Pricing Multi-Dimensional Options by Grid Stretching and High Order Finite Differences

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Joint work with Coen Leentvaar

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Research Concept

Numerical Treatment of Equations for Option Pricing

- Accurate discretization with only a few grid points
- **Example I:** Black-Scholes equation
  - Grid stretching
  - High order discretization
  - European option
- **Example II:** Basket options
  - Increasing problem dimensions ⇒ sparse grids
  - A coordinate transformation
  - Grid stretching and discretization as in the 1D case
  - European basket options
Application: Option pricing

Basic options

- **European Call option**: at maturity time $T$, the holder may **purchase** an asset for the exercise price $K$. The writer must sell the asset, if the holder decides to buy it.

- **European Put option**: The right to **sell** an asset on a certain date at a prescribed amount.

- **Exotic options**: options depending on other functions of the stock price (average stock price, minimum, maximum, a **basket of stocks**)
Options on a Single Asset

Point of Departure

- The asset price follows the lognormal random walk, \( dS_t = \mu S_t dt + \sigma S_t dW_t \), with \( W_t \) a Wiener process, \( \mu \) is drift, \( \sigma \) volatility.
- Interest rate \( r \), dividend yield \( \delta \) and \( \sigma \) are known,
- Transaction costs for hedging are not included,
- There are no arbitrage possibilities.

\[ \Rightarrow \text{Black-Scholes partial differential equation: (for a European option)} \]
\[ \frac{\partial u}{\partial t} + \frac{1}{2} \sigma^2 s^2 \frac{\partial^2 u}{\partial s^2} + (r - \delta) s \frac{\partial u}{\partial s} - ru = 0 \]

- Nobel prize in 1997 for Merton and Scholes (Black died in 1995).
Final/Boundary conditions

Single Asset

- European **Call** option: Right to buy assets at maturity $t = T$ for exercise price $K$.
- Call option: Final condition: $u(s, T) = \max(s - K, 0) = (s - K)^+$

![Graph](image)

- Boundary conditions for $s \approx 0$: $u(s, t) = 0$.
- Boundary conditions at $s \to \infty$: $u(s, t) = se^{-\delta(T-t)} - Ke^{-r(T-t)}$ or: $u_{ss} = 0$.
- The standard European option can be solved exactly (serves as a reference).
Important Quantities

Hedge Parameters

• **Delta**: the rate of change of the option value with respect to $s$. Portfolio with $u \pm \Delta s$ is instantaneously risk neutral.

$$\Delta = \frac{\partial u}{\partial s}$$

• **Gamma**: indicates the change in Delta

$$\Gamma = \frac{\partial^2 u}{\partial s^2}$$

• If Gamma is high, the portfolio results for a very short time in a risk-less scenario.

• There are several other important hedging parameters.
Increasing dimensions

Multi-Asset Options

• The problem dimension increases if the price of an option depends on more than one asset $s_i$ (the so-called multi-asset options).

• Each underlying asset is assumed to follow a geometric (lognormal) diffusion process. It is assumed that the correlation of each asset to all other assets is constant.

• Each additional asset is represented by an extra dimension in the problem:

$$Lu : = \frac{\partial u}{\partial t} + \frac{1}{2} \sum_{i,j=1}^{d} \left[ \sigma_i \sigma_j \rho_{i,j} s_i s_j \frac{\partial^2 u}{\partial s_i \partial s_j} \right] + \sum_{i=1}^{d} \left[ (r - \delta_i) s_i \frac{\partial u}{\partial s_i} \right] - ru = 0 .$$

• The required information to value a basket option is the volatility of each asset $\sigma_i$ and the correlation between each pair of assets $\rho_{i,j}$. 

