_{t+1} \tag{5.17}$$

Using approximation (5.17) for terminal wealth, we can write $J_{t+1}(W_t R^{\mathrm{f}}, Z_{t+1}) = \mathbb{E}_{t+1}[u(\hat{W}_T)]$. Again, because the optimal future portfolio weights have been determined, the maximizer has disappeared.

Now let us evaluate the expression for $\mathbb{E}_t[A_{t+1}]$, which was:

$$\mathbb{E}_t[A_{t+1}] \quad = \quad \mathbb{E}_t[\partial_1 J_{t+1}(W_t R^{\mathrm{f}}, Z_{t+1}) R_{t+1}^{\mathrm{e}}]$$

We subsitute $\partial_1 J_{t+1}(W_t R^{\mathrm{f}}, Z_{t+1})$ by (5.15) with W_{t+1} replaced by $W_t R^{\mathrm{f}}$:

$$\mathbb{E}_t[A_{t+1}] = \mathbb{E}_t[\mathbb{E}_{t+1}[\partial u(W_t R^{\mathrm{f}} \psi_{t+1}) \psi_{t+1}] R_{t+1}^{\mathrm{e}}] \tag{5.18}$$

This is explicitly different than the original article, which mistakenly uses $R^{\mathrm{f}} \psi_{t+1}$ as a chain factor in (5.18). Applying the law of iterated expectations to (5.18) gives:[5]

$$\mathbb{E}_t[A_{t+1}] \quad = \quad \mathbb{E}_t[\partial u(W_t R^{\mathrm{f}} \psi_{t+1}) \psi_{t+1} R_{t+1}^{\mathrm{e}}] \tag{5.19}$$

Analogously we get for $\mathbb{E}_t[B_{t+1}]$:

$$\mathbb{E}_t[B_{t+1}] \quad = \quad \mathbb{E}_t[\partial^2 u(W_t R^{\mathrm{f}} \psi_{t+1}) \psi_{t+1}^2 R_{t+1}^{\mathrm{e}} R_{t+1}^{\mathrm{e}'}] \tag{5.20}$$

The further evaluation of (5.19) and (5.20) will depend on the choice of utility function u. In section 5.7 we consider an investor with exponential or quadratic utility, but first we follow the

[4] The 'true' terminal wealth is $W_T = \hat{W}_T + W_t x_t' R_{t+1}^{'\mathrm{e}} \psi_{t+1}$.

[5] Theoretically the application of the law of iterated expectations is justified. But in practice all conditional expecations are approximated and therefore the law of iterated expectations might not be applicable. This is further discussed in section 5.4.2.

majority of literature and assume an investor with constant relative risk aversion. Then his utility function is either power utility $u(W) = W^{1-\gamma}/(1-\gamma)$ for $\gamma \neq 1$ or log utility $u(W) = \log(W)$ for $\gamma = 1$. For both types we have that $\partial u(W) = W^{-\gamma}$ and $\partial^2 u(W) = -\gamma W^{-\gamma-1}$. We use this in (5.19) and (5.20):

$$\mathbb{E}_t[A_{t+1}] = (W_t R^{\mathrm{f}})^{-\gamma} \mathbb{E}_t[\psi_{t+1}^{1-\gamma} R_{t+1}^{\mathrm{e}}] \tag{5.21}$$

$$\mathbb{E}_t[B_{t+1}] = -\gamma (W_t R^{\mathrm{f}})^{-\gamma-1} \mathbb{E}_t[\psi_{t+1}^{1-\gamma} R_{t+1}^{\mathrm{e}} R_{t+1}^{\mathrm{e}'}] \tag{5.22}$$

We substitute (5.21) and (5.22) in (5.9). Assuming $W_t \neq 0$, simplification gives:

$$\tilde{x}_t = \frac{R^{\mathrm{f}}}{\gamma} \{\mathbb{E}_t[\psi_{t+1}^{1-\gamma} R_{t+1}^{\mathrm{e}} R_{t+1}^{\mathrm{e}'}]\}^{-1} \mathbb{E}_t[\psi_{t+1}^{1-\gamma} R_{t+1}^{\mathrm{e}}] \tag{5.23}$$

For the log utility function (5.23) further simplifies to:

$$\tilde{x}_t = R^{\mathrm{f}} \{\mathbb{E}_t[R_{t+1}^{\mathrm{e}} R_{t+1}^{\mathrm{e}'}]\}^{-1} \times \mathbb{E}_t[R_{t+1}^{\mathrm{e}}] \tag{5.24}$$

We can conclude from (5.23) and (5.24) that under constant relative risk aversion the investment behaviour is indeed independent of current or initial wealth. Moreover, if $\gamma = 1$, the portfolio weight is also independent of the future portfolio return ψ_{t+1}.

The above procedure is for a general step in the backward recursion. Worth mentioning is the first step of the recursion, in which we find the portfolio weight $\tilde{x}_{T-1}$. At time $T-1$ the product in (5.17) is empty and so $\psi_T = 1$. It follows that

$$\begin{aligned} \mathbb{E}_{T-1}[A_T] &= (W_{T-1} R^{\mathrm{f}})^{-\gamma} \mathbb{E}_{T-1}[R_T^{\mathrm{e}}] \\ \mathbb{E}_{T-1}[B_T] &= -\gamma (W_{T-1} R^{\mathrm{f}})^{-\gamma-1} \mathbb{E}_{T-1}[R_T^{\mathrm{e}} R_T^{\mathrm{e}'}] \end{aligned}$$

which leads to

$$\tilde{x}_{T-1} = \frac{R^{\mathrm{f}}}{\gamma} \{\mathbb{E}_{T-1}[R_T^{\mathrm{e}} R_T^{\mathrm{e}'}]\}^{-1} \times \mathbb{E}_{T-1}[R_T^{\mathrm{e}}] \tag{5.25}$$

Expressions (5.23) and (5.24) differ from the formula found by Brandt et al. In their calculations they lose the factor R^{f}, which is not correct. The linear relationship we found between x_t and R^{f} is counterintuitive: if the risk-free rate is higher (and the excess returns remain equal), the allocation to the risky asset will be higher. However, this follows directly from the assumption of constant relative risk aversion. If the risk-free rate is higher and excess returns remain equal, the relative volatility of the risky asset decreases.[6] For an investor with constant relative risk aversion this implies a higher allocation to the risky asset.

5.4 Compute expectations through regressions

The next step is to approximate the two conditional expectations in (5.23). For this we use across-path regression, an idea formulated by Longstaff and Schwartz ([23]). First we explain

[6] The relative volatility is the volatility of the asset return compared to the mean return.

the methodology, then we provide details on the regression methods in subsection 5.4.1. In subsection 5.4.2 some additional notes on this part of the methodology are made.

Recall:

$$\begin{aligned}\tilde{x}_t &= \frac{R^{\mathrm{f}}}{\gamma}\{\mathbb{E}_t[\psi_{t+1}^{1-\gamma}R_{t+1}^{\mathrm{e}}R_{t+1}^{\mathrm{e}'}]\}^{-1}\mathbb{E}_t[\psi_{t+1}^{1-\gamma}R_{t+1}^{\mathrm{e}}] \\ &= \frac{R^{\mathrm{f}}}{\gamma}\{\mathbb{E}_t[B_{t+1}]\}^{-1}\mathbb{E}_t[A_{t+1}]\end{aligned}$$

We still assume that portfolio weights from time $t+1$ up to $T-1$ already have been calculated. Based on the M sample paths of the asset returns, we have M realizations $a_{t+1}^m \in \mathbb{R}^n$ for the vector A_{t+1} and M realizations $b_{t+1}^m \in \mathbb{R}^{n\times n}$ corresponding to matrix B_{t+1}, $m = 1, \ldots, M$.

For each generic element y_{t+1} of A_{t+1} and B_{t+1} we employ an across-path regression. At each time t we use all M paths to fit a linear model between the information at time t, captured by the state variables, and the observations of y_{t+1}. Let X_t denote the matrix with M realizations of the state variables at time t at all simulation path.[7] For simplicity, we assume there is only one state variable Z_t and we use a quadratic polynomial as a basis:[8]

$$X_t = \begin{pmatrix} 1 & Z_{1,t} & Z_{1,t}^2 \\ 1 & Z_{2,t} & Z_{2,t}^2 \\ \vdots & & \\ 1 & Z_{M,t} & Z_{M,t}^2 \end{pmatrix}$$

We assume that for each sample path m there exists a linear relationship between the elements of the corresponding m^{th} row of X_t and the realization $y_{m,t+1}$:

$$y_{m,t+1} = X_{m,t}\beta + \epsilon_m \tag{5.26}$$

for some $\beta \in \mathbb{R}^3$. The ϵ_m's are assumed to be an i.i.d. sequence. With regression we seek an approximation $\hat{\beta}$ of β. The fitted model is:

$$y_{m,t+1} = X_{m,t}\hat{\beta} + e_m$$

where e_m are the residuals.

In our framework, we perform n regressions for $\mathbb{E}_t[a_{t+1}]$ and n^2 regressions for $\mathbb{E}_t[b_{t+1}]$, following the procedure above. Using one of the regression methods of subsection 5.4.1, we can approximate the value of these conditional expectations for each realization of the state variables. Then for each path m the approximate optimal portfolio weight at time t is given by:

$$\hat{x}_{m,t} = \frac{R^{\mathrm{f}}}{\gamma}(\hat{b}_{m,t+1|t})^{-1}\hat{a}_{m,t+1|t}$$

[7] Note that, by only including current values of the state variables, we assume a Markovian problem. The influence of past values of the state variables is assumed to be insignificant.

[8] With more state variables X_t grows larger and cross-sections of the state variables have to be involved.

5.4.1 Regression methods

Different methods exist to fit the regression model (5.26). In this section we consider four methods. Three methods are a part of the same class of M-estimators, the fourth is the so-called least-trimmed squares estimator.

M-estimators

The best known linear estimator is the least-square estimation, which is a specific example of a so-called M-estimator. Generally, an M-estimator minimizes the objective function:

$$\sum_{m=1}^{M} \rho(e_m) = \sum_{m=1}^{M} \rho(y_{m,t+1} - X_{m,t}\hat{\beta})$$

where the function ρ determines the contribution of each residual to the objective function. A reasonable ρ should have the following properties (Fox, [18]):

- $\rho(e) \geq 0$
- $\rho(0) = 0$
- $\rho(e) = \rho(-e)$
- $\rho(e_m) \geq \rho(e_j)$ for $|e_m| > |e_j|$.

Let $\theta = \rho'$ be the derivative of ρ. Differentiating the objective function with respect to the coefficients $\hat{\beta}$ and setting the partial derivatives to zero, produces a system of estimating equations for the coefficients of β. In our example the system consists of three equations:

$$\sum_{m=1}^{M} \theta(y_{m,t+1} - X_{m,t}\hat{\beta})X_{m,t} = 0 \tag{5.27}$$

Define the weight function $w(e) = \theta(e)/e$ and let $w_m = w(e_m)$. Then the system of estimating equations (5.27) can be rewritten as

$$\sum_{m=1}^{M} w_m(y_{m,t+1} - X_{m,t}\hat{\beta})X_{m,t} = 0$$

Solving these equations is a weighted least-squares problem, minimizing $\sum_{m=1}^{M} w_m^2 e_m^2$. An iterative solution is required:

1. Select an initial estimate $\hat{\beta}^{(0)}$, such as the least-squares estimate.
2. At each iteration j calculate residuals $e_m^{(j-1)}$ and associated weights $w_m^{(j-1)} = w(e_m^{(j-1)})$ from the previous iteration.
3. Solve for the new weighted-least-squares estimate:

   $$\hat{\beta}^{(j)} = [X_t' W^{(j-1)} X_t]^{-1} X_t' W^{(j-1)} y_{t+1}$$

   where $W^{(j-1)} = \text{diag}(w_m^{(j-1)})$ is the current weight matrix.

Step 2 and 3 are repeated until the estimated coefficients converge. One can specify a maximum number of iterations or repeat until two subsequent solutions are closer to each other than a certain tolerance level. Supposed the assumption of the underlying linear model is correct, the iterative procedure will converge for real-valued problems. (Bissantz et al, [6]).

We consider three examples of M-estimators: the ordinary least-squares estimator, the Huber estimator and the bisquare estimator.

Least-squares estimator

Under ordinary least-squares estimation we have $\rho_{\text{LS}}(e) = e^2$. Then we get that so $w_{\text{LS}}(e) = 2$. It immediately follows that the least-squares solution has an optimal solution that is obtained without iteration:

$$\hat{\beta} = (X_t' X_t)^{-1} X_t' y_{t+1}$$

Least-squares assigns equal weight to each observation. This can lead to undesirable behaviour if the distribution of the residuals is heavy-tailed.

Huber estimator

We can correct for these heavy tails by considering the Huber and the bisquare estimator. For the Huber estimator all observations have equal weight as long as $|e| < k$. Weights decline when $|e| > k$, eventually tending to zero as $|e|$ tends to infinity. The value k is called the tuning constant.

The corresponding objective and weight function are:

$$\begin{aligned}
\rho_{\text{H}}(e) &= \begin{cases} \frac{1}{2}e^2 & \text{for} \quad |e| \leq k \\ k|e| - \frac{1}{2}k^2 & \text{for} \quad |e| > k \end{cases} \\
w_{\text{H}}(e) &= \begin{cases} 1 & \text{for} \quad |e| \leq k \\ \frac{k}{|e|} & \text{for} \quad |e| > k \end{cases}
\end{aligned}$$

The tuning constant is chosen to give reasonably high efficiency for normal distributions, while still offering protection against outliers. As suggested by Fox, we take $k = 1.345\sigma$. The standard deviation of the errors is approximated by the median absolute deviation, which is a robust measure of spread:

$$\hat{\sigma} = \text{median}_m(|e_m - \text{median}_j(e_j)|)$$

The tuning constant k and the estimate of σ are recalculated at each step of the iteration procedure (Fox, [18]).

Bisquare estimator

The weights assigned by the Tukey bisquare method decline directly as e departs from zero. Moreover, it assigns a weight of zero to all $|e| > k$. Again following Fox's suggestions, we take

$k = 4.685\sigma$. The objective and weight function of the bisquare estimator is

$$\rho_{\text{BS}}(e) = \begin{cases} \frac{k^2}{6}(1 - [1 - (\frac{e}{k})^2]^3) & \text{for} \quad |e| \leq k \\ \frac{k^2}{6} & \text{for} \quad |e| > k \end{cases}$$

$$w_{\text{BS}}(e) = \begin{cases} [1 - (\frac{e}{k})^2]^2 & \text{for} \quad |e| \leq k \\ 0 & \text{for} \quad |e| > k \end{cases}$$

In figure 5.1 the weight functions of the three M-estimators are shown.

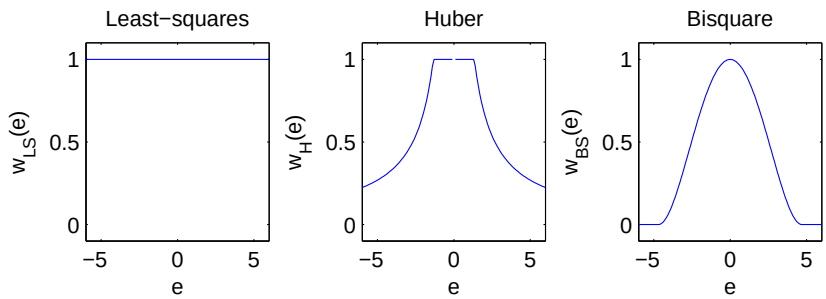

Figure 5.1: *The weight functions for the least-squares, Huber and bisquare estimator, with tuning constants* $k = 1.345$ *for the Huber estimator and* $k = 4.685$ *for the bisquare*

Least-trimmed squares regression

Even M-estimators can be vulnerable to outlying observations. To limit the influence of these observations, one can apply the least-trimmed squares (LTS) regression. The LTS method tries to minimize the sum over the squared residuals over a subset of all M datapoints:

$$LTS(\beta) = \text{argmin}_\beta \sum_{m=1}^{cM} e^2_{(m)} \tag{5.28}$$

where $e^2_{(1)}, e^2_{(2)}, \ldots$ are the order squared residuals and c is the coverage parameter. This parameter can take any value between 0.50 and 1, but typically it is a little more than 0.50.

As we will see, the LTS method has high computations costs. Instead of calculating each regression once (for OLS) or a few times until convergence (Huber and bisquare), the regression has to be calculated for many subsets of the data. Especially when M is large this will be costly. We use the algorithm supplied by Christophe Croux from the Koninklijke Universiteit Leuven, which calculates the regression for a maximum of 500 subsets. (Croux et al., [1])

5.4.2 Additional notes on the regression

Formal justification

Assuming that the linear model (5.26) holds, means that we assume that the value of each conditional expectation can be represented as a linear combination of simple powers of the state variable Z_t.

More generally, we assume that the conditional expectations can be represented as a linear combination of a countable set of $\mathcal{F}_t$-measurable basis functions. This assumption can be formally justified if the conditional expectation is an element of the L^2-space of square-integrable functions relative to some measure. Since L^2 is a Hilbert space[9], it has a countable orthonormal basis and thus we can represent the conditional expectation as a linear function of the elements of the basis. More specifically, let $L_j(Z_t)$ be the jth basis function, so that we can represent each conditional expectation as:

$$\mathbb{E}_t[\cdot] = \sum_{j=0}^{\infty} a_j L_j(Z_t)$$

where the a_j's are constants.

In our algorithm we regress on an ordinary polynomial of order two. This mean we have chosen simple powers of the state variable Z_t as a basis function for the regression: $L_j(Z_t) = Z_t^j$. This choice is not random. Longstaff and Schwartz ([23]) show that the results of the regression method are practically unaffected by the choice of basis functions and reported virtually identical results for the use of ordinary polynomials, Hermite polynomials, Laguerre polynomials and trigonometric series as basis functions. Furthermore, only a few basis functions are needed for a close approximation, which explains why we choose a polynomial of order two.

Value function iteration versus portfolio weight iteration

If we are able to estimate the conditional expectations exactly, applying the law of iterated expectations in (5.18) is permitted, which leads to the solution (5.6) for x_t. In practice we approximate the conditional expectations through regression across simulation paths. The polynomial expansion of the state variables is just an approximation of the true function relation between the state variables and the conditional moments. If this approximation is inaccurate, the law of iterated expectations is not applicable in (5.18) and the conditional expectation of A_{t+1} should be evaluated as:

$$\mathbb{E}_t[A_{t+1}] = (W_t R^{\text{f}})^{-\gamma} \hat{\mathbb{E}}_t[R^{\text{e}}_{t+1} \hat{\mathbb{E}}_{t+1}[\psi_{t+1}^{1-\gamma}]] \tag{5.29}$$

In other words, first $\mathbb{E}_{t+1}[\psi^{1-\gamma}]$ is approximated by regression on Z_{t+1}, afterwards the conditional expectation at t is approximated. This procedure is known as value function iteration.

As shown by Brandt and Van Binsbergen ([8]) portfolio weight iteration and value function iteration are almost equivalent for low values of the investment horizon T and risk aversion γ.

[9] A Hilbert space H is a real or complex inner product space that is also a complete metric space with respect to the distance function induced by the inner product.

For high values of T and γ portfolio weight iteration is advantageous, especially when constraints are imposed on the portfolio weights. Because we want to calculate the asset allocation for a long-term investor with optional constraints on the portfolio weights, we choose to work with portfolio weight iteration.