Introduction: Several option-types

- Final conditions determine the type of the option
  - **Basket Call**, option on a basket of assets:
    \[ u(s, T) = \max \left\{ \sum_{i=1}^{d} n_i s_i - K, 0 \right\} \]
  - Call option on the maximum of several assets
    \[ u(s, T) = \max \left\{ \max\{s_1, s_2, \ldots, s_d\} - K, 0 \right\} \]
  - Exchange option on two assets
    \[ u(s, T) = \max\{s_1 - s_2, 0\} \]
Basket options

- A **basket option** is an option whose payoff depends on the value of a portfolio (or basket) of assets. Basket options are growing in popularity as a means of hedging the risk of a portfolio and are highly interesting for banks nowadays.

- They are attractive because an option on a basket is **cheaper** than buying options on the individual assets. Furthermore, their payoff profile replicates the changes in a portfolio’s value more closely than any combination of options on the underlying assets.
1D Grid Stretching

General

- Consider a general parabolic PDE with non-constant coefficients
  \[
  \frac{\partial u}{\partial t} = \alpha(s) \frac{\partial^2 u}{\partial s^2} + \beta(s) \frac{\partial u}{\partial s} + \gamma(s) u(s, t)
  \]

- Coordinate transformation \( x = \xi(s) \) (one-to-one), inverse \( s = \eta(x) = \xi^{-1}(x) \) and \( \hat{u}(x, t) := u(s, t) \).

- Chain rule, the first and second derivative:
  \[
  \frac{\partial u}{\partial s} = \frac{1}{\eta'(x)} \frac{\partial \hat{u}}{\partial x}, \\
  \frac{\partial^2 u}{\partial s^2} = \frac{1}{(\eta'(x))^2} \frac{\partial^2 \hat{u}}{\partial x^2} - \frac{\eta''(x)}{(\eta'(x))^3} \frac{\partial \hat{u}}{\partial x}.
  \]

Application changes the factors \( \alpha, \beta \) and \( \gamma \) into:

\[
\hat{\alpha}(x) = \frac{\alpha(\eta(x))}{(\eta'(x))^2}, \quad \hat{\beta}(x) = \frac{\beta(\eta(x))}{\eta'(x)} - \alpha(\eta(x)) \frac{\eta''(x)}{(\eta'(x))^3}, \quad \hat{\gamma}(x) = \gamma(\eta(x)).
\]
A coordinate transformation that clusters points in the region of interest, around \( s = K \), the nondifferentiability in the final condition.

Spatial transformation used for Black-Scholes [Clarke-Parrott, Tavella-Randall]:

\[
x = \xi(s) = \sinh^{-1} (\zeta(s - K)) + \sinh^{-1} (\zeta K).
\]

An equidistant grid discretization \((n_x \text{ and } n_t \text{ cells})\) after the analytic transformation
Discretization

Fourth Order Discretization

- Finite differences, based on Taylor’s expansion
- $O(h^2 + k^2)$ is easily achieved by central differencing and Crank-Nicolson discretization
- **Our aim**: High accuracy with only a few grid points

$\Rightarrow$ 4th order “long stencil” discretizations in space and in time “$O(h^4 + k^4)$”

- The 4th order implicit Backward Differentiation Formula, BDF4, time integration is used.
Discretization

- Fourth order in space (long stencils):
  \[
  \frac{\partial \hat{u}_i}{\partial t} = \frac{1}{12h^2} \hat{\alpha}_i (\hat{u}_{i+2} + 16\hat{u}_{i+1} - 30\hat{u}_i + 16\hat{u}_{i-1} - \hat{u}_{i-2}) + \\
  + \frac{1}{12h} \hat{\beta}_i (-\hat{u}_{i+2} + 8\hat{u}_{i+1} - 8\hat{u}_{i-1} + \hat{u}_{i-2}) + \hat{\gamma}_i \hat{u}_i + O(h^4), \quad 2 \leq i \leq N - 2. 
  \]  
  (1)

- Fourth order in time: BDF4 scheme (preceded by CN, BDF3). BDF4 reads
  \[
  \left( \frac{25}{12} I - kL \right) \hat{u}^{j+1} = 4\hat{u}^j - 3\hat{u}^{j-1} + \frac{4}{3} \hat{u}^{j-2} - \frac{1}{4} \hat{u}^{j-3}, 
  \]  
  (2)