5.5 Increasing the order of the Taylor expansion

So far we used a second-order Taylor expansion of the value function, (5.9). To adapt for the effect of a non-zero third and fourth moment of excess return, we use a fourth-order Taylor expansion of the value function:

$$\begin{aligned}
\tilde{J}_t(W_t, Z_t) \quad = \quad & \max_{x_t} \mathbb{E}_t[J_{t+1}(W_t R^{\mathrm{f}}, Z_{t+1}) \\
& +\partial_1 J_{t+1}(W_t R^{\mathrm{f}}, Z_{t+1})(W_t x_t' R^{\mathrm{e}}_{t+1}) \\
& +\frac{1}{2}\partial_1^2 J_{t+1}(W_t R^{\mathrm{f}}, Z_{t+1})(W_t x_t' R^{\mathrm{e}}_{t+1})^2 \\
& +\frac{1}{6}\partial_1^3 J_{t+1}(W_t R^{\mathrm{f}}, Z_{t+1})(W_t x_t' R^{\mathrm{e}}_{t+1})^3 \\
& +\frac{1}{24}\partial_1^4 J_{t+1}(W_t R^{\mathrm{f}}, Z_{t+1})(W_t x_t' R^{\mathrm{e}}_{t+1})^4]
\end{aligned}$$

To obtain the maximum, we take the gradient of the foregoing equation with respect to x_t and set equal to zero:

$$\begin{aligned}
\vec{0} \quad = \quad & W_t \mathbb{E}[\partial_1 J_{t+1}(W_t R^{\mathrm{f}}, Z_{t+1}) R^{\mathrm{e}}_{t+1}] \\
& +W_t^2 \mathbb{E}_t[\partial_1^2 J_{t+1}(W_t R^{\mathrm{f}}, Z_{t+1})(x_t' R^{\mathrm{e}}_{t+1}) R^{\mathrm{e}}_{t+1}] \\
& +\frac{1}{2} W_t^3 \mathbb{E}_t[\partial_1^3 J_{t+1}(W_t R^{\mathrm{f}}, Z_{t+1})(x_t' R^{\mathrm{e}}_{t+1})^2 R^{\mathrm{e}}_{t+1}] \\
& +\frac{1}{6} W_t^4 \mathbb{E}_t[\partial_1^4 J_{t+1}(W_t R^{\mathrm{f}}, Z_{t+1})(x_t' R^{\mathrm{e}}_{t+1})^3 R^{\mathrm{e}}_{t+1}]
\end{aligned}$$

Because of the high orders of x, we can only solve the above equation implicitly. We write x_t in terms of the other orders:

$$\begin{aligned}
\tilde{x}_t \quad = \quad & -\{W_t^2 \mathbb{E}_t[\partial_1^2 J_{t+1}(W_t R^{\mathrm{f}}, Z_{t+1}) R^{\mathrm{e}}_{t+1} R^{\mathrm{e}'}_{t+1}]\}^{-1} \\
& \times\{W_t \mathbb{E}[\partial_1 J_{t+1}(W_t R^{f}, Z_{t+1}) R^{\mathrm{e}}_{t+1}] \\
& \quad +\frac{1}{2} W_t^3 \mathbb{E}_t[\partial_1^3 J_{t+1}(W_t R^{\mathrm{f}}, Z_{t+1})(x_t' R^{\mathrm{e}}_{t+1})^2 R^{\mathrm{e}}_{t+1}] \\
& \quad +\frac{1}{6} W_t^4 \mathbb{E}_t[\partial_1^4 J_{t+1}(W_t R^{\mathrm{f}}, Z_{t+1})(x_t' R^{\mathrm{e}}_{t+1})^3 R^{\mathrm{e}}_{t+1}]\}
\end{aligned}$$

We can write this as:

$$\tilde{x}_t = -\{W_t \mathbb{E}_t[B_{t+1}]\}^{-1} \times \{\mathbb{E}_t[A_{t+1}] + \frac{W_t^2}{2}\mathbb{E}_t[C_{t+1}(x_t)] + \frac{W_t^3}{6}\mathbb{E}_t[D_{t+1}(x_t)]\} \tag{5.30}$$

with A_{t+1} and B_{t+1} as in (5.7) and (5.8), and:

$$\begin{aligned} C_{t+1}(x_t) &:= \partial_1^3 J_{t+1}(W_t R^{\mathrm{f}}, Z_{t+1})(x_t' R_{t+1}^{\mathrm{e}})^2 R_{t+1}^{\mathrm{e}} \\ D_{t+1}(x_t) &:= \partial_1^4 J_{t+1}(W_t R^{\mathrm{f}}, Z_{t+1})(x_t' R_{t+1}^{\mathrm{e}})^3 R_{t+1}^{\mathrm{e}} \end{aligned}$$

Again we continue assuming an investor with constant relative risk aversion. With approximation of terminal wealth (5.17) we can write:

$$\begin{aligned} \mathbb{E}_t[C_{t+1}(x_t)] &= -\gamma(-\gamma-1)(W_t R^{\mathrm{f}})^{-\gamma-2}\mathbb{E}_t[\psi_{t+1}^{1-\gamma}(x_t' R_{t+1}^{\mathrm{e}})^2 R_{t+1}^{\mathrm{e}}] \\ \mathbb{E}_t[D_{t+1}(x_t)] &= -\gamma(-\gamma-1)(-\gamma-2)(W_t R^{\mathrm{f}})^{-\gamma-3}\mathbb{E}_t[\psi_{t+1}^{1-\gamma}(x_t' R_{t+1}^{\mathrm{e}})^3 R_{t+1}^{\mathrm{e}}] \end{aligned}$$

Substituting these expressions in (5.30) gives:

$$\begin{aligned} \tilde{x}_t &= -(-\gamma(W_t R^{\mathrm{f}})^{-\gamma-1})^{-1}\{W_t \mathbb{E}_t[\psi_{t+1}^{1-\gamma} R_{t+1}^{\mathrm{e}} R_{t+1}^{\mathrm{e}'}]\}^{-1} \\ &\quad \times\{(W_t R^{\mathrm{f}})^{-\gamma}\mathbb{E}_t[\psi_{t+1}^{1-\gamma} R_{t+1}^{\mathrm{e}}] \\ &\quad -\frac{W_t^2}{2}\gamma(-\gamma-1)(W_t R^{\mathrm{f}})^{-\gamma-2}\mathbb{E}_t[\psi_{t+1}^{1-\gamma}(x_t' R_{t+1}^{\mathrm{e}})^2 R_{t+1}^{\mathrm{e}}] \\ &\quad -\frac{W_t^3}{6}\gamma(-\gamma-1)(-\gamma-2)(W_t R^{\mathrm{f}})^{-\gamma-3}\mathbb{E}_t[\psi_{t+1}^{1-\gamma}(x_t' R_{t+1}^{\mathrm{e}})^3 R_{t+1}^{\mathrm{e}}]\} \end{aligned}$$

Simplification gives:

$$\begin{aligned} \tilde{x}_t &= \frac{R^{\mathrm{f}}}{\gamma}\{\mathbb{E}_t[\psi_{t+1}^{1-\gamma} R_{t+1}^{\mathrm{e}} R_{t+1}^{\mathrm{e}'}]\}^{-1}\mathbb{E}_t[\psi_{t+1}^{1-\gamma} R_{t+1}^{\mathrm{e}}] \\ &\quad -\{\mathbb{E}_t[\psi_{t+1}^{1-\gamma} R_{t+1}^{\mathrm{e}} R_{t+1}^{\mathrm{e}'}]\}^{-1} \times \{-\frac{1}{2}\frac{(\gamma+1)}{R^{\mathrm{f}}}\mathbb{E}_t[\psi_{t+1}^{1-\gamma}(x_t' R_{t+1}^{\mathrm{e}})^2 R_{t+1}^{\mathrm{e}}] \\ &\quad +\frac{1}{6}\frac{(\gamma+1)(\gamma+2)}{(R^{\mathrm{f}})^2}\mathbb{E}_t[\psi_{t+1}^{1-\gamma}(x_t' R_{t+1}^{\mathrm{e}})^3 R_{t+1}^{\mathrm{e}}]\} \\ &= \frac{R^{\mathrm{f}}}{\gamma}\{\mathbb{E}_t[b_{t+1}]\}^{-1}\mathbb{E}_t[a_{t+1}] - \{\mathbb{E}_t[b_{t+1}]\}^{-1} \\ &\quad \times\{-\frac{1}{2}\frac{(\gamma+1)}{R^{\mathrm{f}}}\mathbb{E}_t[c_{t+1}(x_t)] + \frac{1}{6}\frac{(\gamma+1)(\gamma+2)}{(R^{\mathrm{f}})^2}\mathbb{E}_t[d_{t+1}(x_t)]\} \end{aligned} \tag{5.31}$$

Calculating the conditional expectations of $c_{t+1}(x_t)$ and $d_{t+1}(x_t)$ works as in section 5.4.

Expression (5.31) is an implicit expression, where the first part at the righthand side equals the solution obtained for x_t by second-order expansion, (5.23), which we will write as $x_{t,0}$. We approximate the solution of (5.31) by iteration. The iteration process follows

$$\begin{aligned} \tilde{x}_{t,i+1} = \tilde{x}_{t,0} - \{\mathbb{E}_t[b_{t+1}]\}^{-1} \times \{ & -\tfrac{1}{2}\tfrac{(\gamma+1)}{R^f}\mathbb{E}_t[c_{t+1}(\tilde{x}_{t,i})] \\ & +\tfrac{1}{6}\tfrac{(\gamma+1)(\gamma+2)}{(R^{\mathrm{f}})^2}\mathbb{E}_t[d_{t+1}(\tilde{x}_{t,i})]\} \end{aligned} \tag{5.32}$$

We can either choose to end the iteration process after a fixed number of iterations, or continue the iteration process until the difference between two subsequent values is less than a specified tolerance.

Deriving expressions for x_t using higher-order Taylor expansions is straightforward. The $(k+1)$-order Taylor expansion leads to $(k > 2)$:

$$\tilde{x}_t = -\{W_t \mathbb{E}_t[B_{t+1}]\}^{-1} \times \{\mathbb{E}_t[A_{t+1}] + \sum_{j=2}^{k} \frac{W_t^j (R^{\mathrm{f}})^{j-1}}{j!} (-1)^j \frac{(\gamma+j)!}{(\gamma-1)!} \mathbb{E}_t[\psi_{t+1}^{1-\gamma} (x_t' R_{t+1}^{\mathrm{e}})^j R_{t+1}^{\mathrm{e}}]\}$$

5.5.1 Convergence of the iteration procedure

In the previous section we used iteration scheme (5.32) to approximate the optimal portfolio weight under fourth-order Taylor- expansion. We want to show that this iteration scheme converges for an investor with CRRA utility. For illustrative purposes we show this for the most simple case. Assume that the number of simulations M is 1, such that the estimate of the conditional expectation is equal to the simulated value: $\mathbb{E}_t[y_{t+1}] = y_{t+1}$, and assume the number of assets is $n = 1$. We rewrite (5.31) as:

$$\begin{aligned} x_t &= \frac{R^{\mathrm{f}}}{\gamma} \frac{\psi_{t+1}^{1-\gamma} R_{t+1}^{\mathrm{e}}}{\psi_{t+1}^{1-\gamma} (R_{t+1}^{\mathrm{e}})^2} - \frac{(-\frac{1}{2}\frac{(\gamma+1)}{R^{\mathrm{f}}} \psi_{t+1}^{1-\gamma} (R_{t+1}^{\mathrm{e}})^3 x_t^2 + \frac{1}{6}\frac{(\gamma+1)(\gamma+2)}{(R^{\mathrm{f}})^2} \psi_{t+1}^{1-\gamma} (R_{t+1}^{\mathrm{e}})^4 x_t^3)}{\psi_{t+1}^{1-\gamma} (R_{t+1}^{\mathrm{e}})^2} \\ &= \frac{R^{\mathrm{f}}}{\gamma R_{t+1}^{\mathrm{e}}} + \frac{(\gamma+1)}{2R^{\mathrm{f}}} R_{t+1}^{\mathrm{e}} x_t^2 - \frac{(\gamma+1)(\gamma+2)}{6(R^{\mathrm{f}})^2} (R_{t+1}^{\mathrm{e}})^2 x_t^3 =: T(x_t) \end{aligned}$$

The solution x_t^* of this equation is a so-called fixed point, which we approximated by the iteration procedure $x_{t,i+1} = T(x_{t,i})$. Without loss of generality we transform $x_t = T(x_t)$ by multiplying by $\frac{R^e}{R^f}$ and substituting $z := \frac{R_{t+1}^e}{R^f} x_t$. This gives:

$$z = \frac{1}{\gamma} + \frac{\gamma+1}{2} z^2 - \frac{(\gamma+1)(\gamma+2)}{6} z^3 =: \tilde{T}(z)$$

The iterative procedure used to find the solution of the equation above is $z_{n+1} = \tilde{T}(z_n)$. For this we use Banach's fixed point theorem:

Theorem 5.5.1. *Let $(\mathcal{X}, d)$ be a non-empty complete metric space. If there exists a nonnegative real number $q < 1$ such that $d(\tilde{T}(v), \tilde{T}(w)) \leq q \cdot d(v, w)$ for all $v, w \in \mathcal{X}$, then:*

1. *$\exists! z \in \mathcal{X}$: $z = \tilde{z}$,*
2. *$d(\tilde{T}^n y, z) \to 0$ as $n \to \infty$ $\forall y \in \mathcal{X}$.*

In our simple example the dimension is one, so that the norm is just the absolute value. The condition of Banach's theorem is that:

$$\exists q \in (0,1) \quad s.t. \quad \frac{|\tilde{T}(v) - \tilde{T}(w)|}{|v - w|} \leq q \quad \forall v, w \in \mathcal{X} \tag{5.33}$$

which holds if $|\tilde{T}'(z)| \leq q$ for all $z \in \mathcal{X}$, because $\tilde{T}$ is continuous and differentiable. Thus (5.33) is equivalent to:

$$|\tilde{T}'(z)| = |(\gamma+1)z - \frac{(\gamma+1)(\gamma+2)}{2}z^2| \leq q, \quad q \in (0,1) \tag{5.34}$$

For $q = 1$ (5.34) holds for $z_1 < z < z_2$ with:

$$\begin{aligned} z_1 &= \frac{(\gamma+1) - \sqrt{(\gamma+1)^2 + 2(\gamma+1)(\gamma+2)}}{(\gamma+1)(\gamma+2)} \\ z_2 &= \frac{(\gamma+1) + \sqrt{(\gamma+1)^2 + 2(\gamma+1)(\gamma+2)}}{(\gamma+1)(\gamma+2)} \end{aligned}$$

By defining $\mathcal{X} := [z_1 + \delta, z_2 - \delta]$, with $\delta > 0$ some arbitrary small number, for all $v, w \in \mathcal{X}$ we have $d(\tilde{T}(v), \tilde{T}(w)) < d(v, w)$.

In our procedure the starting point for the iteration is $x_{t,0} = \frac{R_f}{\gamma R^e_{t+1}}$. After transformation this is equivalent to $z_0 = \frac{1}{\gamma}$. For $\gamma \geq 2$ we have:

$$z_0 = \frac{1}{\gamma} \leq \frac{2}{\gamma+2} = \frac{2(\gamma+1)}{(\gamma+1)(\gamma+2)} \leq z_2$$

Also $z_0 > z_1$. Since $z_0 \in \mathcal{X}$, the iterative procedure will converge.

Actually, as long as the polynomial of state variables on which we regress has more terms than the number of simulations M, the procedure above holds, because then $\mathbb{E}_t[y_{t+1}] = y_{t+1}$. Of course we usually take a number M which is much larger. However, in various applications we have seen convincing convergence and therefore we might assume that the iteration procedure also works when the conditional expectation is approximated by regression.

5.6 Imposing constraints on portfolio weights

In practice, almost every investor will face constraints on the portfolio weights. If we do not allow borrowing or going short, this means that $0 \leq x_t \leq 1$. If constraints are imposed, the region of portfolio weights is bounded and thus the error is bounded as well.

With only one risky asset imposing constraints is easy. At each step in the recursion process the constrained portfolio weights are:

$$x_t^{\text{constr}} = \max(0, \min(x_t^{\text{unconstr}}, 1))$$

For the case of multiple risky assets, constraining x_t is not that straightforward: the optimum of the constrained problem does not necesarilly follow explicitly from (5.6). Instead we have to use an optimization algorithm for each timestep t and sample path m, which greatly increases computation time. Within this thesis we used the medium-scaled `fmincon` routine in `MATLAB` with default settings. Under a second-order Taylor expansion we get:

$$\max_{x_t} x_t' \mathbb{E}_t[A_{t+1}] - \frac{1}{2}\frac{\gamma}{R^{\text{f}}} x_t' \mathbb{E}_t[B_{t+1}] x_t, \quad s.t.: 0 \leq x_t \leq 1$$

Higher-order Taylor expansions cannot be calculated directly with the `fmincon` routine, because the x_t is inside a conditional expectation. Therefore we first apply the second-order constrained algorithm to find a first solution and apply the iteration procedure (5.32) afterwards. If the iterated solution violates any constraints, we choose to work with the solution obtained by a second-order constrained optimization.

5.7 Alternative utility functions

Until now we assumed an investor with constant relative risk aversion. In this section we derive the equation of the optimal portfolio weights for an investor with constant absolute risk aversion, and for an investor with a quadratic utility function.

5.7.1 Exponential utility

As shown in chapter 3, the utility of an investor with constant absolute risk aversion can be represented by the exponential utility function, which has derivatives $\partial^k u(W) = -(-\xi)^k e^{-\xi W}$. Then (5.19) and (5.20) become:

$$\mathbb{E}_t[A_{t+1}] = \xi \mathbb{E}_t[e^{-\xi(W_t R^{\mathrm{f}} \psi_{t+1})} \psi_{t+1} R^{\mathrm{e}}_{t+1}] \tag{5.35}$$

$$\mathbb{E}_t[B_{t+1}] = -\xi^2 \mathbb{E}_t[e^{-\xi(W_t R^{\mathrm{f}} \psi_{t+1})} \psi^2_{t+1} R^{\mathrm{e}}_{t+1} R^{\mathrm{e}'}_{t+1}] \tag{5.36}$$

Substitution of (5.35) and (5.36) in (5.9) gives:

$$\tilde{x}_t = \frac{1}{\xi W_t} \{\mathbb{E}_t[e^{-\xi W_t R^{\mathrm{f}} \psi_{t+1}} \psi^2_{t+1} R^{\mathrm{e}}_{t+1} R^{\mathrm{e}'}_{t+1}]\}^{-1} \times \mathbb{E}_t[e^{-\xi W_t R^{\mathrm{f}} \psi_{t+1}} \psi_{t+1} R^{\mathrm{e}}_{t+1}] \tag{5.37}$$

At the first step of the recursion process, where $t = T - 1$, we have that $\psi_T = 1$, so that (5.37) reduces to:

$$\tilde{x}_{T-1} = \frac{1}{\xi W_{T-1}} \{\mathbb{E}_{T-1}[R^{\mathrm{e}}_{t+1} R^{\mathrm{e}'}_{t+1}]\}^{-1} \times \mathbb{E}_t[R^{\mathrm{e}}_{t+1}] \tag{5.38}$$

For a Taylor expansion of order four we get the additional terms:

$$\mathbb{E}_t[C_{t+1}(x_t)] = \xi^3 \mathbb{E}_t[e^{-\xi W_t R^{\mathrm{f}} \psi_{t+1}} \psi^3_{t+1} (x'_t R^{\mathrm{e}}_{t+1})^2 R^{\mathrm{e}}_{t+1}]$$

$$\mathbb{E}_t[D_{t+1}(x_t)] = -\xi^4 \mathbb{E}_t[e^{-\xi W_t R^{\mathrm{f}} \psi_{t+1}} \psi^4_{t+1} (x'_t R^{\mathrm{e}}_{t+1})^3 R^{\mathrm{e}}_{t+1}]$$

We substitute this in (5.30) and obtain:

$$\begin{aligned}
\tilde{x}_t = & \frac{1}{\xi W_t} \{\mathbb{E}_t[e^{-\xi W_t R^{\mathrm{f}} \psi_{t+1}} \psi^2_{t+1} R^{\mathrm{e}}_{t+1} R^{\mathrm{e}'}_{t+1}]\}^{-1} \\
& \times \{\mathbb{E}_t[e^{-\xi W_t R^{\mathrm{f}} \psi_{t+1}} \psi_{t+1} R^{\mathrm{e}}_{t+1}] + \frac{\xi W_t}{2} \mathbb{E}_t[e^{-\xi W_t R^{\mathrm{f}} \psi_{t+1}} \psi^3_{t+1} (x'_t R^{\mathrm{e}}_{t+1})^2 R^{\mathrm{e}}_{t+1}] \\
& - \frac{\xi^2 W_t^2}{6} \mathbb{E}_t[e^{-\xi W_t R^{\mathrm{f}} \psi_{t+1}} \psi^4_{t+1} (x'_t R^{\mathrm{e}}_{t+1})^3 R^{\mathrm{e}}_{t+1}]\}
\end{aligned}$$

We see that at each time t the portfolio weights still depend on the endogenous variable W_t, which is not known yet at time t. This complicates the process and asks for a different approach than for an investor with CRRA. This is discussed in section 5.7.3.

5.7.2 Quadratic utility

The third and final type of utility which we consider is the quadratic utility function (3.8), with derivatives $\partial u(W) = 1 - \zeta W$ and $\partial u(W) = -\zeta$.[10] Then (5.19) and (5.20) become:

$$\mathbb{E}_t[A_{t+1}] = \mathbb{E}_t[(1 - \zeta W_t R^{\mathrm{f}} \psi_{t+1})\psi_{t+1} R^{\mathrm{e}}_{t+1}] \tag{5.39}$$

$$\mathbb{E}_t[B_{t+1}] = -\zeta \mathbb{E}_t[\psi^2_{t+1} R^{\mathrm{e}}_{t+1} R^{\mathrm{e}'}_{t+1}] \tag{5.40}$$

Substitution of (5.39) and (5.40) in (5.9) gives the equation for portfolio weights:

$$\tilde{x}_t = \frac{1}{\zeta W_t}\{[\psi^2_{t+1} R^{\mathrm{e}}_{t+1} R^{\mathrm{e}'}_{t+1}]\}^{-1} \times \mathbb{E}_t[(1 - \zeta W_t R^{\mathrm{f}} \psi_{t+1})\psi_{t+1} R^{\mathrm{e}}_{t+1}] \tag{5.41}$$

The first step of the recursion process is:

$$\tilde{x}_{T-1} = \frac{1}{\zeta W_{T-1}}\{[R^{\mathrm{e}}_T R^{\mathrm{e}'}_T]\}^{-1} \times \mathbb{E}_{T-1}[(1 - \zeta W_{T-1} R^{\mathrm{f}}) R^{\mathrm{e}}_{t+1}] \tag{5.42}$$

Again we notice that portfolio weights explicitly depend on wealth.

5.7.3 Constructing a grid for W_t

If we assume exponential or quadratic utility, portfolio weights x_t dependend of the endogenous variable wealth, which is unknown at time t. To be able to calculate the portfolio weights, we follow the next procedure:

- At each time t we construct a grid of size K for the endogenous variable W_t.The kth gridpoint is denoted as $W_{t,k}$. This grid is constructed as follows:

  $$W_1 \cdot (R^{\mathrm{f}} \cdot (1-d))^{t-1}, W_1 \cdot (R^{\mathrm{f}} \cdot (1-d+\Delta))^{t-1}, \ldots, W_1 \cdot (R^{\mathrm{f}} \cdot (1+h))^{t-1}$$

  where parameters h and d define the upper and lower limit of the grid and Δ is the stepsize.
- Using backward recursion, suppose that for $t+1$ we have already calculated the sequence of optimal portfolio weights $\{\hat{x}_{s,k}\}_{s=t+1}^{T-1}$ and thus we know the values of $\psi_{t+1,k}$ for all gridpoints $W_{t+1,k}$. Note that we have M sequences of optimal portfolio weights and M realizations of $\psi_{t+1,k}$ at each gridpoint of W_{t+1}.
- The next step is to calculate the optimal portfolio weights at time t, which has to be done for each gridpoint $W_{t,k}$ separately. Let $W_{t,k}$ be fixed. To be able to calculate the current optimal weights, we need the optimal future weights.

[10] Because all the third and higher derivatives of the quadratic utility function are zero, there is no need for a fourth-order Taylor expansion.

- We approximate W_{t+1} by $W_{t,k}R^{\mathrm{f}}$. The point $W_{t,k}R^{\mathrm{f}}$ is not necessarily exactly on the grid of W_{t+1}. To obtain the sequence of future optimal portfolio weights we use linear interpolation. First we find a grid point g such that $W_{t+1,g} \leq W_{t,k} \leq W_{t+1,g+1}$. Then for gridpoint $W_{t,k}$ the sequence of optimal future portfolio weights is approximated by:

$$\begin{aligned}\{\hat{x}_s\}_{s=t+1}^{T-1} \approx \quad & \tfrac{W_{t,k}R^{\mathrm{f}}-W_{t+1,g}}{W_{t+1,g+1}-W_{t+1,g+1}}\{\hat{x}_{s,g+1}\}_{s=t+1}^{T-1} \\ & + \tfrac{W_{t+1,g+1}-W_{t,k}R^{\mathrm{f}}}{W_{t+1,g+1}-W_{t+1,g+1}}\{\hat{x}_{s,g}\}_{s=t+1}^{T-1}\end{aligned}$$

  Now we are able to calculate the M values of ψ_{t+1} that correspond to $W_{t,k}$.

- This gives us sufficient information to estimate the optimal portfolio weights at time t for gridpoint $W_{t,k}$.

Chapter 6

Implementation in a simple setting

For a first insight into the behaviour of our methodology, we use a simple data generating model with only one risky asset next to the risk-free asset. The use of a simple model gives the opportunity to test the method with various extensions. The set-up of this model is discussed in section 6.1. To compare different strategies performance measures are introduced in section 6.2. In section 6.3 we calculate actual optimal asset allocations and in the sections thereafter we investigate the effect of various extensions and the sensitivity towards parameters under the assumption of constant relative risk aversion. For the three discussed types of utility we do a sensitivity analysis in section 6.8. Finally the dynamic strategies are compared with the static strategy of chapter 2 in section 6.9.

6.1 A simple VAR-model

In econometrics, the most common way to model the economy is as a vector autoregressive model, better known as a VAR-model. A VAR(p)-model describes the evolution of a vector $y_t \in \mathbb{R}^k$ over the same sample period $t = 1, \ldots, T$ as a linear function of its past evolution. Formally:

Definition The sequence $\{y_t\}_{t=1,\ldots,T}$ follows a *VAR(p)-process* if its dynamics can be written as:

$$y_t = c + A_1 y_{t-1} + A_2 y_{t-2} + \ldots + A_k y_{t-p} + \epsilon_t \tag{6.1}$$

with $c \in \mathbb{R}^k$ (the intercept), $A_i \in \mathbb{R}^{k \times k}$ and $\epsilon_t \in \mathbb{R}^k$ is a vector of error terms satisfying:

- $\mathbb{E}(\epsilon_t) = 0$
- $\mathbb{E}(\epsilon_t \epsilon_t') = \Sigma$
- $\mathbb{E}(\epsilon_t \epsilon_{t-k}') = 0 \quad \forall k \neq 0$

This means all error terms have mean zero, its contemporaneous covariance matrix Σ is constant in time and there is no correlation of error terms across time. (Brockwell and Davis, [10])

We assume a riskfree asset and only one risky asset or stock. We use a VAR(1)-model to model the log dividend yield d_t[1] and the log excess return of a stock index, which is the logarithm of the stock index return divided by the risk-free rate: $r_t^{\mathrm{e}} = \log \frac{R_t}{R^{\mathrm{f}}}$.

The model is on a quarterly basis and calibrated on the value-weighted CRSP-index.[2] The coefficients are equal to those used by Brandt et al. ([7]). The model is given by:

$$\begin{bmatrix} r_{t+1}^{\mathrm{e}} \\ d_{t+1} \end{bmatrix} = \begin{bmatrix} 0.227 \\ -0.155 \end{bmatrix} + \begin{bmatrix} 0 & 0.060 \\ 0 & 0.958 \end{bmatrix} \begin{bmatrix} r_t^{\mathrm{e}} \\ d_t \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t+1} \\ \epsilon_{2,t+1} \end{bmatrix} \tag{6.2}$$

where the quarterly innovations are binormally distributed:

$$\begin{bmatrix} \epsilon_{1,t+1} \\ \epsilon_{2,t+1} \end{bmatrix} \sim N(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.0060 & -0.0051 \\ -0.0051 & 0.0049 \end{bmatrix}) \tag{6.3}$$

This implies an almost perfect negative correlation ($\rho = -0.95$) between r_{t+1}^{e} and d_{t+1}. This is explained by the fact that if few or no dividend payments are made, this value remain intrinsic in the stock, which will cause a rise in stock prices.

To find a seed for the log dividend yield, we assume it is a stationary process of which the expectation is the same at each step. Then $d_0 = -0.155/(1 - 0.958) = -3.69$.

We assume a constant risk-free rate at 6% per year. Since the VAR-system (6.2) is defined quarterly, this translates to a risk-free rate of $R^{\mathrm{f}} = 1.06^{0.25} = 1.0147$. Under this set-up of the VAR-model, the assumption on the risk-free rate does not influence the portfolio weights, only the terminal wealth.

In the methodology we used the excess return R_t^{e}, while the VAR-model simulates the log excess return r_t^{e}. To calculate the excess return from the log excess return, we apply the following transformation:

$$R_t^{\mathrm{e}} = R^{\mathrm{f}}(e^{r_t^{\mathrm{e}}} - 1) \tag{6.4}$$

Throughout the whole chapter we use Monte Carlo samples of size 10,000.

6.2 Performance measures

To see the advantages of a dynamic strategies, we compare the strategy with several static strategies. Five descriptive statistics in terms of terminal wealth W_T are used to compare performance. We will display mean $\hat{\mu}$ and standard deviation $\hat{\sigma}$ of the realizations of W_T, as well as the (empirical) probability of underperforming the risk-free strategy:

$$P_{\mathrm{upf}} = P(W_T < W_1 \cdot (R^{\mathrm{f}})^{T-1})$$

[1] The log dividend yield is the logarithm of a company's annual dividend payments divided by the number of stocks issued.

[2] CRSP stands for Centre for Research of Security Prices. Its database contains daily price quotations for common stocks traded on New York and American stock exchanges and NASDAQ. The value-weighted market index is a weighted average of all stock returns, with the weights given by the market value of the stock issue (price times shares outstanding) at the end of the previous trading period.

where W_1 denotes the investor's initial capital.

Furthermore we apply two risk measures which have become popular lately: the Value-at-Risk (VaR) and the expected shortfall, also known as conditional Value-at-Risk (cVaR). Usually Value-at-Risk is defined in terms of loss. A higher loss meaning a higher risk, the VaR is calculated from the upper quantile from the loss distribution. Because we work with wealth instead of loss, the VaR and cVaR are calculated from the lower quantile of the wealth distribution. Nevertheless, the concept of the two statistics remains the same.

In this thesis VaR and cVaR with confidence level $\alpha \in (0, 1)$ are defined as:

$$\begin{aligned} \mathrm{VaR}_\alpha &= \sup(y \in \mathbb{R} : P(W_T \geq y) = \alpha) \\ \mathrm{cVaR}_\alpha &= \mathbb{E}[W_T | W_T < \mathrm{VaR}_\alpha] \end{aligned}$$

In words, a VaR_α of V says we are α percent certain that final wealth will not be lower than V. In the case that final wealth will be lower than V, the cVaR_α is the conditional expectation of final wealth in those $(1 - \alpha)\%$ worst scenarios. (Hull, [22])

Since we work with simulations, all performance measures will be approximated. Throughout this thesis we will use $\alpha = 0.975$ unless stated otherwise.

6.3 A first look at the gains of a dynamic strategy

We generated 10,000 simulations from the VAR-model (6.2) and applied the unconstrained algorithm based on a second-order Taylor-expansion for an investor with constant relative risk aversion and an investment horizon of 20 quarters. The investor uses a dynamic strategy and rebalances quarterly. For the regression we use ordinary least-squares.

Under our methodology the allocation to stocks is different for each simulation path. To visualize the strategy, we calculated the mean allocation that followed from all these individual strategies, as well as the trend and the standard deviation. In figure 6.1 the mean allocation to stocks is shown against the investment horizon $T - t$ for this investor. The dashed lines represent the allocation to stocks one standard deviation under and above the mean allocation. Even though we look at the mean of a large number of simulations, there are still intertemporal fluctuations, which relate directly to the underlying data process. If we would plot the mean of R^{e}_{t+1}, the same fluctuations are observed.

Nevertheless, behind this fluctuations we can see a trend of increasing allocation to stock for a longer investment horizon. This is comparable to the results we find in most literature: the longer the investment horizon $T - t$ still is, the larger the allocation to stock is.

In table 6.1 performance measures are shown for an initial wealth of $W_1 = 100$. As mentioned, the dynamic strategy is path-dependent. We also considered the use of a path-independent dynamic strategy, which is either computed as the mean or as the underlying trend of all individual dynamic strategies.

The gains of a dynamic strategy are clear. The mean of terminal wealth is high, only slightly underperforming the strategy that invests fully in stocks, but all other performance measures

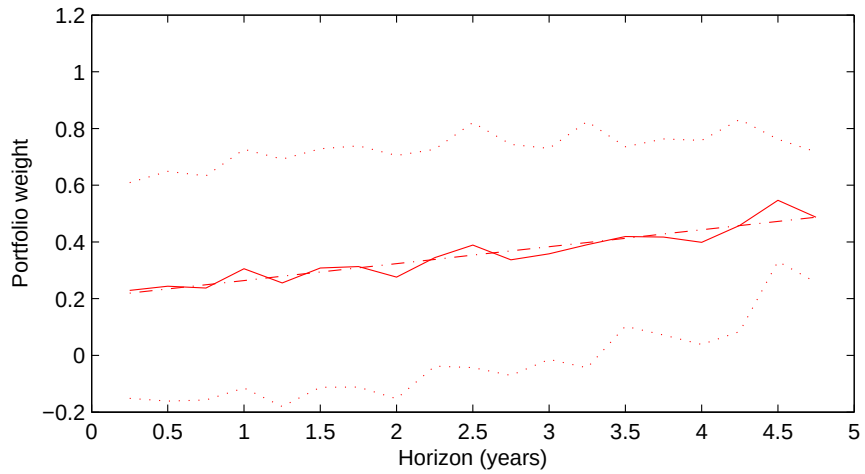

Figure 6.1: *Mean allocation to stocks (solid line) and its underlying trend, plus/minus one standard deviation (dashed lines) under a second-order Taylor expansion,* $T = 20$, $\gamma = 5$.

Strategy	$\hat{\mu}$	$\hat{\sigma}$	P_{upf}	$\text{VaR}_{0.975}$	$\text{cVaR}_{0.975}$
100 % risk-free asset	131.8	0	0	131.8	131.8
100 % stock	150.4	36.0	0.33	91.6	84.7
Path-dependent dynamic	149.4	16.1	0.12	114.3	104.6
Dynamic trend	139.2	11.9	0.28	117.2	113.8
Dynamic mean	139.2	12.0	0.28	117.0	113.8

Table 6.1: *Performance measures of different portfolio strategies under the basic VAR-model,* $T = 20$, $\gamma = 5$.

have much more favourable values. The probability of underperforming the risk-free strategy is less than half the value of the 100% stock strategy. Furthermore we see that the idea to use a general, path-independent dynamic strategy based on either mean or trend of all dynamic strategies is not advisable.

These results are visualized in figure 6.2, where histograms of W_T are displayed under two portfolio strategies. The red vertical line is the terminal wealth obtained by investing fully in the risk-free asset, which grows at 6% each year: the 'risk-free' strategy.

6.4 Effect of constraining the portfolio weights

With constraints on $\{x_t\}_{\{t=1,\ldots,T\}}$, the allocation to stocks is slightly higher on average for the same sample paths of r^{e}_{t+1} and d_t. However, in terms of the statistics of W_T the differences are not significant, as is shown in table 6.2. Results are almost equal after imposing constraints on x_t.

Furthermore, we find that the constraining of portfolio weights happens more often in the first

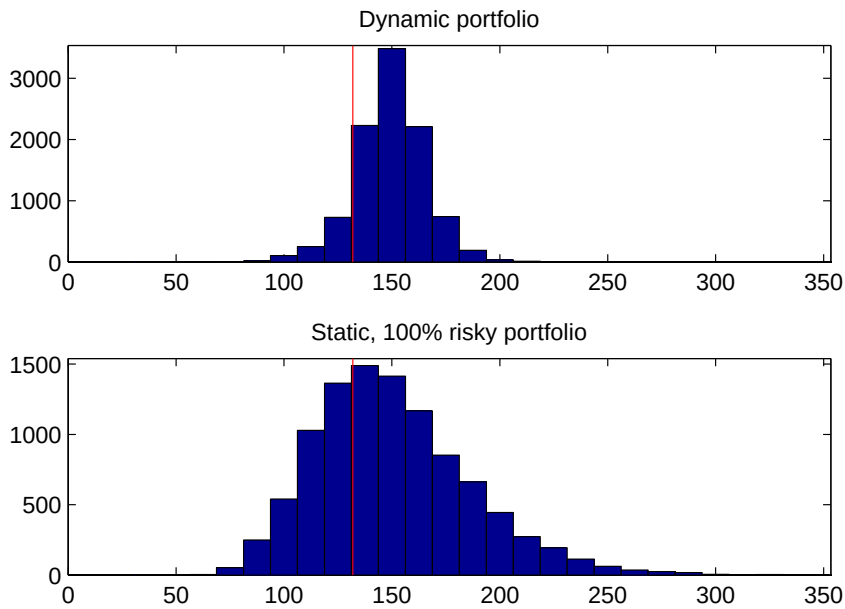

Figure 6.2: *Histograms of W_T under different portfolio strategies, $T = 20$, $\gamma = 5$.*

Strategy	$\hat{\mu}$	$\hat{\sigma}$	P_{upf}	$\text{VaR}_{0.975}$	$\text{cVaR}_{0.975}$
Unconstrained	149.4	16.1	0.14	114.3	104.6
Constrained	147.6	15.1	0.14	114.4	104.3

Table 6.2: *The differences between constrained and unconstrained portfolio weights, $T = 20$, $\gamma = 5$.*

steps in the recursion process. However, this might relate to the fact that once weights in earlier steps have been constrained, it becomes less likely to generate extreme portfolio weights in the following steps. We have to mention though, that for other data generating processes or values of γ, results between the constrained and unconstrained process might differ significantly.

Because constraints on x_t give a more realistic setting, it is advisable to use them.

6.5 Effect of a higher-order Taylor expansion

We extend the unconstrained algorithm with both a fourth, sixth and eighth-order Taylor expansion. The iteration process (5.32) uses a tolerance level of 10^{-3}, with a maximum of 20 iteration steps. For numerical reasons we take a lower value of γ and T and we generate new sample paths.

One can see in figure 6.3 that the mean allocation on stocks is changed slightly by a higher-order expansion. On average portfolio weights become higher for a higher-order expansion. This effect

is especially present for the change of a second-order to a fourth-order expansion.

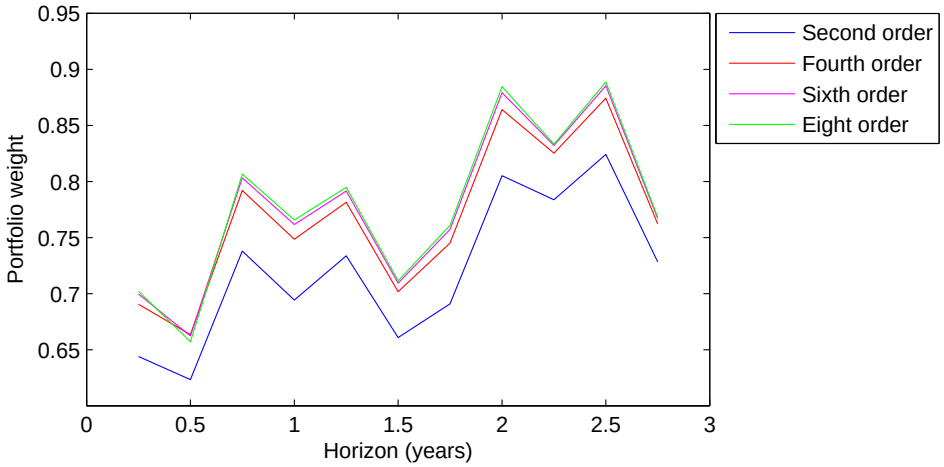

Figure 6.3: *Mean allocation to stocks under a second, fourth and eighth-order Taylor expansion,* $T = 12$, $\gamma = 2$.

In table 6.3 the improvements by using a fourth, sixth and eighth-order Taylor-expansion are quantified in terms of W_T-statistics. Comparing the higher order expansions to the second-order expansion, little improvements are observed on mean and probability of underperforming the risk-free strategy, while $\text{VaR}_{0.975}$ and $\text{cVaR}_{0.975}$ decline slightly. While the difference between the second and fourth-order expansion is still significant, the influence of including more terms is neglible.

Strategy	$\hat{\mu}$	$\hat{\sigma}$	P_{upf}	$\text{VaR}_{0.975}$	$\text{cVaR}_{0.975}$
Dynamic second-order	132.6	24.5	0.25	78.4	65.5
Dynamic fourth-order	133.5	25.9	0.24	76.3	62.7
Dynamic sixth-order	133.6	26.2	0.24	76.0	62.1
Dynamic eighth-order	133.7	26.3	0.24	75.7	61.9

Table 6.3: *Performance measures of portfolio strategies based on a different order of the Taylor-expansion,* $T = 12$, $\gamma = 2$.

Using the constraints on x_t, the difference between a second and a fourth-order expansion is still present, but under the fourth, sixth and eight-order expansion the portfolio weights are almost equal. Differences between statistics of W_T are insignificant. We can conclude that both with or without constraints a fourth-order expansion will be sufficient.

6.6 Numerical issues for higher values of γ and T

The gains of a dynamic strategy are clear and under a second-order expansion, a horizon of 20 quarters, risk aversion $\gamma = 5$, ordinary least-squares regression and no constraints the portfolio

weights have acceptable values. Unfortunately the method does not work properly for high values of T or γ. An increase of the horizon to $T = 30$ quarters gives figure 6.4. The big peaks in the standard deviation are caused by extreme portfolio weights of 9,500,000%, which is obviously unrealistic. The location and magnitude of such extreme portfolio weights differ

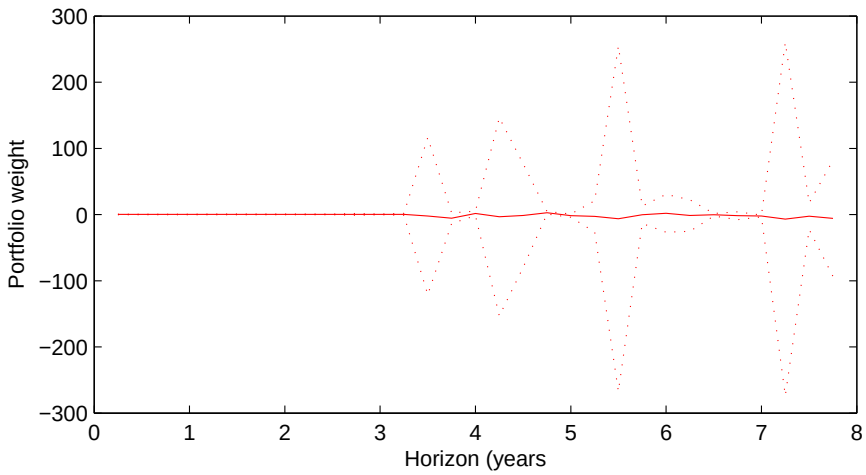

Figure 6.4: *Mean allocation to stocks (solid line), plus/minus one standard deviation (dashed lines) under a second-order Taylor expansion,* $T = 32$, $\gamma = 5$.

between simulation runs, but their presence is persistent. In this section we deal with possible causes and solutions.

6.6.1 Explanation of the cause

When the horizon T or risk version γ is increased, the value of $\psi_{t+1}^{1-\gamma}$, which functions as a scaling factor in the conditional expectations, takes unfavorable values. Recall that:

$$\psi_{t+1}^{1-\gamma} = (\prod_{s=t+1}^{T-1} (\hat{x}_s' R_{s+1}^{\mathrm{e}} + R^{\mathrm{f}}))^{1-\gamma}$$

We give a numerical example to illustrate this. Suppose that for one sample path we have a portfolio return of 1.02 on quarterly basis. For another sample path we succeed to obtain a portfolio weight of 1.20 each quarter. The corresponding values of $\psi_{t+1}^{1-\gamma}$ are $1.02^{(1-\gamma)(T-1-t)}$ and $1.20^{(1-\gamma)(T-1-t)}$.

Now suppose that $\gamma = 5$ and that ten recursion steps have been evaluated, so $T - 1 - t = 10$. For the two sample paths, $\psi_{t+1}^{1-\gamma}$ takes values of respectively $1.02^{-40} = 4.5 \cdot 10^{-1}$ and $1.20^{-40} = 6.8 \cdot 10^{-4}$. Continuing in the recursion process, after thirty timesteps we get $1.02^{-120} = 9.2 \cdot 10^{-2}$ and $1.20^{-120} = 3.1 \cdot 10^{-10}$. In other words, as we get further in the recursion process, $\psi_{t+1}^{1-\gamma}$ tends to zero. More important, the spread of all realizations of $\psi_{t+1}^{1-\gamma}$ becomes very large not even mentioning the effect of γ being one unit higher.

In `MATLAB` we immediately find that these two effects lead to the extreme portfolio weights as observed in figure 6.4. As γ tends to zero too fast, the matrix $\mathbb{E}_t[\psi_{t+1}^{1-\gamma} R_{t+1}^{\mathrm{e}} R_{t+1}^{\mathrm{e}'}]$ becomes singular. But, more persistent is the spread in $\psi_{t+1}^{1-\gamma}$, resulting in an inability of the ordinary least-squares regression to properly estimate the conditional expectations.

This is underlined by the fact that the algorithm works perfectly for all magnitudes and combinations of parameters T and γ if the factor $\psi_{t+1}^{1-\gamma}$ is absent in the conditional expectations. Unfortunately eliminating the factor $\psi_{t+1}^{1-\gamma}$ is only allowed if the intertemporal returns are independent, $\mathrm{Cov}(R_t^{\mathrm{e}}, R_{t+1}^{\mathrm{e}}) = 0$. For realistic models of the asset excess returns this is generally not true.[3]

Before we discuss various adjustments of the methodology, aimed at reduction of these implementational issues, we need to make another remark. Even though the malperformance manifests itself by numerical issues, the real cause may be more fundamental. At some point, the state variables do not contain enough information for a proper prediction of the future optimal portfolio return. For $t = T - 1$, the regression matrix X_{T-1} only has to predict the value of R_T^{e}, while for a small value of t, X_t is a predictor for both R_{t+1}^{e} and ψ_{t+2}, which contains R_{t+2}^{e} up to R_T^{e} and all future optimal portfolio weights $\hat{x}_{t+1}$ to $\hat{x}_{T-1}$. It should be clear that as we get too far in the recursion process, it is unlikely that the value of the state variables at one single moment will contain enough information to properly predict the whole evolution of future asset returns, being R_{t+1}^{e} up to R_T^{e}.[4]

6.6.2 Adjustment of the scaling factor $\psi_{t+1}^{1-\gamma}$

If the number of terms in ψ_{t+1} increase, its spread will increase as well. Therefore we try to limit the spread by limiting the number of terms in ψ_{t+1} by k, so that after k recursion steps the spread of $\psi_{t+1}^{1-\gamma}$ will stay of the same order.