- No stability complications observed

- Well-suited for linear complementarity problems (for American options)
Accuracy

European option pricing experiment, dividend yield

- Error in $u_h$ and hedge parameters $\Delta_h, \Gamma_h$ (comparison with analytic solution $u_{ex}$).
- $K = 15$, $\sigma = 0.3$, $r = 0.05$, $\delta = 0.03$, $T = 0.5$.

| Scheme          | Grid         | $||u - u_{ex}||_\infty$ | rate | $||\Delta - \Delta_{ex}||_\infty$ | rate | $||\Gamma - \Gamma_{ex}||_\infty$ | rate |
|-----------------|--------------|--------------------------|------|------------------------------------|------|------------------------------------|------|
| $O(h^2 + k^2)$  | $20 \times 20$ | $3.6 \times 10^{-2}$    | 4.7  | $1.0 \times 10^{-2}$              | 3.0  | $6.2 \times 10^{-3}$              | 5.2  |
|                 | $40 \times 40$ | $8.6 \times 10^{-3}$    | 4.2  | $2.8 \times 10^{-3}$              | 3.6  | $1.6 \times 10^{-3}$              | 4.0  |

| Scheme          | Grid         | $||u - u_{ex}||_\infty$ | rate | $||\Delta - \Delta_{ex}||_\infty$ | rate | $||\Gamma - \Gamma_{ex}||_\infty$ | rate |
|-----------------|--------------|--------------------------|------|------------------------------------|------|------------------------------------|------|
| $O(h^4 + k^4)$  | $20 \times 20$ | $1.1 \times 10^{-3}$    | 10.1 | $3.1 \times 10^{-3}$              | 7.6  | $1.3 \times 10^{-3}$              | 4.8  |
| grid stretching | $40 \times 40$ | $9.4 \times 10^{-5}$    | 11.2 | $2.9 \times 10^{-4}$              | 10.8 | $9.7 \times 10^{-5}$              | 13.6 |
European pricing experiment on stretched grid

\[ u_h, \quad \Delta h, \quad \Gamma_h \]
Higher Dimensions: Sparse Grids

Combination Technique
Sparse Grids, Combination of solutions

Zenger, Griebel (1990/1991)

The combination equation for the sparse grid solutions in 2D reads:

$$u_{n}^{\text{comb}} = \sum_{|I|=n+1} u^{\text{sparse}} - \sum_{|I|=n} u^{\text{sparse}}$$

where $|I|$ corresponds to the number of grid points in each direction. If $n = 4$ then, all combinations of $|I| = 5$ are: $(16, 2), (8, 4), (4, 8), (2, 16)$.

The combination equation for the sparse grid solutions in in general dimensions reads:

$$u_{n}^{\text{comb}} = \sum_{k=0}^{d-1} (-1)^{k} \binom{d-1}{k} \sum_{|I|=n+d-1-k} u^{\text{sparse}}$$

Sparse Grid Techniques converge nicely if mixed second derivatives in the problem are bounded.
Higher Dimensions: Sparse Grids

Overall Grid

- Number of points processed $O(N(\log N)^{d-1})$ versus $O(N^d)$ (full grid)
- Accuracy of solutions $O(N^{-2}(\log N)^{d-1})$ versus $O(N^{-2})$ (full grid, $h = 1/N$)
Sparse Grid Test Case

5D Reference Equation

- Second order Accurate Discretization of

\[ \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2} + \sum_{i=1}^{d} \frac{\partial u}{\partial x_i} - 5u = 0, \text{ with solution: } u(x_1, \ldots, x_d) = e^{\sum_{i=1}^{d} (-1)^i + 1 x_i} \]

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- Asymptotic convergence factors sparse grid: 2D: 3.61, 3D: 3.13, 4D: 2.86, 5D: 2.61
Higher Dimensional B-S

Transformation, Stretching and Sparse Grids!