Under this adjustment the algorithm works well for higher values of γ and T, but the portfolio weights differs significantly from the portfolio weights obtained by the algorithm that uses all terms for ψ_{t+1}. This is visualized in figure 6.5. It does not need explanation that notable changes in portfolio weights goes hand in hand with significant differences in final wealth.

In fact, the differences in portfolio weights are caused by a change of the approximation of W_T. Limiting the number of terms in ψ_{t+1} by k means W_T is approximated by:

$$\hat{W_T} = W_t (R^{\mathrm{f}})^{\max(0, T-1-(t+k))} \prod_{s=t+1}^{\min(t+k, T-1)} (\hat{x}_s' R_{s+1}^{\mathrm{e}} + R^{\mathrm{f}}) \tag{6.5}$$

We see in formula (6.5) that the approximation of W_T will be the same as originally in (5.17) as long as the horizon is shorter than $k + 1$. Corresponding to this, we see in figure 6.5 that for these horizons the portfolio weights coincide with the portfolio weights obtained by the full

[3] In the case that $\mathrm{Cov}(R_t^{\mathrm{e}}, R_{t+1}^{\mathrm{e}}) = 0$, we get that $\mathbb{E}_t[\psi_{t+1}^{1-\gamma} R_{t+1}^{\mathrm{e}}] = \mathbb{E}_t[\psi_{t+1}^{1-\gamma}]\mathbb{E}_t[R_{t+1}^{\mathrm{e}}]$. Assuming that the conditional of expectation of ψ_{t+1} will not be zero, we get that (5.23) simplifies to $\tilde{x}_t = \frac{R^{\mathrm{f}}}{\gamma}\mathbb{E}_t[R_{t+1}^{\mathrm{e}} R_{t+1}^{\mathrm{e}'}]^{-1}\mathbb{E}_t[R_{t+1}^{\mathrm{e}}]$.

[4] Moreover, if we set the volatility of the underlying VAR-model (6.2) to zero, perfect prediction is possible and the mentioned issues are not observed anymore. This emphasizes the trade-off between a more realistic, volatile underlying model and a good performance of the dynamic portfolio choice.

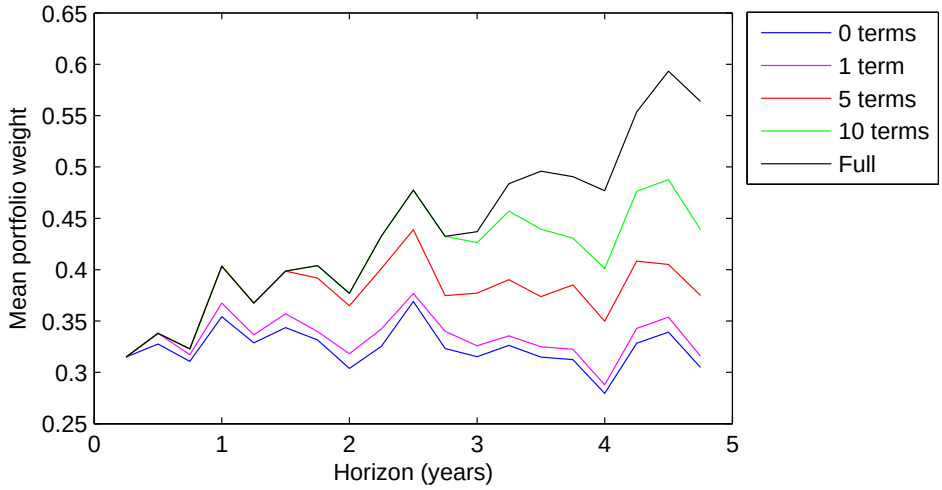

Figure 6.5: *Mean allocation to stocks for different compositions of* ψ_{t+1}, $T = 20$, $\gamma = 5$.

algorithm. Once the horizon gets larger than $k + 1$, the approximation of W_T will be too low, resulting in lower and suboptimal portfolio weights.

Actually, by diminishing the number of terms of ψ_{t+1} to k, from timestep $t = T - k - 1$ to $t = 1$ we change the problem to that of an investor with constant horizon k instead of increasing horizon $T - t$. In figure 6.5 this is clear for the blue line, where ψ_{t+1} contains zero terms and thus $\psi_{t+1} \equiv 1$. The underlying trend of this line is a constant weight of 0.32. The same holds for all other lines: the red line (5 terms) fluctuates around a constant portfolio weight of 0.39 after 5 iteration steps. Thus changing the number of terms in ψ_{t+1} also changes the problem definition, which is not acceptable, as in results in suboptimal results.

Another idea is to scale $\psi_{t+1}^{1-\gamma}$ to avoid too small values, either by dividing by its maximum or minimum value or by using a logarithmic scale. However, by a small test in `MATLAB` we immediately see that this does not work. Manually removing extremes is another possibility, but this reduces the randomness of the sample.

6.6.3 Alternative regression methods

A better idea is the replace the ordinary least-squares regression by alternative regression methods, which we discussed in section 5.4. This was also suggested by Brandt et al. ([7]).

We use a fourth-order Taylor expansion of the value function and put constraints on the portfolio weights for a more realistic setting. The same dataset is used that underlies figure 6.1, but γ is increased to 20. For this value of γ the least-square regression that we used until now is affected by outliers, as is visualized in figure 6.6. Therefore we try both the more robust Huber and the bisquare estimator, both with a maximal number of 20 iterations and a tolerance level of 10^{-3}. The least-trimmed squares estimator is applied with a second-order Taylor expansion and coverage parameter $c = 0.95$.

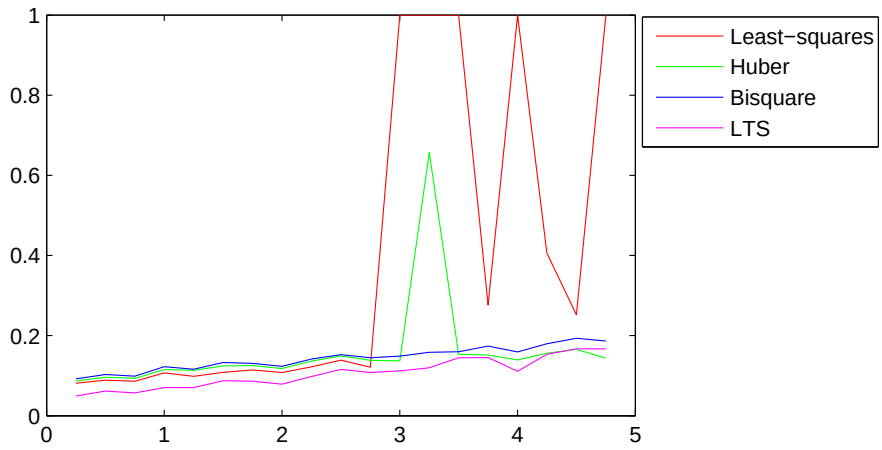

Figure 6.6: *Mean allocation under different regression methods,* $M = 10000$, $T = 20$, $\gamma = 20$

We see that the Huber estimator gives a more stable solution than the least-squares estimator, but still has one peak caused by an outlier. The bisquare method, which puts zero weight on outliers, has no peaks, as is the case for the LTS estimator. The weights for the LTS-method are lower on average, which is caused by the use of a second-order instead of a fourth-order Taylor expansion.

Method	$\hat{\mu}$	$\hat{\sigma}$	P_{upf}	$\text{VaR}_{0.975}$	$\text{cVaR}_{0.975}$	**Speed**
Least-squares	141.3	21.0	0.35	105.4	99.1	0.9 sec
Huber	137.5	9.4	0.24	119.3	115.4	3.8 sec
Bisquare	137.6	5.0	0.10	126.5	122.8	5.8 sec
LTS	137.9	34.4	0.09	127.0	104.2	859.1 sec

Table 6.4: *Performance measures for different regression methods,* $T = 20$, $\gamma = 20$

From the results in table 6.4 we can conclude that the bisquare estimator performs best, since the computation time is still acceptable, while the LTS regression comes with very high computation costs. Compared to the least-squares regression there is a small loss in mean terminal wealth, but this comes with a much lower standard deviation and probability of underperforming. Actually, the high mean terminal wealth and standard deviation of the least-squares method is a result of the malperformance.

6.7 Varying the rebalancing frequency

Until now we used quarterly rebalancing. Assuming there are no transaction costs, one could question whether rebalancing more frequent is favourable or not. Let us apply the dynamic strategy with different rebalancing periods: quarterly, half-yearly and yearly.

From figure 6.7 it is clear that the intertemporal fluctuations in portfolio weights become smaller.

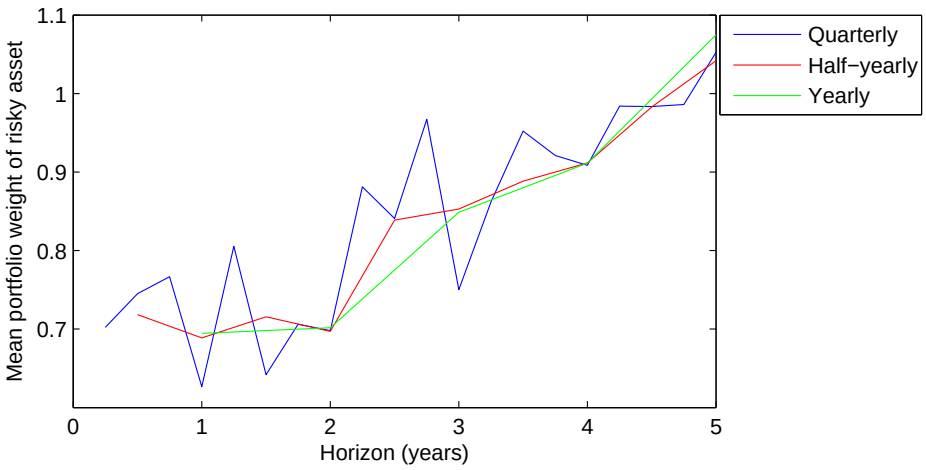

Figure 6.7: *Mean allocation to the risky asset for different rebalancing frequencies,* $T = 21$, $\gamma = 2$.

Rebalancing frequency	$\hat{\mu}$	$\hat{\sigma}$	P_{upf}	$\text{VaR}_{0.975}$	$\text{cVaR}_{0.975}$
Quarterly	178.4	45.0	0.15	84.5	67.1
Half-yearly	175.3	44.7	0.16	87.0	70.3
Yearly	170.8	42.9	0.17	88.5	74.9

Table 6.5: *Performance measures of different rebalancing frequencies,* $T = 21$, $\gamma = 2$.

But as we can see from table 6.5, rebalancing less often gives a loss in terms of mean terminal wealth and increases the probability of underperforming. Therefore it is advisable to maintain a high rebalancing frequency.

6.8 Sensitivity analysis of the parameter of risk aversion

6.8.1 Constant relative risk aversion

Having already discussed the numerical issues that can accompany an increase of γ, we would like to investigate the sensitivity of the results towards parameter γ. Using the algorithm with a Taylor expansion of order four, we visualize the sensitivity of results with respect to the value of γ in figure 6.8.

Under the unconstrained algorithm, we see a perfect hyperbolic relation between γ and the mean and standard deviation of W_T, as well as between γ and x_1, the mean allocation to the risky asset at time 1. As the relative risk aversion γ grows larger, the allocation on the risky asset approaches zero, which was already intuitively clear and which also followed from the formula (5.6). Moreover, for $\gamma = \infty$, the mean of W_T will equal the wealth generated by the risk-free strategy and the standard deviation will become zero. If $\gamma = 0$, the investor will allocate an

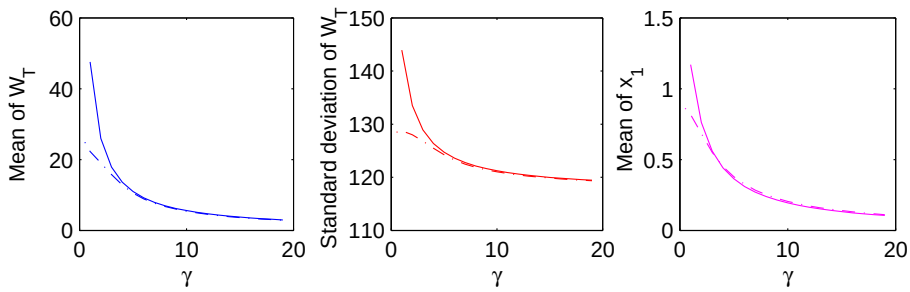

Figure 6.8: *Mean and standard deviation of W_T and mean of x_1 for varying values of γ, $T = 12$, unconstrained (solid lines) and constrained (dashed lines).*

infinite amount to the risky asset and borrow an infinite amount of the risk-free asset, or vice versa if the expectation of excess returns is negative.

Imposing the usual set of constraints does not have a significant impact if γ is larger than five. For smaller values, implying an investor who is willing to take more risk, we see differences with the unconstrained algorithm. For γ approaching zero the allocation to the risky asset will tend to a 100 percent unless expectation of excess returns is negative. Therefore the limiting strategy as γ tends to zero is close, but not exactly equal to the strategy that invests fully in stocks.

6.8.2 Constant absolute risk aversion

All results until now were under the assumption of an investor with CRRA. In this section we change this assumption towards an investor with constant absolute risk aversion ξ and investigate the sensitivity of results towards the parameters of a CARA-investor. The horizon we use is $T = 10$, the algorithm is based on a fourth-order Taylor expansion. Initial wealth is chosen 1 and $\xi = 2$.

First we analyze the influence of the construction of the grid for W_t. We fix $\Delta = 0.01$ and assume $d = h$ and let d vary from 0.03 to 0.30. If we evaluate the optimal strategies in terms of the mean and standard deviation of W_T, surprisingly, these statistics are independent of parameter d. In the remainder of this section we choose $d = h = 0.05$ and $\Delta = 0.005$.

Next we evaluate the sensitivity of the mean and standard deviation of W_T towards ξ. We let ξ vary from 0.5 to 10 with a step size of 0.5 and show results in figure 6.9. Without constraints we observe a hyperbolic relation between ξ and both mean and standard deviation of W_T. Moreover, as ξ tends to infinity, it follows directly from formula (5.37) that the optimal strategy will be equal to the risk-free strategy with and without constraints. Limiting behaviour for ξ tending to zero equals that of γ tending to zero.

Analogue reasoning holds for the relation between mean and standard deviation of W_T and W_1, which is visualized in figure 6.10. Again we observe a hyperbolic relation under the unconstrained algorithm. As W_1 tends to infinity, the optimal allocation to the risky asset will equal zero. This also follows from (5.37), where we see that x_t approaches zero as W_t grows. Because W_t and W_1 obviously have a direct relationship, there exists an inverse relation between W_1 and x_t as

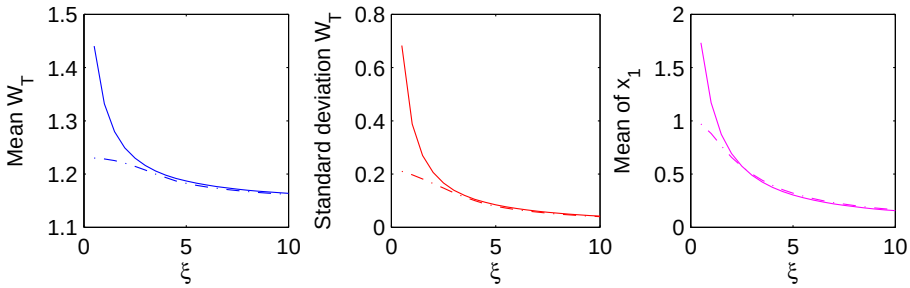

Figure 6.9: *Mean and standard deviation of W_T and mean of x_1 for varying values of ξ, $W_1 = 1$, $T = 10$, unconstrained (solid lines) and constrained (dashed lines).*

well. Of course, it is not desirable that an investor with $\xi = 2$ and initial wealth higher than 10 will follow the risk-free strategy. However, this is avoided by normalizing the initial wealth to magnitude 1.

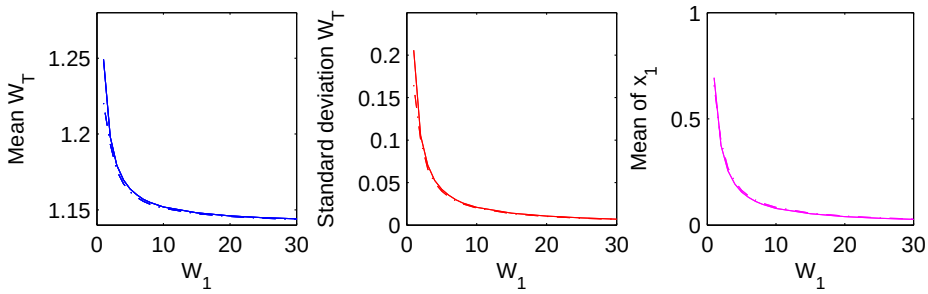

Figure 6.10: *Mean and standard deviation of W_T and mean of x_1 for varying values of W_1, $\xi = 2$, $T = 10$, unconstrained (solid lines) and constrained (dashed lines).*

Sensitivity analysis of the optimal strategy for a investor with CARA utility did not give unexpected information but did learn us to normalize the initial wealth.

6.8.3 Quadratic utility

For an investor with quadratic utility the sensitivity of the results towards the parameter ξ is shown in figure 6.11. Without loss of generality we took $W_1 = 1$. We note that the dynamic strategy tends to the risk-free strategy as ξ tends to 1. It is not visible in figure 6.11, but for $\xi < 0.1$, `MATLAB` is not able to give reliable results.

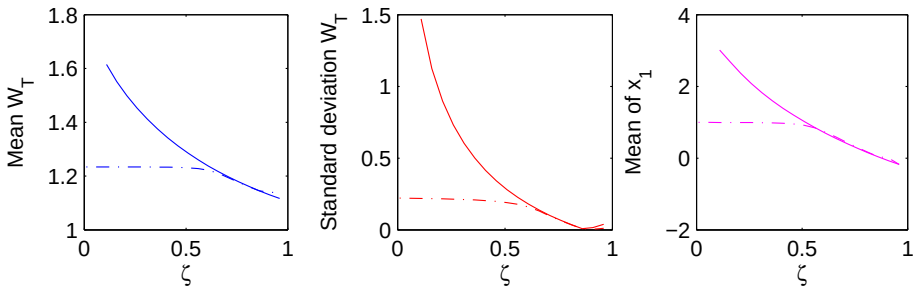

Figure 6.11: *Mean and standard deviation of W_T and mean of x_1 for varying values of ζ, $W_1 = 1$, $T = 10$, unconstrained (solid lines) and constrained (dashed lines).*

6.9 Comparison with mean-variance portfolios

6.9.1 Comparison in terms of mean portfolio return and variance

We are particularly interested in the gains of the dynamic strategy compared to the static mean-variance strategy. They are compared graphically by drawing the efficient frontier. The construction of the efficient frontier for the static strategy was discussed in chapter 2. Because there is a risk-free asset and we forbid borrowing and going short, the static efficient frontier will be a straight line, with the risk-free strategy and the strategy investing fully in stock as boundary points.

For an investor who behaves dynamically, we construct the efficient frontier using simulations for an investor with log or power utility, exponential and quadratic utility. By varying γ, ξ respectively ζ we obtain different attainable combinations of mean and standard deviation of W_T. As seen in section 6.8 on the limiting behaviour of γ, the starting point of the frontier of the dynamic investor with constant relative risk aversion will be the risk-free strategy. The same holds for the investor with constant absolute risk aversion or quadratic utility as ξ respectively ζ tend to infinity.

We use a fourth-order expansion, with constraints and quarterly rebalancing. Without loss of generality, initial wealth W_1 is chosen equal to 1. Then terminal wealth W_T and the cumulative portfolio return $R_\mathrm{p} = W_T/W_1$ are equal.

In figure 6.12 the dynamic efficient frontiers are compared with the mean-variance efficient frontier. The gain of using any dynamic strategy compared to a static Markowitz strategy is obvious. The same mean terminal wealth is obtained with a lower standard deviation. Furthermore, we see that in terms of mean and variance, the efficient frontier of an investor with constant relative risk aversion is almost exactly equal to the efficient frontier of an investor with constant absolute risk aversion.[5] An investor with quadratic utility has an efficient frontier which is slightly less favourable.[6]

[5]Empirically we observed that for a very risk averse investor with $\gamma > 5$ or $\xi > 10$ we get exactly the same optimal strategy if $\gamma = 2.18\xi$.

[6]Moreover, empirical findings show that this is completely caused by the first step of the recursion process. While for a CRRA and a CARA-investor portfolio weights are equal in this step, compare (5.25) and (5.38), an

Despite the efficient frontiers being equal, there are differences in terms of speed. For one set of parameters evaluation of the CRRA-problem takes only a few seconds, versus over ninety seconds for an investor with CARA or quadratic utility.