- See also Reisinger (2003/2004)
- Coordinate transformation

\[
\frac{\partial u}{\partial t} = \sum_{i=1}^{d} \sum_{j=1}^{d} \alpha_{ij} \frac{\partial^2 u}{\partial s_i \partial s_j} + \sum_{i=1}^{d} \beta_i \frac{\partial u}{\partial s_i} - ru
\]

with

\[
\alpha_{ij} = \frac{1}{2} \rho_{ij} \sigma_i \sigma_j s_i s_j, \quad \beta_i = (r - \delta_i) s_i
\]  \hspace{1cm} (3)

- Linear transformation:

\[
X = \Gamma S, \quad S = \Gamma^{-1} X = ZX
\]  \hspace{1cm} (4)

\[
x_i = \sum_{m=1}^{d} \gamma_{im} s_m, \quad s_i = \sum_{m=1}^{d} z_{im} x_m, \quad \frac{\partial x_i}{\partial s_k} = \gamma_{ik}
\]  \hspace{1cm} (5)
• Transformed equation:

\[
\frac{\partial u}{\partial t} = \sum_{i=1}^{d} \sum_{j=1}^{d} \hat{\alpha}_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} \hat{\beta}_i \frac{\partial u}{\partial x_i} - ru
\]

and

\[
\hat{\alpha}_{ij} = \sum_{k=1}^{d} \sum_{\ell=1}^{d} \alpha_{k\ell} \frac{\partial x_i}{\partial s_k} \frac{\partial x_j}{\partial s_\ell}, \quad \hat{\beta}_i = \sum_{k=1}^{d} \beta_k \frac{\partial x_i}{\partial s_k}
\]  

(6)

• Final Result

\[
\frac{\partial u}{\partial t} = \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \sum_{k=1}^{d} \sum_{\ell=1}^{d} \sum_{m=1}^{d} \sum_{n=1}^{d} \rho_{k\ell} \sigma_k \sigma_\ell \gamma_{ik} \gamma_{j\ell} z_{km} z_{\ell n} x_m x_n \frac{\partial^2 u}{\partial x_i \partial x_j} + \\
+ \sum_{i=1}^{d} \sum_{k=1}^{d} \sum_{m=1}^{d} (r - \delta_k) \gamma_{ik} z_{km} x_m \frac{\partial u}{\partial x_i} - ru
\]  

(7)
Actual Coordinate Transformation

• Transformation used (Tavella, Randall):

\[
2D : \begin{align*}
x_1 &= n_1s_1 + n_2s_2 \\
x_2 &= -n_1s_1 + n_2s_2
\end{align*}
\]

\[
3D : \begin{align*}
x_1 &= n_1s_1 + n_2s_2 + n_3s_3 \\
x_2 &= -n_1s_1 + n_2s_2 + n_3s_3 \\
x_3 &= -n_1s_1 - n_2s_2 + n_3s_3
\end{align*}
\]
Grid Stretching

- Stretching coordinate $x_i$ with function $x_i = x_i(y_i)$, gives a solution $u = u(y_1, y_2, \ldots, y_d)$. The equation then reads

$$
\frac{\partial u}{\partial t} = \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \sum_{k=1}^{d} \sum_{\ell=1}^{d} \sum_{m=1}^{d} \sum_{n=1}^{d} \rho_{k\ell} \sigma_k \sigma_\ell \gamma_{ik} \gamma_{j\ell} \gamma z_{km} z_{ln} x_m x_n \frac{1}{J_i J_j \partial y_i \partial y_j} \partial^2 u + \\
- \frac{1}{2} \sum_{i=1}^{d} \sum_{k=1}^{d} \sum_{\ell=1}^{d} \sum_{m=1}^{d} \sum_{n=1}^{d} \rho_{k\ell} \sigma_k \sigma_\ell \gamma_{ik} \gamma_{i\ell} \gamma z_{km} z_{ln} x_m x_n \frac{H_i}{J_i^3} \partial y_i \partial^2 u + \\
+ \sum_{i=1}^{d} \sum_{k=1}^{d} \sum_{m=1}^{d} (r - \delta_k) z_{km} x_m \frac{\gamma_{ik}}{J_i} \partial u - ru
$$

- With $J_i$ the first, $H_i$ the second derivative of the stretching function: $J_i = dx_i/dy_i, H_i = d^2x_i/dy_i^2$.