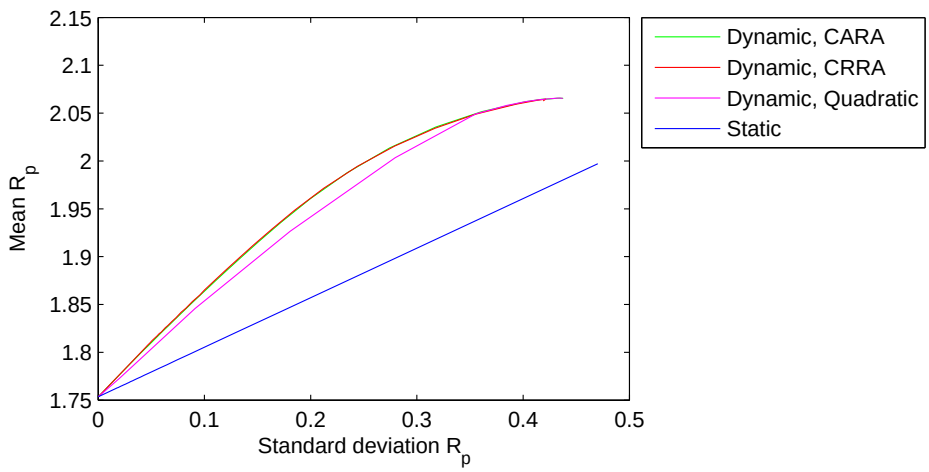

Figure 6.12: *Efficient frontier for static versus dynamic strategies,* $T = 12$.

The equality of the efficient frontier of a CRRA and CARA-investor is somewhat surprising. Empirically we find more remarkable results. While we noticed that the portfolio weights are affected by the order of the Taylor expansion, the efficient frontier is not. A second-order expansion leads to exactly the same frontier as a fourth-order expansion, the only difference being that the same point at the frontier corresponds to a slightly different value of γ or ξ.

The same holds for the factor R^{f}, with which we multiplied the portfolio weight in constrast with the original article by Brandt et al. Using the algorithm with or without this factor does not influence the efficient frontier, but again, a different value of γ or ξ corresponds to another point at this frontier.

6.9.2 Comparison in terms of mean portfolio return and Value-at-Risk

Variance is not the only risk measure we can apply. But based on our simulation we can easily construct a mean-VaR efficient frontier as well, which is done in figure 6.13. The dynamic strateges also realizes the same mean terminal wealth with a higher Value-at-Risk than the static strategy. The CARA and CRRA mean-VaR frontier only differ slightly, while the mean-VaR frontier of an investor with quadratic utility is again suboptimal.

Note that one can also use the mean-VaR$_\alpha$ efficient frontier to solve the problem of an investor who wants to reach a target wealth with a certain probability $1-\alpha$. This investor has to choose the strategy on the mean-VaR$_\alpha$ efficient frontier with mean terminal wealth equal to his target wealth.

investor with quadratic utility has a different portfolio weight at time $T-1$, see (5.42).

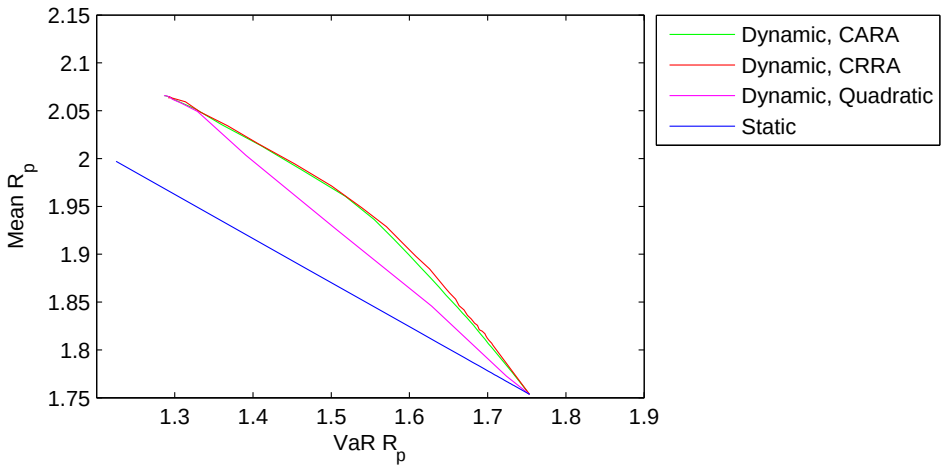

Figure 6.13: *Mean-VaR efficient frontier for* $\alpha = 0.95$, $T = 12$.

6.10 Summary of chapter 6

From the evaluation of the methodology in a simple setting we can conclude that applying a dynamic strategy has clear gains over applying a static strategy. By a path-dependent rebalancing strategy we can greatly increase the mean portfolio return with a significant lower downside risk in terms of variance, probability of underperforming the risk-free asset, Value-at-Risk and expected shortfall. Analogue to the common findings in literature we find that as the investment horizon becomes shorter, the allocation to risky asset becomes lower on average.

A fourth-order Taylor expansion is sufficient to approximate the optimal portfolio weights. Imposing constraints makes the problem more realistic without having a drastic impact on the optimal strategy. A high rebalancing frequency further improves the strategy and it is therefore advisable to rebalance as often as the underlying economic model permits.

For a long investment horizon and especially for high relative risk aversion γ the method faces numerical issues, but in this simple setting we could still obtain a stable solution by application of the bisquare robust regression method. However, we did remark that the presence of these numerical issues might have a fundamental basis. Our methodology assumes that the current value of the state variables can predict next period's asset returns, scaled by a factor that contains *all* future returns and the optimal portfolio weights. This asks a great amount of predicting power once we are far in the recursion process.

Furthermore we came to the important insight that the assumption of either CARA or CRRA leads to the same efficient frontier. An investor with quadratic utility behaves suboptimal. Since, by the independence of endogenous variable wealth, the dynamic strategy is easier and faster to evaluate for an investor with constant relative risk aversion, it is a natural choice to continue with the common assumption of CRRA. The equality of efficient frontiers justifies this.

In the next chapter we will use these findings to calculate the portfolio for an investor with multiple assets.

Chapter 7

Results for multiple risky assets

Under the simple VAR-model the gains of using a dynamic, path-dependent portfolio strategy are clear. In a real investor's problem, the investor has multiple risky assets to consider and the number of state variables is much larger. In this chapter we will apply the method of BGSS on a setting with five risky assets and five exogenous state variables. The investor's attitude towards risk is modeled by constant relative risk aversion.

In section 7.1 we first explain how the scenarios are constructed with the generator of Ortec Finance. Afterwards the set-up of the portfolio optimization is discussed. The resulting asset allocation is discussed in section 7.2, as well as the distribution of wealth, performance measures. Finally the efficient frontier is visualized.

7.1 Set-up of the model

7.1.1 Modeling the economy

The evolution of state variables and the asset returns are simulated by ALS (Assets & Liabilities Scenario model), which is the generator Ortec Finance developed to simulate economic scenarios. It is an extensive VAR(1)-model with more than 40 variables, based on data from 1970 to 2009. All variables are generated on a yearly basis.

Figure 7.1 illustrates how the output of the scenario generator looks like for two variables, being the GDP (Gross Domestical Product) of Europe and of the USA. Up to 2009 the blue line presents the actual values of the GDP. The 'yellow cloud', which diverges from 2010 and on, represents all scenarios. The red line is the mean of all scenarios, which reaches a steady state after ten years. The blue line from 2010 and further highlights the evolution of the two variables under one specific scenario.

We consider five asset classes:

- Risk-free asset with a constant yearly return of 3%, denoted as cash.
- Bond with a duration of 10 years, traded anually to keep the maturity of the portfolio constant.

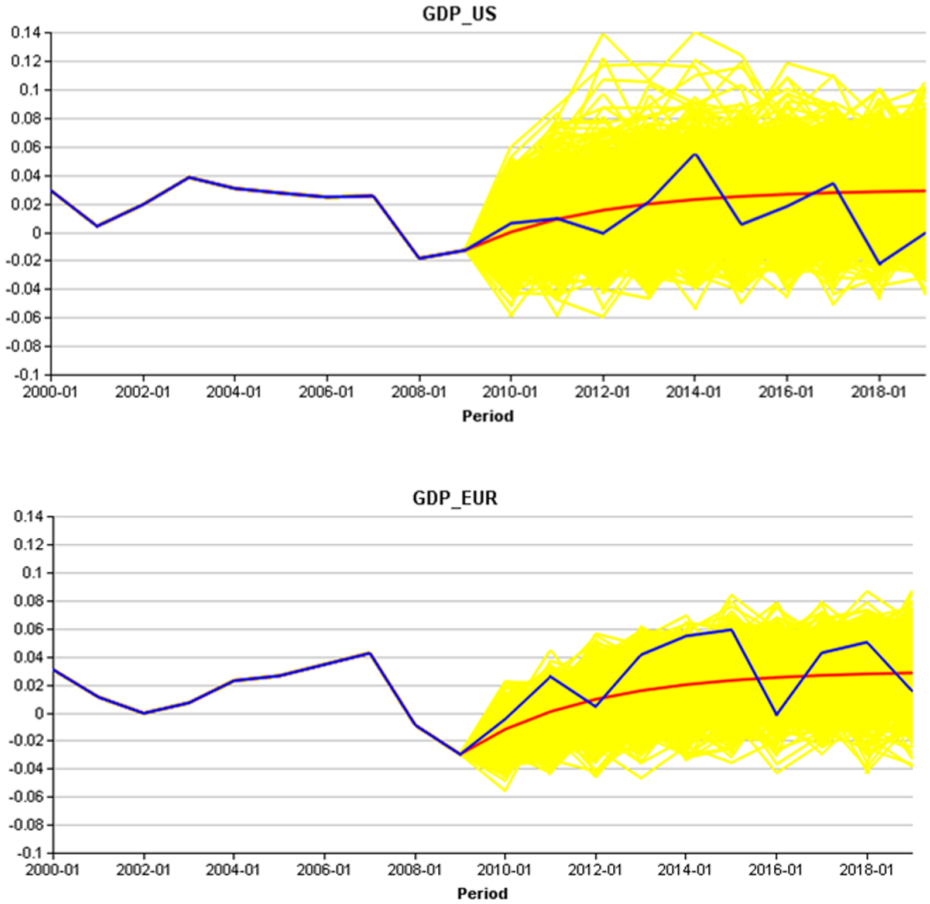

Figure 7.1: *An example of the output of the Ortec Finance scenario generator*

- US Hedge fund, completely hedged for currency risk.
- Indirect Real Estate (RE) Europe.
- Stocks of European MSCI-index.[1]
- Stocks of emerging markets (EMM).

All consider asset classes are considered to have enough liquidity to trade in each desirable amount.

To avoid numerical issues and overfitting we did not include all variables, but we selected the five most important drivers as state variables. Even though the main assets are mostly non-American, the main drivers of the VAR-model are primarly American:

- 3 months US nominal interest rate.

[1]MSCI is short for Morgan Stanley Capital International. The index is a weighted stock market index, comparable to the CRSP-index used in chapter 6.

- 10 years US nominal interest rate.
- Stocks of North-American MSCI-index.
- 10 years break-even inflation US.
- Return on direct Real Estate Europe.

The risk-free cash return is subtracted from the risk asset's returns to obtain the excess returns. Basis statistics of the excess returns are in table 7.1.[2] We see that in terms of standard deviation, the bond and hedge fund are relative safe. The stocks EMM have the highest mean excess return but are also highly volatile. We use this generator to sample 10,000 economic scenarios. The

Asset class	**Mean** R_t^e	**Standard deviation** R_t^e
Bond	-0.01 - 0.01	0.03 - 0.04
Hedge Fund	0.03	0.10
Indirect RE	0.04	0.23
Stock EU	0.05	0.21
Stock EMM	0.09	0.34

Table 7.1: *Description of asset classes*

starting point of the simulation is 2010.[3]

7.1.2 Details on the portfolio optimization

In the dynamic portfolio optimization algorithm we assume rebalancing is employed once a year, as often as the scenario model permits. Next to borrowing and short sale constraints, an additional set of constraints is introduced to avoid large changes in the portfolio. From year to year, the absolute change of each portfolio weight cannot be more than 20 percent. Because with such constraints imposed, it is unlikely that there will still be convergence towards the fourth-order Taylor solution, we use a second-order Taylor expansion. This also decreases computation time. Moreover, we noticed in the foregoing chapter that the resulting efficient frontier was the same for a second-order or a fourth-order Taylor expansion.

A polynomial of degree two is used as a basis for the regression. The regression matrix includes cross-terms of the five state variables. Moreover we assume that the excess returns at $t+1$ have a correlation with the excess return at time t. Then each row of regression matrix X_t contains 26 items. For the regression we use the bisquare method.

Running the algorithm as described in this section is quite fast considering the number of variables. With 10,000 simulations, 5 asset classes and 5 state variables, the constrained algorithm using the `fmincon`-optimization routine takes about 25 for each value of γ if $T = 10$, or 55 minutes if $T = 20$. We ran the algorithm on a 1.83 GHz laptop with 1.99 GB RAM.

[2]Due to the 2008-2009 crisis, the bond mean excess return and its standard deviation take a few years to reach a stationary solution

[3]We choose 2010 instead of 2009 in order to avoid the rather typical year 2009.

7.2 Results

7.2.1 Mean asset allocation and performance measures

Figures 7.2 up to 7.7 show the mean asset allocation against the remaining horizon and the distribution of terminal wealth for an investor with an investment horizon of respectively 10, 20 and 30 years. For each value of T we show results for four values of the parameter of risk aversion: $\gamma = 1$, 3, 5 and 10.

Comparing the asset allocations in figures 7.2, 7.4 and 7.6, we notice that for a short (remaining) horizon, which relates to the first steps of the recursion process, the method works quite well. If the risk aversion is higher, this results in a higher allocation to safer assets like bonds (about 40% for $\gamma = 10$ versus approximately 8% for $\gamma = 1$), while the allocation to the most risky asset, stocks EMM, is lower (less than 5% versus about 40%).

However, as the horizon increases, the method starts performing badly. Once the combination of γ and horizon $T - t$ reaches a certain threshold, the regression does not work properly. The allocation to all risky assets tends to zero and the complete wealth is invested in cash, which cannot be optimal. Undesirable behaviour for high risk aversion and a long investment horizon was also observed in the previous chapter, but there it could solve this by implementing the bisquare method. Here we see that the use of the bisquare method is not sufficient.

Investors with $\gamma = 1$ are an exception. Since their portfolio weights at each time do not depend on the factor $\psi_{t+1}^{1-\gamma}$,[4] the algorithm works for all horizons. This follows directly from the figures. In fact, we found that the algorithm behaves well for each $\gamma \in (0, 1]$. This means that, if γ is within this range, the found dynamic strategy is the best we can obtain under this set of scenarios and, not unimportantly, under this choice of state variables. For $\gamma > 1$ the found strategy becomes suboptimals as T increases.

In figures 7.3, 7.5 and 7.7 the distribution of terminal wealth for four values of risk aversion γ is shown for $T = 10$, 20 and 30 respectively. The red line in each histogram corresponds to the risk-free strategy. The outer right bar corresponds to the frequency of the right tail values of terminal wealth. We notice that for a lower γ, the distribution of terminal wealth is more heavy-tailed. Especially the righter tail becomes more profounded, an effect which is leveraged by an increase of T.

[4] Recall formula (5.24).

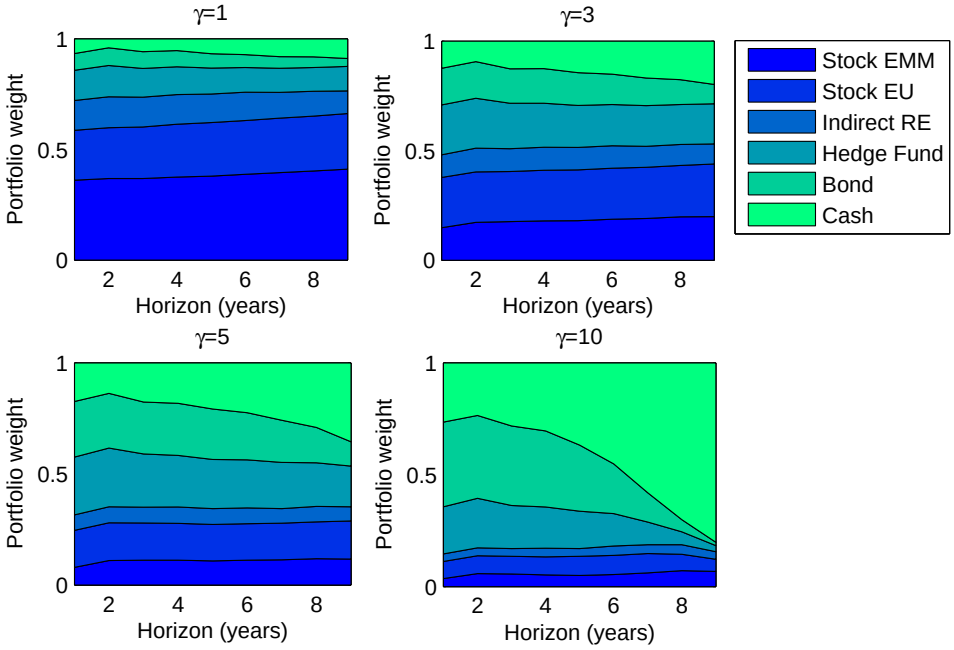

Figure 7.2: *Mean asset allocation for* $T = 10$, $\gamma = \{1, 3, 5, 10\}$.

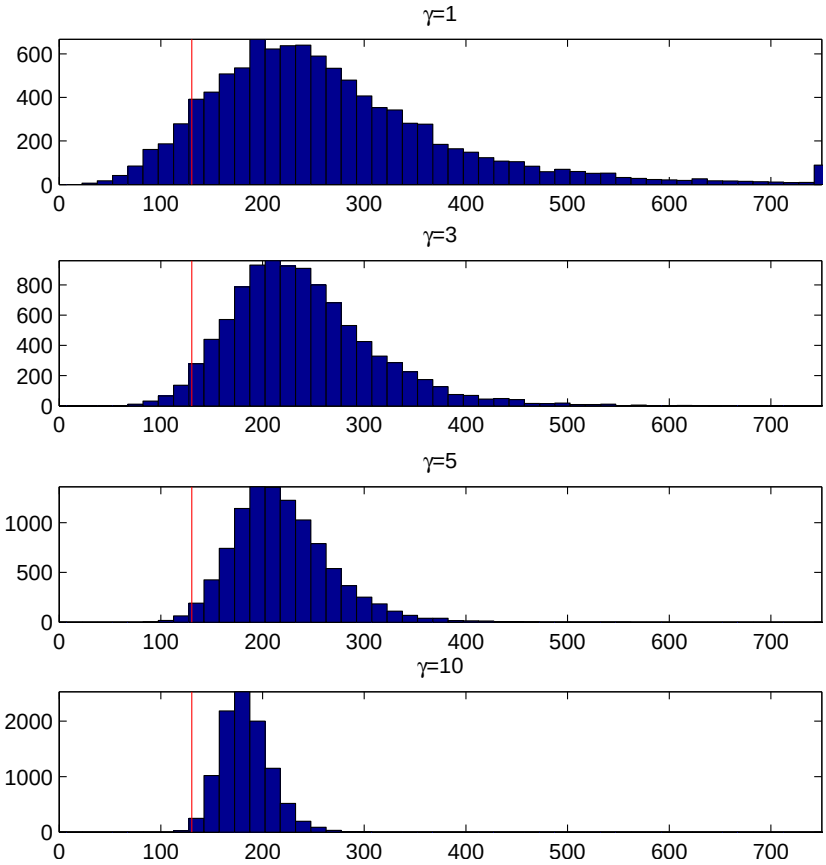

Figure 7.3: *Histograms of* W_T *for* $T = 10$, $\gamma = \{1, 3, 5, 10\}$.

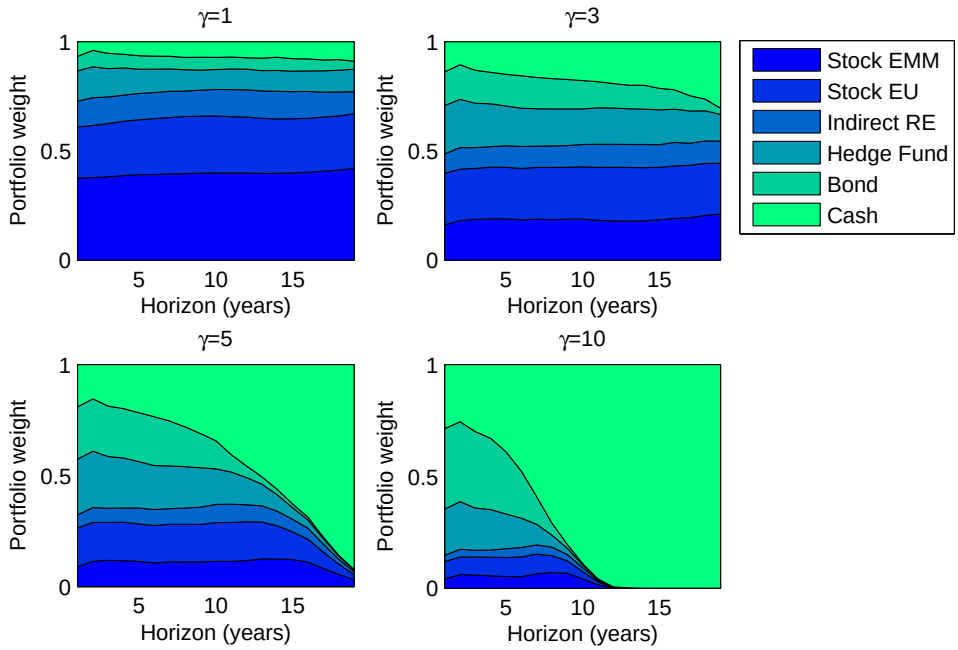

Figure 7.4: *Mean asset allocation for* $T = 20$, $\gamma = \{1, 3, 5, 10\}$.

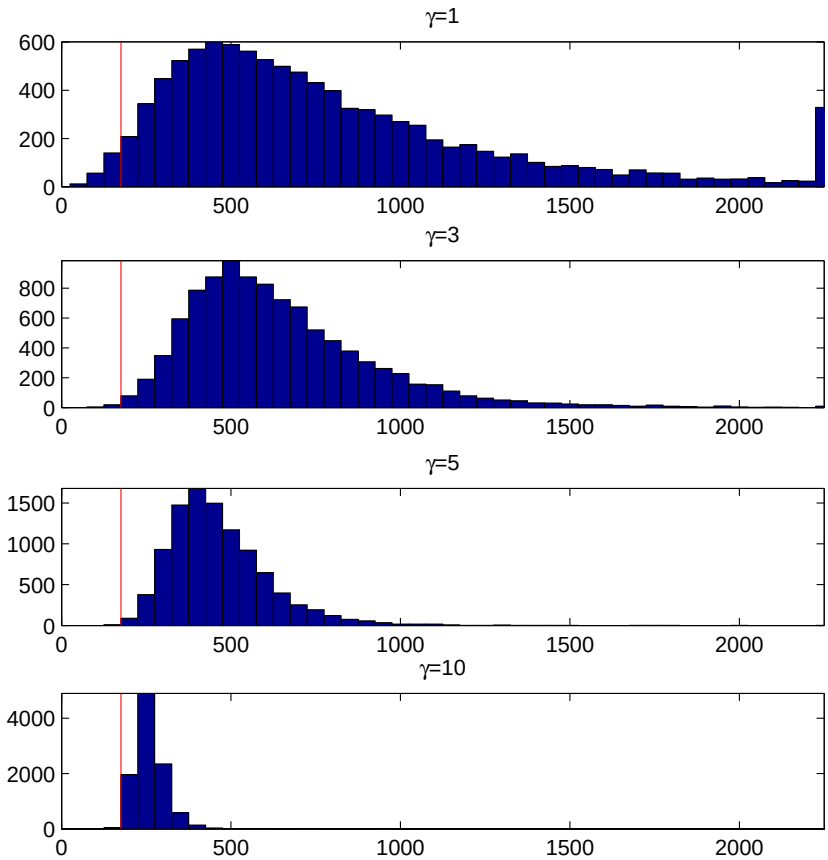

Figure 7.5: *Histograms of* W_T *for* $T = 20$, $\gamma = \{1, 3, 5, 10\}$.

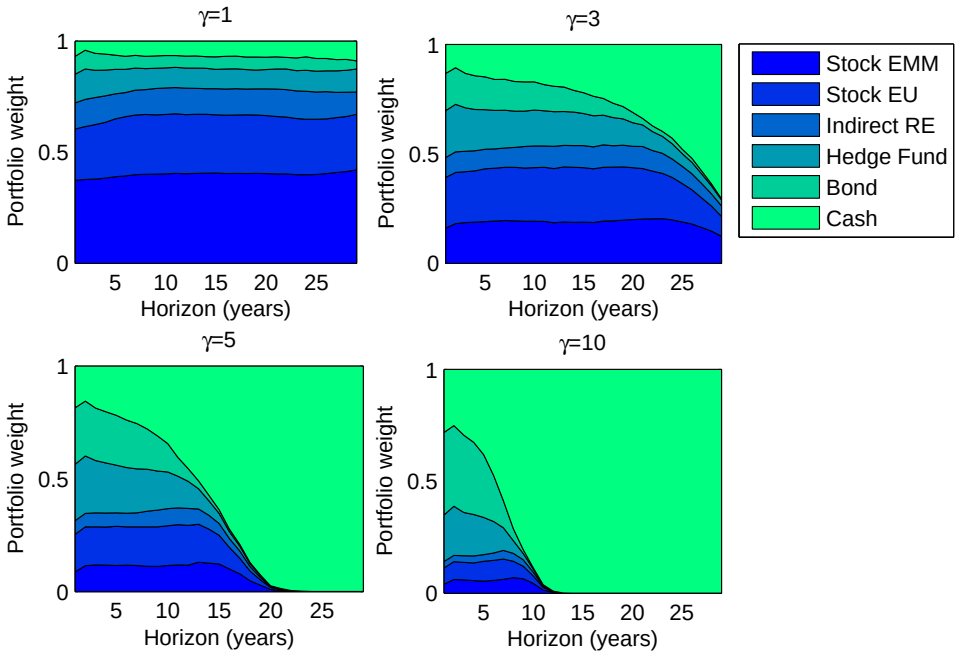

Figure 7.6: *Mean asset allocation for* $T = 30$, $\gamma = \{1, 3, 5, 10\}$.

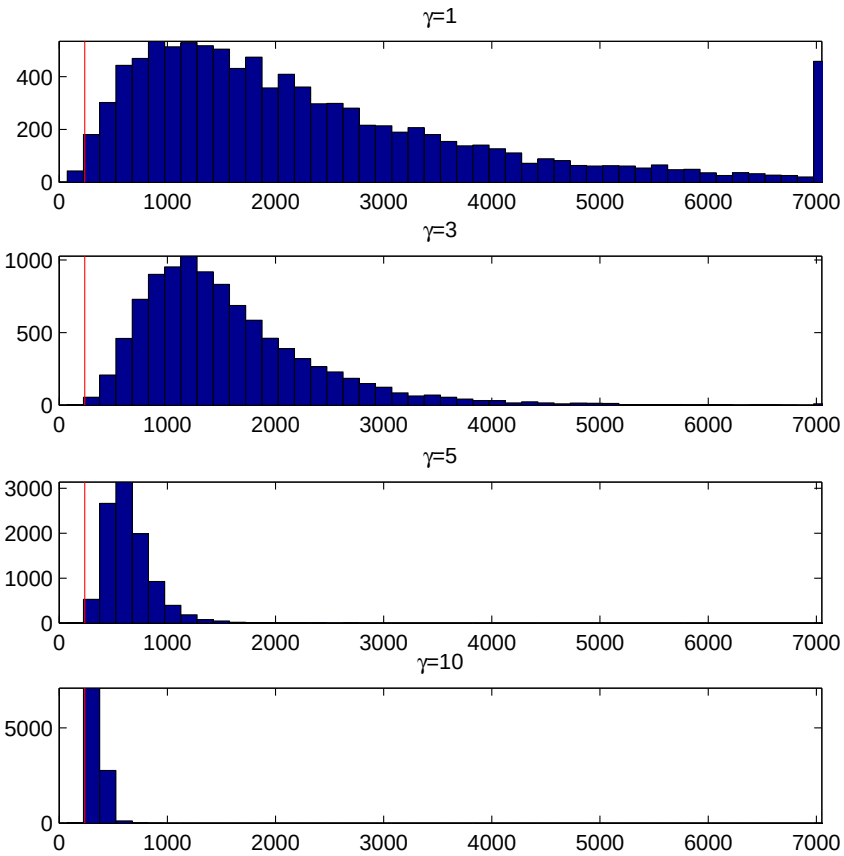

Figure 7.7: *Histograms of* W_T *for* $T = 30$, $\gamma = \{1, 3, 5, 10\}$.

Additional to these figures, tables 7.2, 7.3 and 7.4 show the values of the performance measures for the different values of risk aversion and lengths of the investment horizon. The strategy with $\gamma = \infty$ is the strategy that invests fully in the risk-free asset.

Risk aversion γ	$\hat{\mu}$	$\hat{\sigma}$	P_{upf}	$\text{VaR}_{0.975}$	$\text{cVaR}_{0.975}$
1	266.3	128.6	0.085	91.2	74.7
3	239.5	71.8	0.030	127.8	111.3
5	219.8	48.5	0.011	140.8	129.3
10	183.5	24.2	0.005	141.8	135.1
∞	130.5	0	0	130.5	130.5

Table 7.2: *Performance measures for multiple assets,* $T = 10$.

Risk aversion γ	$\hat{\mu}$	$\hat{\sigma}$	P_{upf}	$\text{VaR}_{0.975}$	$\text{cVaR}_{0.975}$
1	820.2	609.5	0.021	186.7	141.6
3	643.5	276.6	0.002	266.3	224.4
5	464.4	149.1	0.001	253.2	226.4
10	260.4	44.3	0.004	192.8	183.2
∞	175.4	0	0	175.4	175.4

Table 7.3: *Performance measures for multiple assets,* $T = 20$.

Risk aversion γ	$\hat{\mu}$	$\hat{\sigma}$	P_{upf}	$\text{VaR}_{0.975}$	$\text{cVaR}_{0.975}$
1	2564.6	2618.0	0.005	393.8	289.1
3	1563.8	829.6	0.001	519.1	427.0
5	643.8	221.0	0.001	334.0	297.4
10	350.8	60.5	0.005	256.8	242.7
∞	235.7	0	0	235.7	235.7

Table 7.4: *Performance measures for multiple assets,* $T = 30$.

The three tables again show us the high gains of dynamic strategies. The mean final wealth under a dynamic strategy with $\gamma = 1$ is twice as high as the mean final wealth under the risk-free strategy for an investor with $T = 10$. As T increases to 30 years, we observe a *tenfold* increase in the mean final wealth if we switch from the risk-free strategy to a strategy with $\gamma = 1$.

The standard deviation of final wealth also grows very fast as T increases. But, the probability of underperforming the risk-free strategy reaches a reasonable maximum 8.5% for $\gamma = 1$ and $T = 10$, while being even less than 0.5% for all values of γ at $T = 30$. Moreover, as Tincreases, both the Value-at-Risk and the conditional Value-at-Risk at $\alpha = 0.975$ become even higher than the mean final wealth of the risk-free strategy.

7.2.2 Efficient frontier

In the foregoing section, we compared the dynamic strategies mainly with the risk-free strategy. Here we will compare them to the optimal static strategies. By varying the risk aversion parameter γ, we construct the static and dynamic efficient frontier for $T = 10$, 20 and 30. We known that both the static and dynamic efficient frontier start at the risk-free strategy, corresponding to $\gamma = \infty$. The right end-point is reached by letting γ approach zero. Figures 7.8, 7.9 and 7.10 show the dynamic versus static efficient frontier for $T = 10$, 20 and 30.

We note that the static efficient frontier is not a straight line anymore. This is caused by the increased number of risky assets and the imposed constraints. As said, the static frontier starts at the risk-free strategy for $\gamma = \infty$. It ends for $\gamma = 0$ at the strategy that invests fully in the asset with the highest mean return, which is stock EMM.

Of course we are more interested in the dynamic efficient frontier. As could be expected from earlier results, the three figures show that the dynamic efficient frontier is more favourable than the static efficient frontier for each value of T. While both the static and the dynamic frontier start at the risk-free strategy, the dynamic efficient frontier has a higher slope. It is remarkable that for all values of T, the dynamic and static efficient frontier become parallel to each other as γ decreases.

This parallel part corresponds to low risk aversion, being γ lower than two. In the foregoing section we noticed that for these values of γ there are no methodological or numerical issues. So, unless the predictive power of the state variables changes, this part of the efficient frontier is 'as best as it gets'.

As for the left part of the dynamic efficient frontier, which corresponds to values of γ higher than two, there is still room for improvement. We already noticed that for higher values of γ, combined with an increasing value of T, the found dynamic strategy becomes suboptimal. We suspect that the frontier which follows from the real optimal dynamic strategy for all values of γ is even steeper at the start. As remarked, it will end in the same parallel part at the right.

Despite the suboptimality for increasing values of γ and T, the dynamic efficient frontier outperforms the static frontier convincingly.

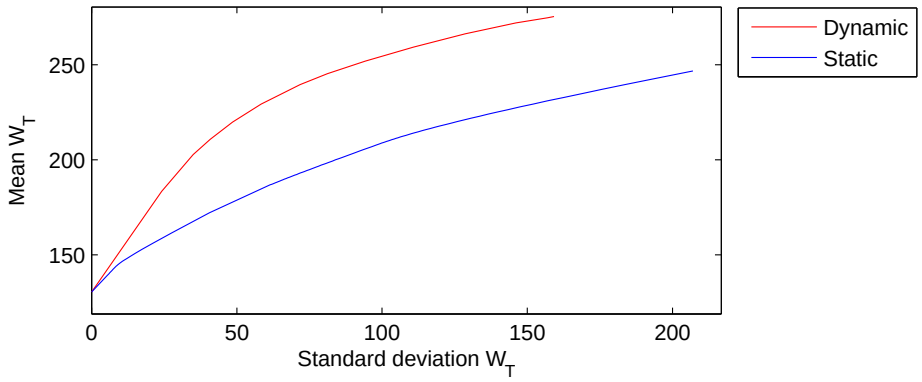

Figure 7.8: *Efficient frontier with multiple assets for static versus dynamic strategies,* $T = 10$.

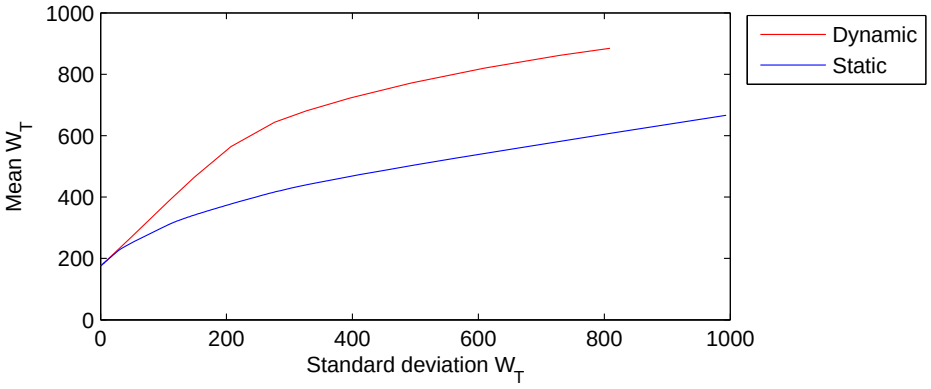

Figure 7.9: *Efficient frontier with multiple assets for static versus dynamic strategies,* $T = 20$.

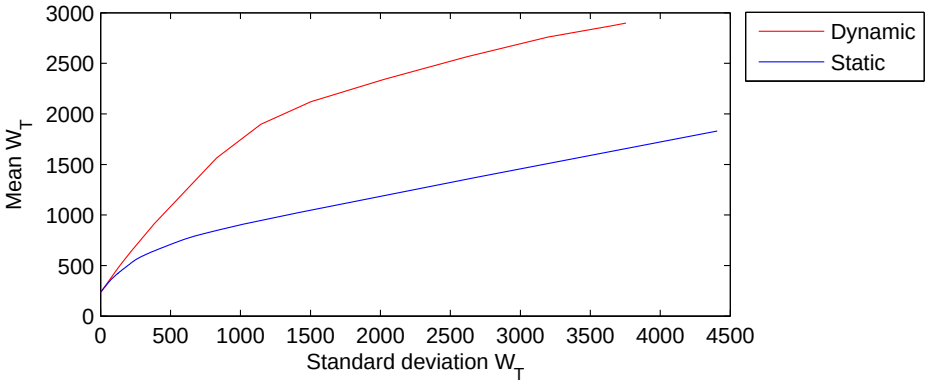

Figure 7.10: *Efficient frontier with multiple assets for static versus dynamic strategies,* $T = 30$.

Chapter 8

Conclusions, discussion and recommendations

In the past few years, dynamic portfolio choice has become a popular subject of research. Unfortunately closed-form, analytic solutions are only unavailable under strict assumptions of return dynamics and numerical approaches are needed. By now many methods have been published, but almost all suffer from inflexibility towards the number of assets or ask for a very specific structure of the asset's return dynamics.

An exception is the method published by Brandt, Goyal, Santa-Clara and Stroud. They developed a simulation-based method with as main features a Taylor-expansion, backward recursion and regression analysis to predict returns. In their own words their method is fast, accurate and very flexible towards the number of assets and the way in which return dynamics are modeled. However, they only showed results for a situation with one risky asset, one state variable and an investor with constant relative risk aversion.

In this master thesis the goal was to apply this methodology and investigate the dynamic portfolio choice for a long-term investor in a realistic environment with multiple assets without strict assumptions on the return dynamics, various state variables and optional portfolio constraints. But, before we were able to apply the model in such a complex environment, we used a simple setting with only one risky asset and one state variable to investigate behaviour and various extensions of the methodology itself, as well as the impact of different ways of modeling the investor's attitude towards risk. This gave three new and important insights, which are given in section 8.1. In section 8.2 the conclusions are given that followed from evaluation of the dynamic portfolio choice with multiple assets. In sections 8.3 and 8.4 thereafter, there is a short discussion and recommendations are made.

8.1 Conclusions from a simple setting

First of all, we discovered an error in the original formulation of the algorithm. Correction of this error leads to a higher asset allocation with a factor exactly equal to the risk-free rate, resulting in higher terminal wealth. More specifically, under the assumption of constant relative risk aversion and if excess returns are modeled independent of the risk-free rate, the error correction led to a linear relationship between the portfolio weight and the risk-free rate.

At first glance a higher rate of return on the risk-free asset implying a higher allocation to the risky asset might be counterintuitive. However, we can explain this relationship by the assumption of constant relative risk aversion. If the risk-free rate is higher and if absolute excess returns with respect to this rate remain equal, the relative volatility of the risky asset decreases. Under constant relative risk aversion a decrease of the relative volatility implies an increase of the portfolio weight.

Second, under constant relative risk aversion we discovered numerical issues for high risk aversion or a long horizon. As we are further in the backward recursion process, the regression starts to malfunction. Brandt et al. suggested to use robust regression methods. In this thesis we showed that in a simple setting especially the bisquare estimator performs well. Its use succesfully eliminates the observed numerical issues, although we did remark that the real cause of the numerical issues might be more fundamental.

Finally and third, we put our attention to the way in which the investor's attitude towards the trade-off between risk and return is modeled. Because empirical results do not agree on the most realistic way to model and calibrate the investor's risk aversion, we calculated the efficient frontier for an investor with constant absolute risk aversion (CARA), constant relative risk aversion (CRRA) and quadratic utility. The comparison of different assumptions on utility by means of the efficient frontier led to interesting results. It was not mentioned in any literature and is therefore a valuable addition.

From the efficient frontiers we concluded that under all three types of utility a dynamic strategy outperforms a static strategy. Assuming quadratic utility leads to a suboptimal dynamic portfolio, while the assumption of CARA or CRRA leads to exactly the same efficient frontier and mean-VaR frontier. This justifies the common choice to model utility as CRRA, which is usually done because CRRA seems intuitively correct and has convenient analytical properties. Furthermore, the form and location of the efficient frontier remains equal regardless the order of the Taylor expansion or the correction of the error which was mentioned in the first conclusion.

In general, we can conclude that in a simple setting the dynamic strategy has clear gains over a static strategy. Similar to known results from literature, the allocation to the risky asset is on average lower as the investment horizon becomes shorter.

8.2 Conclusions from a complex investment environment

With the knowledge obtained in a simple setting we evaluated the asset allocation for a CRRA-investor who faces a more realistic environment, with a risk-free asset and five types of risky assets with different mean return and volatility. Furthermore five state variables were included that were supposed to contain enough predicting power and the portfolio weights were constrained. Returns of all variables were generated by the ALS, the scenario generator developed by Ortec Finance.

Also in a multivariate investment environment dynamic strategies outperform the optimal static strategies, which was both quantified in performance measures, like mean final wealth and the Value-at-Risk, and shown visually by comparing efficient frontiers. As the investment horizon becomes shorter, there is on average a higher allocation to assets with less volatility, which again agrees with the findings in literature. Moreover, we observed that this effect is more present for investors with a higher risk aversion.

Unfortunately, for a moderate risk aversion and a long investment horizon we still observe numerical issues which are not sufficiently reduced anymore by robust regression. These issues lead to strategies that invest fully in the risk-free asset for years, which is clearly suboptimal. But, even with this suboptimal behaviour for some parameter values, the dynamic strategies continue to outperform the static strategies.

8.3 Discussion

Having sufficiently reduced the numerical issues in a simple setting by applying robust regression, the recurrence of the numerical issues in complex setting brings us back to our earlier remarks about the method having a more fundamental problem for a long investment horizon. The severity of this problem is further leveraged by the parameter of risk aversion. Because the methodology is based at backward recursion, at each time the state variables do not only have to predict asset returns after one timestep, but also the portfolio return generated by the sequence of optimal future portfolio weights. During the first steps of the recursion process this does not give any problems yet. However, as the time until the horizon increases, it is questionable if the current values of the state variables contain enough information for a decent prediction.

To what extent prediction is possible and thus how severe the suboptimality of the strategy is, will depend mostly on the underlying scenarios. If the scenarios do not contain any randomness, perfect prediction is possible. But in that case, it is not likely that the scenarios represent the economy very well. At the other hand, if the scenarios reflect the volatility of returns very well and includes unexpected events, this reduces the predicting power, especially for a sequence of future returns. This means that there is always a trade-off between the realism of the economic model and the performance portfolio model under this economic model.

8.4 Recommendations and future research

The methodology considered in this thesis was clearly capable of dealing with a realistic investor's environment and delivered convincingly better results than the static strategies Ortec Finance uses nowadays. Implementation is therefore recommended, although additional research is needed.

For general future research we would advise to investigate whether other methods than robust regression are capable of reducing the numerical issues which where still observed in a complex investing environment. If this does not lead to improvements, a stopping criterium should be developed, which defines acceptable combinations of risk aversion and investment horizon. The stopping criterium will probably depend on the underlying scenarios.

Before Ortec Finance can implement this dynamic strategy and use it in their daily activities, they face additional areas of future research. In this thesis we assumed an investor with only an initial capital and without any financial liabilities, intertemporal consumption or additional injections of investment capital. Moreover, we only specified utility of final wealth. Such a 'simple' investor is not the general client of Ortec Finance. Therefore they should extend the model such that it can be applied to an investor with liabilities, injections of investment capital and other objectives than terminal wealth, such as funding ratio, surplus or inflation effects.

It is also important to quantify the client's risk aversion. Because the exact estimation of an individual's risk aversion parameter is hard, we would recommend to use the visualization of the efficient frontier as a mean to decide on the level of risk aversion. Based on this frontier, the investor can decide which combination of mean and standard deviation of final wealth (or which combination of mean final wealth and Value-at-Risk) suits him best, and thus define the parameter of risk aversion that corresponds to this combination.

Next to that, Ortec Finance is on the point of introducing a new, improved economic scenario generator. In this generator the economy is modeled in cycles and there is a higher correlation between current values and future values of all variables. The presence of cycles and the existence of long-term correlation might reduce the earlier mentioned numerical issues and improve the predicting power. This should be investigated once the new generator is completely up and running. If our intuitions about the economic scenarios are correct, the methodology of this thesis would be a very suitable candidate for implementation.

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Appendix A

Additional methodology and models

In the whole thesis the focus of the investor was always on final wealth. However, he can also decide to optimize on a target wealth or so-called funding ratio, which is the subject of section A.1. Next to that, we consider an investor who injects (or consumes) investment capital during the investment period in section A.2. Finally in section **??** we limited the number of variables in ALS, to see whether this improves predicting power.

A.1 Application to an investor with a target wealth

Suppose we have an investor with initial wealth W_1. He wants to invest such that his terminal wealth W_T will equal W_{target}. Ideally his wealth grows each period by a constant factor:

$$R_{\text{target}} = (\frac{W_{\text{target}}}{W_1})^{\frac{1}{T-1}}$$

At each time t we can measure how far the investor is from realizing his objectives by quantifying the 'funding ratio' F_t. If the funding ratio is 1, the portfolio has a return which exactly matches the target return. If $F_t < 1$, the return on his portfolio is less than targeted and if $F_t > 1$ the investor made more return than planned. The funding ratio is calculated as:

$$F_t = \frac{W_t}{W_1 \cdot R_{\text{target}}^{t-1}}, \quad t \geq 1 \tag{A.1}$$

Instead of optimizing on utility of terminal wealth, we optimize on the utility of the funding ratio, such that our value function is:

$$J_t(F_t, Z_t) = \max_{\{x_s\}_{s=t}^{T-1}} \mathbb{E}_t[u(F_T)] \tag{A.2}$$

The corresponding sequence of constraints follows if we substitute the budget constraint (4.9) for W_{t+1} in (A.1):

$$F_{t+1} = \frac{W_t(x_t' R_{t+1}^{\text{e}} + R^{\text{f}})}{W_1 \cdot R_{\text{target}}^{t-1} \cdot R_{\text{target}}} = \frac{F_t}{R_{\text{target}}}(x_t' R_{t+1}^{\text{e}} + R^{\text{f}})$$

Following the same steps as in chapter 5, a second-order Taylor expansion leads to the following expression for x_t, which is the equivalent of (5.6):

$$\begin{aligned}\tilde{x}_t &= -\{\frac{F_t}{R_{\text{target}}}\mathbb{E}_t[\partial_1^2 J_{t+1}(\frac{F_t R^{\text{f}}}{R_{\text{target}}}, Z_{t+1})R^{\text{e}}_{t+1}R^{\text{e}'}_{t+1}]\}^{-1} \\ &\quad \times\mathbb{E}_t[\partial_1 J_{t+1}(\frac{F_t R^{\text{f}}}{R_{\text{target}}}, Z_{t+1})R^{\text{e}}_{t+1}] \end{aligned} \tag{A.3}$$

which we write as:

$$\tilde{x}_t = -\{\frac{F_t}{R_{\text{target}}}\mathbb{E}_t[\tilde{B}_{t+1}]\}^{-1} \times \mathbb{E}_t[\tilde{A}_{t+1}] \tag{A.4}$$

Supposed that the optimal portfolio weights at $s = t+1, \ldots, T-1$ are already known, we approximate the terminal funding ratio by:

$$\hat{F}_T = \frac{F_t R^{\text{f}}}{R_{\text{target}}}\frac{\psi_{t+1}}{R_{\text{target}}^{T-(t+1)}}$$

Under constant relative risk aversion the two conditional expectations in (A.4) become:

$$\mathbb{E}_t[\tilde{A}_{t+1}] = \Big(\frac{F_t R^{\text{f}}}{R_{\text{target}}}\Big)^{-\gamma}\mathbb{E}_t[\Big(\frac{\psi_{t+1}}{R_{\text{target}}^{T-(t+1)}}\Big)^{1-\gamma}R^{\text{e}}_{t+1}] \tag{A.5}$$

$$\mathbb{E}_t[\tilde{B}_{t+1}] = -\gamma\Big(\frac{F_t R^{\text{f}}}{R_{\text{target}}}\Big)^{-\gamma-1}\mathbb{E}_t[\Big(\frac{\psi_{t+1}}{R_{\text{target}}^{T-(t+1)}}\Big)^{1-\gamma}R^{\text{e}}_{t+1}R^{\text{e}'}_{t+1}] \tag{A.6}$$

Let us substitute (A.5) and (A.6) in (A.4).

$$\tilde{x}_t = \frac{R^{\text{f}}}{\gamma}\{\mathbb{E}_t[\Big(\frac{\psi_{t+1}}{R_{\text{target}}^{T-(t+1)}}\Big)^{1-\gamma}R^{\text{e}}_{t+1}R^{\text{e}'}_{t+1}]\}^{-1}\mathbb{E}_t[\Big(\frac{\psi_{t+1}}{R_{\text{target}}^{T-(t+1)}}\Big)^{1-\gamma}R^{\text{e}}_{t+1}] \tag{A.7}$$

We use expression (A.7) for the optimal portfolio weights, vary γ and construct the efficient frontier for $T = 10$. It is shown in figure A.1, together with the static frontier and the frontier obtained by optimizing on final wealth W_T instead of F_T.

We notice that the obtained dynamic efficient frontiers are equal. Theoretically this also follows. Because $R_{\text{target}}^{T-(t+1)}$ is deterministic, we can take it out of both expectations in (A.7), so that we get:

$$\tilde{x}_t = \frac{R^{\text{f}}}{\gamma}\{\mathbb{E}_t[\psi_{t+1}^{1-\gamma}R^{\text{e}}_{t+1}R^{\text{e}'}_{t+1}]\}^{-1}\mathbb{E}_t[\psi_{t+1}^{1-\gamma}R^{\text{e}}_{t+1}] \tag{A.8}$$

We notice that (A.8) is exactly equal to (5.23). In other words, optimizing on terminal funding ratio instead of terminal wealth leads to the same portfolio weights.[1] Moreover, the optimal strategy will be independent of $R_{\text{target}}^{T-(t+1)}$.

[1] This was also found by Burger ([11]).

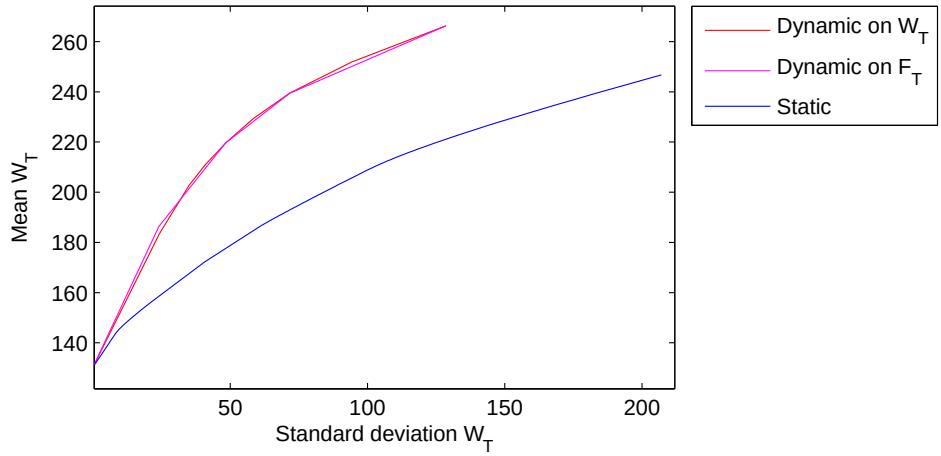

Figure A.1: *Efficient frontier with multiple assets for static versus dynamic strategies obtained by optimizing on W_T (red line) or F_T (pink line), $T = 10$.*

A.2 Adding intertemporal capital injections

A.2.1 Absolute injections

Until now we assumed an investor with initial capital W_1, who did not inject (or consume) a part of his investment capital between time 1 and T. Now let us assume that the investor injects I_s to his investment capital at each time s. I_s is deterministic and independent of his wealth.[2].

The problem is now as follows. The investor starts with initial wealth $W_1 = 0$ and adds an initial injection of I_1. At time T he does not inject any money, $I_T = 0$. The budget constraint 4.9 changes to:

$$W_{s+1} = (W_s + I_s)(x_s' R_{s+1}^{\mathrm{e}} + R^{\mathrm{f}}) \tag{A.9}$$

With recursion we can show that the terminal wealth can be expressed in terms of current wealth as follows:

$$W_T = (W_t + I_t) \prod_{s=t}^{T-1} (x_s' R_{s+1}^{\mathrm{e}} + R^{\mathrm{f}}) + \sum_{s=t+1}^{T-1} I_s \prod_{j=s}^{T-1} (x_j' R_{j+1}^{\mathrm{e}} + R^{\mathrm{f}})$$

which simplifies to:

$$W_T = W_t \psi_t + \sum_{s=t}^{T-1} I_s \psi_s = W_t \psi_t + \chi_t$$

with ψ_t as in (5.13) and χ_t defined as:

$$\chi_t = \sum_{s=t}^{T-1} I_s \psi_s \tag{A.10}$$

[2]To adapt for consumption, I_t will equal the net effect of consumption and injection at time t

The Bellman equation changes to:

$$J_t(W_t + I_t, Z_t) = \max_{\{x_s\}_{s=t}^{T-1}} \mathbb{E}_t[u(W_T)] \tag{A.11}$$

$$= \max_{x_t} \mathbb{E}_t[J_{t+1}(W_{t+1} + I_{t+1}, Z_{t+1})] \tag{A.12}$$

For notational convenience we do not show the second argument, Z_t of the value function. A second-order Taylor expansion of $J_{t+1}(W_{t+1} + I_{t+1})$ around $(W_t + I_t)R^{\mathrm{f}} + I_{t+1}$ gives:

$$\begin{aligned}\tilde{J}_{t+1}(W_{t+1} + I_{t+1}) &= J_{t+1}((W_t + I_t)(x_t' R^{\mathrm{e}}_{t+1} + R^{\mathrm{f}}) + I_{t+1}) \\ &\quad + \partial_1 J_{t+1}((W_t + I_t)R^{\mathrm{f}} + I_{t+1})(W_t + I_t)(x_t' R^{\mathrm{e}}_{t+1}) \\ &\quad + \frac{1}{2}\partial_1^2 J_{t+1}((W_t + I_t)R^{\mathrm{f}} + I_{t+1})(W_t + I_t)^2 (x_t' R^{\mathrm{e}}_{t+1})^2 \end{aligned} \tag{A.13}$$

which leads to an explicit solution for the approximate optimal portfolio weight $\tilde{x}_t$:

$$\begin{aligned}\tilde{x}_t &= -\{(W_t + I_t)\mathbb{E}_t[\partial_1^2 J_{t+1}((W_t + I_t)R^{\mathrm{f}} + I_{t+1})R^{\mathrm{e}}_{t+1} R^{\mathrm{e}'}_{t+1}]\}^{-1} \\ &\quad \times \mathbb{E}_t[\partial_1 J_{t+1}((W_t + I_t)R^{\mathrm{f}} + I_{t+1})R^{\mathrm{e}}_{t+1}] \end{aligned} \tag{A.14}$$

The corresponding approximation of terminal wealth is:

$$\begin{aligned}\hat{W}_T &= (W_t + I_t)R^{\mathrm{f}}\psi_{t+1} + \chi_{t+1} \\ &= ((W_t + I_t)R^{\mathrm{f}} + I_{t+1})\psi_{t+1} + \chi_{t+2}\end{aligned}$$

Supposing that optimal weights from time $t+1$ until $T-1$ have already been determined and assuming a CRRA-investor, the two conditional expectations in (A.14) give:

$$\begin{aligned}&\mathbb{E}_t[\partial_1 J_{t+1}((W_t + I_t)R^{\mathrm{f}} + I_{t+1})R^{\mathrm{e}}_{t+1}] \\ &= \mathbb{E}_t[\partial u(\hat{W}_T)\psi_{t+1} R^{\mathrm{e}}_{t+1}] \\ &= \mathbb{E}_t[((W_t + I_t)R^{\mathrm{f}}\psi_{t+1} + \chi_{t+1})^{-\gamma}\psi_{t+1} R^{\mathrm{e}}_{t+1}]\end{aligned}$$

and:

$$\begin{aligned}&\mathbb{E}_t[\partial_1^2 J_{t+1}((W_t + I_t)R^{\mathrm{f}} + I_{t+1})R^{\mathrm{e}}_{t+1} R^{\mathrm{e}'}_{t+1}] \\ &= -\gamma\mathbb{E}_t[((W_t + I_t)R^{\mathrm{f}}\psi_{t+1} + \chi_{t+1})^{-\gamma-1}\psi_{t+1}^2 R^{\mathrm{e}}_{t+1} R^{\mathrm{e}'}_{t+1}]\end{aligned}$$

By the independency of $\{I_s\}_{s=t}^{T-1}$ and W_t, we cannot take $(W_t + I_t)$ out of the conditional expectations. In that case x_t still depends on the endogenous and unknown variable W_t. This can be solved by using a grid for W_t, as explained in section 5.7.3.

A.2.2 Proportional injections

However, we can significantly simplify the situation of the last section by assuming that the investor always injects or consumes a fixed proportion of wealth. For injection this might not

be very realistic, for consumption it is. Suppose at each time t the investor injects an additional cW_t, $c \in \mathbb{R}\backslash\{-1\}$. Then the budget constraint becomes

$$W_{t+1} = (1+c)W_t(x_t' R_{t+1}^{\mathrm{e}} + R^{\mathrm{f}})$$

and terminal wealth is

$$W_T = (1+c)^{T-t} W_t \psi_t$$

We get

$$J_{t+1}(W_{t+1} + I_{t+1}) = J_{t+1}((1+c)W_{t+1}) = J_{t+1}((1+c)^2 W_t(x_t' R_{t+1}^{\mathrm{e}} + R^{\mathrm{f}}))$$

Under a second-order expansion this will lead to:

$$\begin{aligned} \tilde{x}_t &= -\{(1+c)^2 W_t \mathbb{E}_t[\partial_1^2 J_{t+1}((1+c)^2 W_t R^{\mathrm{f}}) R_{t+1}^{\mathrm{e}} R_{t+1}^{\mathrm{e}'}]\}^{-1} \\ &\quad \times \mathbb{E}_t[\partial_1 J_{t+1}((1+c)^2 W_t R^{\mathrm{f}}) R_{t+1}^{\mathrm{e}}] \end{aligned} \tag{A.15}$$

Furthermore, under the assumption of constant relative risk aversion, we have that:

$$\begin{aligned} &\mathbb{E}_t[\partial_1 J_{t+1}((1+c)^2 W_t R^{\mathrm{f}}) R_{t+1}^{\mathrm{e}}] \\ &= \mathbb{E}_t[((1+c)^{T-t} W_t R^{\mathrm{f}} \psi_{t+1})^{-\gamma} (1+c)^{T-t-2} \psi_{t+1} R_{t+1}^{\mathrm{e}}] \\ &= (W_t R^{\mathrm{f}})^{-\gamma} (1+c)^{(1-\gamma)(T-t)-2} \mathbb{E}_t[\psi_{t+1}^{1-\gamma} R_{t+1}^{\mathrm{e}}] \end{aligned}$$

and

$$\begin{aligned} &\mathbb{E}_t[\partial_1^2 J_{t+1}((1+c)^2 W_t R^{\mathrm{f}}) R_{t+1}^{\mathrm{e}} R_{t+1}^{\mathrm{e}'}] \\ &= \mathbb{E}_t[((1+c)^{T-t} W_t R^{\mathrm{f}} \psi_{t+1})^{-\gamma-1} ((1+c)^{T-t-2} \psi_{t+1})^2 R_{t+1}^{\mathrm{e}} R_{t+1}^{\mathrm{e}'}] \\ &= (W_t R^{\mathrm{f}})^{-1-\gamma} (1+c)^{(1-\gamma)(T-t)-4} \mathbb{E}_t[\psi_{t+1}^{1-\gamma} R_{t+1}^{\mathrm{e}} R_{t+1}^{\mathrm{e}'}] \end{aligned} \tag{A.16}$$

Substituting (A.16) and (A.16) in (A.15) and simplifying gives the well-known expression:

$$\tilde{x}_t = \frac{R^{\mathrm{f}}}{\gamma} \{\mathbb{E}_t[\psi_{t+1}^{1-\gamma} R_{t+1}^{\mathrm{e}} R_{t+1}^{\mathrm{e}'}]\}^{-1} \mathbb{E}_t[\psi_{t+1}^{1-\gamma} R_{t+1}^{\mathrm{e}}]$$

which is the same as (5.23) and (A.8). The optimal strategy is not changed by proportional injection or consumption of investment capital. However, it should be clear that the realizations of W_T do depend on the value of c. In fact, the problem definition of proportional injection is exactly the same as that of the funding ratio, where $(1+c)$ equals $\frac{1}{R_{\mathrm{target}}}$.

A.3 A scenario model in ALS with a limited number of variables

In chapter 7 we used the Ortec Finance scenario generator with all variables included to simulate sample paths, while we only included 5 assets and 5 state variables to calculate optimal portfolios.

In this appendix we show the results for scenarios which were only generated by the variables that were also used as state variables in the portfolio optimization process. Our intuition was that this would improve predicting power and thus give more optimal portfolios.

In table A.1 we show the performance measures for various values of γ. Even though strategies that invest fully in one asset have higher mean returns than under the complete scenarios, we do not observe notable improvements in terms of the dynamic strategies. This is even more clear if we compare the efficient frontiers, which is done in figure A.2. The efficient frontier of the scenarios that were only created with a limited set of variables is less favourable for the investor.

The results of this section underline the importance as well as the dependence of the portfolio strategy on the underlying scenarios.

Risk aversion γ	$\hat{\mu}$	$\hat{\sigma}$	P_{upf}	$\text{VaR}_{0.975}$	$\text{cVaR}_{0.975}$
1	786.6	592.8	0.046	134.8	95.7
3	665.0	360.2	0.008	229.2	186.3
5	480.2	193.0	0.004	230.4	201.5
10	262.4	57.0	0.008	187.4	176.9
∞	175.4	0	0	175.4	175.4

Table A.1: *Descriptive statistics for multiple returns,* $T = 20$

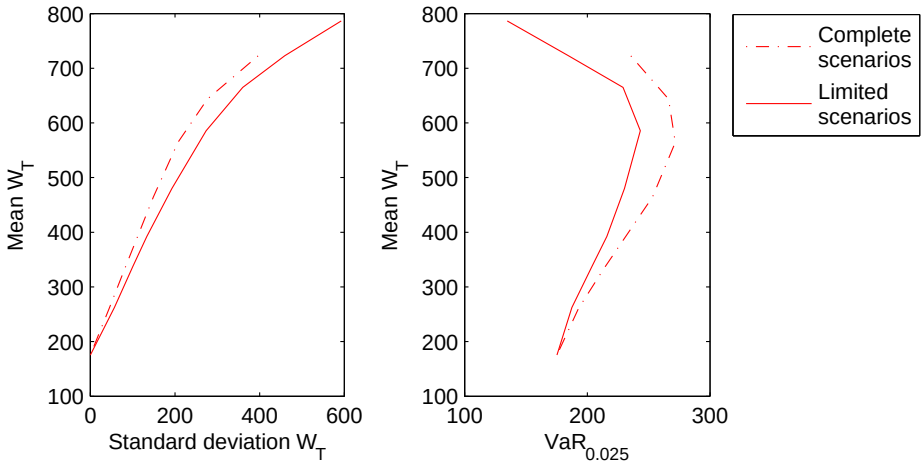

Figure A.2: *Dynamic efficient frontiers with complete scenario generation versus stripped scenario generation,* $T = 20$.

Appendix B

MATLAB-code

B.1 Model for single risky asset

Core model

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% SPECIFY INITAL PARAMETERS & READ DATA
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

M=1000;          % Number of simulations
T=20;            % Horizon
gamma = 5;       % Risk aversion
r_f=(1.06)^0.25;% Risk-free rate


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% GENERATE DATA
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

[R_e, Z] = VAR_CRSP(M,T,r_f);


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%CALCULATE PORTFOLIO WEIGHTS
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

x1 = Portfolioweights_it_bs(R_e,Z,gamma,r_f,0,0);
```

Scenario generation

```
function [R_e, dp] = VAR_CRSP(M,T,r_f);

epsilon = zeros(M,T,2);
r = zeros(M,T);
dp= zeros(M,T);

for m=1:M
    epsilon(m,:,:) = mvnrnd([0;0], [0.006,-0.0051;-0.0051,0.0049],T);
end

for m=1:M
    dp0=-0.155/(1-0.958);
    r(m,1) = 0.227+0.060*dp0 + epsilon(m,1,1);
    dp(m,1)=-0.155+0.958*dp0 + epsilon(m,1,2);
    for i = 1:T-1
         r(m,i+1) =  0.227+0.060*dp(m,i) + epsilon(m,i+1,1);
        dp(m,i+1) = -0.155+0.958*dp(m,i) + epsilon(m,i+1,2);
    end
end

% Create excess return out of log excess return
R_e = r_f*exp(r)-r_f;
disp('Excess return created out of log excess return')
```

Calculating portfolio weights for CRRA-investor

```
function Weights = Portfolioweights_it_bs(r, z, gamma, r_f, constr_ON, it_ON)
% Calculates portfolio weight for a model with one risky asset and one state
% variable
% r contains M scenarios of excess returns for the risky asset
% z contains M scenarios of state variable
% gamma is the parameter of relative risk aversion
% r_f is the risk-free rate
% constr_ON==1: portfolio weights are between 0 and 1
% it_ON == 1: 4th order Taylor expansion is used

N_max = 10;
% maximum number of iterations to obtain 4th order solution
k=1;
% tuning constant for bisquare method

[M,T] = size(r);
x = zeros(M,T-1);
```

```
%%%%%%
t=T-1;
%%%%%%
X = [ones(M,1), z(:,t), z(:,t).^2];
abeta = bisquare(X, r(:,t+1),k);
ahat = X*abeta;
bbeta = bisquare(X, r(:,t+1).^2,k);
bhat = X*bbeta;
x(:,t) = r_f/gamma * ahat./bhat;
if it_ON==1
    disp('4th order Taylor iteration used')
    x_it = zeros(M,N_max);
    i=1;
    x_it(:,1) = x(:,t);
    cbeta = bisquare(X, (x_it(:,i).*r(:,t+1)).^2.*r(:,t+1),k);
    chat = X*cbeta;
    dbeta = bisquare(X, (x_it(:,i).*r(:,t+1)).^3.*r(:,t+1),k);
    dhat = X*dbeta;
    x_it(:,i+1) = x(:,t)- (bhat).^(-1).*...
           (0.5*(-gamma-1)/r_f*chat+1/6*(-gamma-1)/r_f^2*(-gamma-2)*dhat);
    while ( max(abs(x_it(:,i+1)-x_it(:,i)))>0.01 && i<N_max)
       i=i+1;
        cbeta = bisquare(X, (x_it(:,i).*r(:,t+1)).^2.*r(:,t+1),k);
        chat = X*cbeta;
        dbeta = bisquare(X, (x_it(:,i).*r(:,t+1)).^3.*r(:,t+1),k);
        dhat = X*dbeta;
       x_it(:,i+1) = x(:,t)- (bhat).^(-1).*...
           (0.5*(-gamma-1)/r_f*chat+1/6*(-gamma-1)/r_f^2*(-gamma-2)*dhat);
    end
    if i==N_max
        disp('Maximum number of iteration steps used')
    end
    x(:,t)=x_it(:,i+1);
else
    disp('2nd order Taylor iteration used')
end

if constr_ON==1
    x(:,t)=max(0,min(x(:,t),1));
    disp('Constraints on x')
end

for tt=2:T-1
    t=T-tt;
    X = [ones(M,1), z(:,t), z(:,t).^2];
    Psi = prod(x(:,t+1:T-1).*r(:,t+2:T)+r_f,2).^(1-gamma);
    abeta  = bisquare(X, r(:,t+1).*Psi,k);
```

```
    bbeta = bisquare(X, r(:,t+1).^2.*Psi,k);
    ahat = X*abeta;
    bhat = X*bbeta;
    x(:,t) = r_f/gamma * ahat./bhat;
    if it_ON ==1
        x_it = zeros(M,N_max);
        i=1;
        x_it(:,1) = x(:,t);
        cbeta = bisquare(X, (x_it(:,i).*r(:,t+1)).^2.*r(:,t+1).*Psi,k);
        chat = X*cbeta;
        dbeta = bisquare(X, (x_it(:,i).*r(:,t+1)).^3.*r(:,t+1).*Psi,k);
        dhat = X*dbeta;
        x_it(:,i+1) = x(:,t)- (bhat).^(-1).*...
               (0.5*(-gamma-1)/r_f*chat+1/6*(-gamma-1)/r_f^2*(-gamma-2)*dhat);
        while ( max(abs(x_it(:,i+1)-x_it(:,i)))>0.001 && i<N_max)
           i=i+1;
            cbeta = bisquare(X, (x_it(:,i).*r(:,t+1)).^2.*r(:,t+1).*Psi,k);
            chat = X*cbeta;
            dbeta = bisquare(X, (x_it(:,i).*r(:,t+1)).^3.*r(:,t+1).*Psi,k);
            dhat = X*dbeta;
           x_it(:,i+1) = x(:,t)- (bhat).^(-1).*...
               (0.5*(-gamma-1)/r_f*chat+1/6*(-gamma-1)/r_f^2*(-gamma-2)*dhat);
        end
        if i==N_max
            disp('Maximum number of iteration steps used')
        end
        x(:,t)=x_it(:,i+1);
        if i==N_max
            disp('Maximum number of iteration steps used')
        end
    end
    if constr_ON==1
        x(:,t)=max(0,min(x(:,t),1));
    end
end

Weights = x;
```

Robust regression: bisquare estimator

```
function BS = bisquare(X, y, k)
%X is regression matrix
%y are values of dependent variables

N_max = 20;
% maximum number of iterations for bisquare optimization
```

```
[M,N]=size(X);
beta = zeros(N,N_max);

beta(:,1)=lscov(X,y);
e = y-X*beta(:,1);
sigmahat = mad(e)/0.6745;
k=sigmahat*4.685;

j=1;
beta(:,j+1)= inv(X'*spdiags(W_BS(e,k),0,M,M)*X)*X'*spdiags(W_BS(e,k),0,M,M)*y;
while (max(abs(beta(:,j+1)-beta(:,j)))>0.001 && j<N_max)
    e = y-X*beta(:,j+1);
    sigmahat = mad(e)/0.6745;
    k=sigmahat*4.685;
    j=j+1;
    beta(:,j+1)= inv(X'*spdiags(W_BS(e,k),0,M,M)*X)*X'*spdiags(W_BS(e,k),0,M,M)*y;

end
if j==N_max
    disp('Bisquare uses maximal number of iterations')
end
BS = beta(:,j+1);


function w_bs = W_BS(e,k)
w_bs = (1-(e/k).^2).^2.*(abs(e)<= k)+0*(abs(e)>k);
```

CARA-investor

```
function [Weights,Pret] = Portfolioweights1_it_cara(r, z, xi, r_f, W_1, constr_ON,...
    it_ON,d,u,delta);

N_max = 20;
tic


[M,T] = size(r);
vec = ((r_f*(1-d)):delta:(r_f*(1+u)));

% Define grid for wealth
Gridsize=length(vec);
W_grid=zeros(T-1, length(vec));
for t=1:T-1
    W_grid(t,:)=(W_1*vec.^(t-1));
end
```

```
x = zeros(M,T-1, Gridsize);
Psi = zeros(M,T-1,Gridsize);

%%%%%%
t=T-1;
%%%%%%
X = [ones(M,1), z(:,t), z(:,t).^2];
abeta = lscov(X, r(:,t+1));
ahat = X*abeta;
bbeta = lscov(X, r(:,t+1).^2);
bhat = X*bbeta;
for j = 1:Gridsize
    x(:,t,j) = 1./(xi*W_grid(t,j)).*ahat./bhat;
end
    if it_ON ==1
        disp('Taylor order four')
        x_it = zeros(M,N_max);
        i=1;
        x_it(:,1) = squeeze(x(:,t,j));
        cbeta  = lscov(X, (x_it(:,1).*r(:,t+1)).^2.*r(:,t+1));
        chat = X*cbeta;
        dbeta  = lscov(X, (x_it(:,1).*r(:,t+1)).^3.*r(:,t+1));
        dhat = X*dbeta;
        x_it(:,i+1) = x(:,t,j)+ (bhat).^(-1).*...
               (0.5*xi*W_grid(t,j)*chat-xi^2*W_grid(t,j)^2/6*dhat);
        while ( max(abs(x_it(:,i+1)-x_it(:,i)))>0.001 && i<N_max)
            i=i+1;
            cbeta  = lscov(X, (x_it(i,1).*r(:,t+1)).^2.*r(:,t+1));
            chat = X*cbeta;
            dbeta  = lscov(X, (x_it(i,1).*r(:,t+1)).^3.*r(:,t+1));
            dhat = X*dbeta;
            x_it(:,i+1) = x(:,t,j)+ (bhat).^(-1).*...
                   (0.5*xi*W_grid(t,j)*chat-xi^2*W_grid(t,j)^2/6*dhat);
        end
        x(:,t,j)=x_it(:,i+1);
    end
if constr_ON==1
    x(:,t,:)=max(0,min(x(:,t,:),1));
end
for j=1:Gridsize
    Psi(:,t,j) = (x(:,t,j).*r(:,t+1)+r_f);
end

for tt=2:T-1
    t=T-tt;
```

```
    X = [ones(M,1), z(:,t), z(:,t).^2];
    What = W_grid(t,:).*r_f;
    Psihat = (interp1(W_grid(t+1,:)',squeeze(Psi(:,t+1,:))',What'))';
    for j=1:Gridsize
        abeta  = lscov(X, exp(-xi*W_grid(t,j)*Psihat(:,j)).*r(:,t+1).*Psihat(:,j));
        bbeta  = lscov(X, exp(-xi*W_grid(t,j)*Psihat(:,j)).*r(:,t+1).^2.*...
                 (Psihat(:,j).^2));
        ahat = X*abeta;
        bhat = X*bbeta;
        x(:,t,j) = 1/(W_grid(t,j)*xi) * ahat./bhat;
        if it_ON ==1
            x_it = zeros(M,N_max);
            i=1;
            x_it(:,1) = squeeze(x(:,t,j));
            cbeta  = lscov(X, exp(-xi*W_grid(t,j)*Psihat(:,j)).*(x_it(:,1).*r(:,t+1)).^2.
                     Psihat(:,j).^3);
            chat = X*cbeta;
            dbeta  = lscov(X, exp(-xi*W_grid(t,j)*Psihat(:,j)).*(x_it(:,1).*r(:,t+1)).^3.
                     Psihat(:,j).^4);
            dhat = X*dbeta;
            x_it(:,i+1) = x(:,t,j)+ (bhat).^(-1).*...
                   (0.5*xi*W_grid(t,j)*chat-xi^2*W_grid(t,j)^2/6*dhat);
            while ( max(abs(x_it(:,i+1)-x_it(:,i)))>0.001 && i<N_max)
               i=i+1;
                cbeta  = lscov(X, exp(-xi*W_grid(t,j)*Psihat(:,j)).*(x_it(i,1).*r(:,t+1))
                         r(:,t+1).*Psihat(:,j).^3);
                chat = X*cbeta;
                dbeta  = lscov(X, exp(-xi*W_grid(t,j)*Psihat(:,j)).*(x_it(i,1).*r(:,t+1))
                         r(:,t+1).*Psihat(:,j).^4);
                dhat = X*dbeta;
                x_it(:,i+1) = x(:,t,j)+ (bhat).^(-1).*...
                   (0.5*xi*W_grid(t,j)*chat-xi^2*W_grid(t,j)^2/6*dhat);
            end
            x(:,t,j)=x_it(:,i+1);
        end
    end
    if constr_ON==1
        x(:,t,:)=max(0,min(x(:,t,:),1));
    end
    for j=1:Gridsize
        Psi(:,t,j)=Psihat(:,j).*(x(:,t,j).*r(:,t+1)+r_f);
    end
end
toc

% Indici: M, T, Gridsize
%[mean(mean(Psi(:,1,:),3)') ; std(mean(Psi(:,1,:),3)')]
```

```
Weights = x;
Pret = Psi;
```

Quadratic utility

```
function [Weights,Pret] = Portfolioweights1_it_quad(r, z, b, r_f, W_1, constr_ON,d,u,delta);
% For an investor with quadratic utility
tic

[M,T] = size(r);
vec = ((r_f*(1-d)):delta:(r_f*(1+u)));

% Define grid for wealth
Gridsize=length(vec);
W_grid=zeros(T-1, length(vec));
for t=1:T-1
    W_grid(t,:)=(W_1*vec.^(t-1));
end

x = zeros(M,T-1, Gridsize);
Psi = zeros(M,T-1,Gridsize);

%%%%%%
t=T-1;
%%%%%%
X = [ones(M,1), z(:,t), z(:,t).^2];

bbeta = lscov(X, r(:,t+1).^2);
bhat = X*bbeta;
for j = 1:Gridsize
    abeta = lscov(X, (1-b*W_grid(t,j)*r_f)*r(:,t+1));
    ahat = X*abeta;
    x(:,t,j) = 1/b*1/W_grid(t,j)*ahat./bhat;
end
if constr_ON==1
    x(:,t,:)=max(0,min(x(:,t,:),1));
end
for j=1:Gridsize
    Psi(:,t,j) = (x(:,t,j).*r(:,t+1)+r_f);
end

for tt=2:T-1
    t=T-tt;
    X = [ones(M,1), z(:,t), z(:,t).^2];
```

```
    What = W_grid(t,:).*r_f;
    Psihat = (interp1(W_grid(t+1,:)',squeeze(Psi(:,t+1,:))',What'))';
    for j=1:Gridsize
        abeta  = lscov(X, (1-b*W_grid(t,j)*r_f*Psihat(:,j)).*r(:,t+1).*Psihat(:,j));
        bbeta  = lscov(X, r(:,t+1).^2.*(Psihat(:,j).^2));
        ahat = X*abeta;
        bhat = X*bbeta;
        x(:,t,j) = 1/b*1/W_grid(t,j) * ahat./bhat;
    end
    if constr_ON==1
        x(:,t,:)=max(0,min(x(:,t,:),1));
    end
    for j=1:Gridsize
        Psi(:,t,j)=Psihat(:,j).*(x(:,t,j).*r(:,t+1)+r_f);
    end
end
toc

Weights = x;
Pret = Psi;
```

B.2 Model for multiple risky assets

```
function Weights = Portfolio_fmincon_multiple_it_state_vb_bs(r,z,gamma,r_f,constr_ON,...
                   it_ON, N_max)
% Calculates portfolio weights for multiple assets
% Constraining weights between 0 and 1 and only changing portfolio weights with 0.20
% at each time step
% Possibility to use a fourth-order Taylor expansion
% Bisquare robust regression

[M,T,N_assets] = size(r);
x = zeros(M,T-1,N_assets);
x_0 = zeros(N_assets,1);
options = optimset('Display','off','TolFun',1e-6,'LargeScale','off');

if constr_ON==1
    LB = zeros(N_assets,1);
    UB = ones(N_assets,1);
else
    LB = [];
    UB = [];
end
```

```
tic

%%%%%%
t=T-1;
%%%%%%

    X = [ones(M,1), r(:,t,1), r(:,t,2), r(:,t,3), r(:,t,4), r(:,t,5),...
        z(:,t,1), z(:,t,1).^2, z(:,t,2), z(:,t,2).^2, z(:,t,3),...
        z(:,t,3).^2, z(:,t,4), z(:,t,4).^2, z(:,t,5), z(:,t,5).^2, ...
        z(:,t,1).*z(:,t,2), z(:,t,1).*z(:,t,3),z(:,t,1).*z(:,t,4),...
        z(:,t,1).*z(:,t,5), z(:,t,2).*z(:,t,3), z(:,t,2).*z(:,t,4),...
        z(:,t,2).*z(:,t,5), z(:,t,3).*z(:,t,4), z(:,t,3).*z(:,t,5),...
        z(:,t,4).*z(:,t,5)];

ahat = zeros(M,N_assets);
B    = zeros(M,N_assets,N_assets);
Bhat = zeros(M,N_assets,N_assets);
chat = zeros(M,N_assets);
dhat = zeros(M,N_assets);


for i=1:N_assets
    abeta = bisquare(X, r(:,t+1,i),1);
    ahat(:,i) = X*abeta;
end

for m=1:M
    B(m,:,:) = squeeze(r(m,t+1,:))*squeeze(r(m,t+1,:))';
end

for i=1:N_assets
    for j=1:N_assets
        beta = bisquare(X, B(:,i,j),1);
        Bhat(:,i,j)=X*beta;
    end
end

% Optimize with second-order Taylor-expansion
for m=1:M
    x(m,t,:) = fmincon(@(y) -(y'*ahat(m,:)'-0.5*gamma/r_f*y'*squeeze(Bhat(m,:,:))*y), ...
        x_0, ones(1,N_assets), 1, [],[],LB,UB,[], options);
end

% Fourth-order Taylor expansion
if it_ON ==1
    disp('4th order Taylor iteration used - if optimal')
    x_it = zeros(M,N_assets,N_max+1);
```

```
    x_it(:,:,1)=squeeze(x(:,t,:));
    xrprod=zeros(M,1);
    for i=1:N_max
        for m=1:M
            xrprod(m,1) = squeeze(x(m,t,:))'*squeeze(r(m,t+1,:));
        end
        for j=1:N_assets
            cbeta = bisquare(X, xrprod.^3.*squeeze(r(:,t+1,j)),1);
            chat(:,j) = X*cbeta;
            dbeta = bisquare(X, xrprod.^4.*squeeze(r(:,t+1,j)),1);
            dhat(:,j) = X*dbeta;
        end
        for m = 1:M
            x_it(m,:,i+1)=(x_it(m,:,1)'-squeeze(Bhat(m,:,:))^(-1)*...
                          (0.5*(-gamma-1)/r_f*chat(m,:)'+...
                           1/6*(-gamma-1)*(-gamma-2)/r_f^2*dhat(m,:)'))';
        end
    end
    for m=1:M
            if sum(x_it(m,:,N_max+1)>=0)==N_assets && sum(x_it(m,:,i+1)<=1)==N_assets &&
                x(m,t,:)=x_it(m,:,i+1);
            end
   end
end

%%%%%%
t=T-2;
%%%%%%
    X = [ones(M,1), r(:,t,1), r(:,t,2), r(:,t,3), r(:,t,4), r(:,t,5),...
        z(:,t,1), z(:,t,1).^2, z(:,t,2), z(:,t,2).^2, z(:,t,3),...
        z(:,t,3).^2, z(:,t,4), z(:,t,4).^2, z(:,t,5), z(:,t,5).^2, ...
        z(:,t,1).*z(:,t,2), z(:,t,1).*z(:,t,3),z(:,t,1).*z(:,t,4),...
        z(:,t,1).*z(:,t,5), z(:,t,2).*z(:,t,3), z(:,t,2).*z(:,t,4),...
        z(:,t,2).*z(:,t,5), z(:,t,3).*z(:,t,4), z(:,t,3).*z(:,t,5),...
        z(:,t,4).*z(:,t,5)];

Psi = zeros(M,1);
for m=1:M
    Psi(m,1)= (squeeze(x(m,t+1,:))'*squeeze(r(m,t+2,:))+r_f).^(1-gamma);
end
    for i=1:N_assets
       abeta = bisquare(X, Psi.*r(:,t+1,i),1);
       ahat(:,i) = X*abeta;
    end

    for m=1:M
        B(m,:,:) = squeeze(r(m,t+1,:))*squeeze(r(m,t+1,:))';
```

```
    end
    for i=1:N_assets
        for j=1:N_assets
            bbeta = bisquare(X, Psi.*B(:,i,j),1);
            Bhat(:,i,j)= X*bbeta;
        end
    end
LB = max(0,x(:,t+1,:)-0.2);
UB = min(1,x(:,t+1,:)+0.2);
for m=1:M
    x(m,t,:) = fmincon(@(y) -(y'*ahat(m,:)'-0.5*gamma/r_f*y'*squeeze(Bhat(m,:,:))*y), ...
        x_0, ones(1,N_assets), 1, [],[],LB(m,:)',UB(m,:)',[], options);
end
if it_ON ==1
    disp('4th order Taylor iteration used')
    x_it = zeros(M,N_assets,N_max+1);
    x_it(:,:,1)=squeeze(x(:,t,:));
    xrprod=zeros(M,1);
    for i=1:N_max
        for m=1:M
            xrprod(m,1) = squeeze(x(m,t,:))'*squeeze(r(m,t+1,:));
        end
        for j=1:N_assets
            cbeta = bisquare(X, Psi.*xrprod.^3.*squeeze(r(:,t+1,j)),1);
            chat(:,j) = X*cbeta;
            dbeta = bisquare(X, Psi.*xrprod.^4.*squeeze(r(:,t+1,j)),1);
            dhat(:,j) = X*dbeta;
        end
        for m = 1:M
            x_it(m,:,i+1)=(x_it(m,:,1)'-squeeze(Bhat(m,:,:))^(-1)*...
                           (0.5*(-gamma-1)/r_f*chat(m,:)'+...
                           1/6*(-gamma-1)*(-gamma-2)/r_f^2*dhat(m,:)'))';
        end
    end
    for m=1:M
        if sum(x_it(m,:,N_max+1)>=0)==N_assets && ...
                 sum(x_it(m,:,i+1)<=1)==N_assets && sum(x_it(m,:,i+1))<=1
                x(m,t,:)=x_it(m,:,i+1);
        end
   end
end

%%%%%

for tt=3:T-1
    t=T-tt;
```

```
X = [ones(M,1), r(:,t,1), r(:,t,2), r(:,t,3), r(:,t,4), r(:,t,5),...
    z(:,t,1), z(:,t,1).^2, z(:,t,2), z(:,t,2).^2, z(:,t,3),...
    z(:,t,3).^2, z(:,t,4), z(:,t,4).^2, z(:,t,5), z(:,t,5).^2, ...
    z(:,t,1).*z(:,t,2), z(:,t,1).*z(:,t,3),z(:,t,1).*z(:,t,4),...
    z(:,t,1).*z(:,t,5), z(:,t,2).*z(:,t,3), z(:,t,2).*z(:,t,4),...
    z(:,t,2).*z(:,t,5), z(:,t,3).*z(:,t,4), z(:,t,3).*z(:,t,5),...
    z(:,t,4).*z(:,t,5)];

for m=1:M
    Psi(m)= Psi(m).*((squeeze(x(m,t+1,:))'*squeeze(r(m,t+2,:))+r_f).^(1-gamma));
end

for i=1:N_assets
   abeta = bisquare(X, Psi.*r(:,t+1,i),1);
   ahat(:,i) = X*abeta;
end

for m=1:M
    B(m,:,:) = squeeze(r(m,t+1,:))*squeeze(r(m,t+1,:))';
end
for i=1:N_assets
    for j=1:N_assets
        bbeta = bisquare(X, Psi.*B(:,i,j),1);
        Bhat(:,i,j)= X*bbeta;
    end
end
LB = max(0,x(:,t+1,:)-0.2);
UB = min(1,x(:,t+1,:)+0.2);
for m=1:M
    x(m,t,:) = fmincon(@(y) -(y'*ahat(m,:)'-0.5*gamma/r_f*y'*squeeze(Bhat(m,:,:))*y),
        x_0, ones(1,N_assets), 1, [],[],LB(m,:)',UB(m,:)',[], options);
end

if it_ON ==1
    disp('4th order Taylor iteration used')
    x_it = zeros(M,N_assets,N_max+1);
    x_it(:,:,1)=squeeze(x(:,t,:));
    xrprod=zeros(M,1);
    for i=1:N_max
        for m=1:M
            xrprod(m,1) = squeeze(x(m,t,:))'*squeeze(r(m,t+1,:));
        end
        for j=1:N_assets
            cbeta = bisquare(X, Psi.*xrprod.^3.*squeeze(r(:,t+1,j)),1);
            chat(:,j) = X*cbeta;
            dbeta = bisquare(X, Psi.*xrprod.^4.*squeeze(r(:,t+1,j)),1);
            dhat(:,j) = X*dbeta;
```

```
            end
            for m = 1:M
                x_it(m,:,i+1)=(x_it(m,:,1)'-squeeze(Bhat(m,:,:))^(-1)*...
                               (0.5*(-gamma-1)/r_f*chat(m,:)'+...
                               1/6*(-gamma-1)*(-gamma-2)/r_f^2*dhat(m,:)'))';
            end
        end
        for m=1:M
            if sum(x_it(m,:,N_max+1)>=0)==N_assets && ...
                        sum(x_it(m,:,i+1)<=1)==N_assets && sum(x_it(m,:,i+1))<=1
                    x(m,t,:)=x_it(m,:,i+1);
            end
       end
    end

end

toc

Weights = x;
```