- In practice: Stretching only in $x_1$
Discretization and Kronecker products

- Each derivative must be taken apart and summed up. To simplify this, we use Kronecker products, which combine the 1D discretization stencils to the \( d \)--dimensional case.

- The second derivative of the assets is in the 2D case:

\[
\begin{bmatrix}
\frac{\partial^2 u}{\partial^2 s_2}
\end{bmatrix}_2 = \begin{bmatrix}
\frac{\partial^2 u}{\partial^2 s_2}
\end{bmatrix}_1 \otimes I_1 \Rightarrow \begin{bmatrix}
1 \\
-2 \\
1
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{\partial^2 u}{\partial^2 s_1}
\end{bmatrix}_2 = I_2 \otimes \begin{bmatrix}
\frac{\partial^2 u}{\partial^2 s_1}
\end{bmatrix}_1 \Rightarrow \begin{bmatrix}
1 & -2 & 1
\end{bmatrix}
\]
Kronecker products

General

• Let $A$ be a matrix of size $k \times \ell$ and $B$ a matrix of size $m \times n$. Then the Kronecker product $A \otimes B$ is a matrix of size $k \times m \times \ell$ with

$$A \otimes B = \begin{pmatrix}
a_{11}B & a_{12}B & \ldots & a_{1\ell}B \\
a_{21}B & a_{22}B & \ldots & a_{2\ell}B \\
\vdots & \vdots & \ddots & \vdots \\
a_{k1}B & a_{k2}B & \ldots & a_{k\ell}B
\end{pmatrix}$$

• Assume that the matrices are of appropriate size to calculate the products.

• The multiple Kronecker product is given by

$$A_1 \otimes A_2 \otimes \ldots \otimes A_N = \bigotimes_{i=1}^{N} A_i$$

and the following expression:

$$B \otimes \bigotimes_{i=1}^{N} A_i = B \otimes (A_1 \otimes A_2 \otimes \ldots \otimes A_N)$$
Discretization and Kronecker products

- The second derivative of the $i$-th asset in $d$-dimensions can be written in the same way as the 2D case:

$$\left[ \frac{\partial^2 u}{\partial^2 s_i} \right]_d = \bigotimes_{n=0}^{d-i-1} I_{d-n} \bigotimes_{n=1}^{i-1} I_{i-n}$$

- Also non-constant coefficients can easily be implemented and it also usable for the first derivative.
Kronecker products: Correlation

- To use Kronecker products with the correlation term, we rewrite this term as:

\[
\frac{\partial^2 u}{\partial s_1 \partial s_2} = \frac{\partial}{\partial s_2} \left( \frac{\partial u}{\partial s_1} \right)
\]

- It follows that:

\[
\left[ \frac{\partial^2 u}{\partial s_1 \partial s_2} \right]_2 = \left[ \frac{\partial u}{\partial s_2} \right]_1 \otimes \left[ \frac{\partial u}{\partial s_1} \right]_1
\]
Kronecker products: Correlation

• In general, for $d$ dimensions, the correlation matrix reads:

$$\left[ \frac{\partial^2 u}{\partial s_i \partial s_j} \right]_d = \bigotimes_{n=0}^{i-1} \mathbf{I}_{d-n} \otimes \left[ \frac{\partial u}{\partial s_j} \right]_1 \otimes \bigotimes_{n=i+1}^{j-1} \mathbf{I}_{d-n} \otimes \left[ \frac{\partial u}{\partial s_i} \right]_1 \otimes \bigotimes_{n=j+1}^{d-1} \mathbf{I}_{d-n}$$

• This can also be used with non-constant coefficients and therefore it is usable for the high-D Black-Scholes equation.
Results: 2D

Two-asset European basket call option (from Tavella’s book).

- $K = 100$
- $r = 4.5\%$
- $\delta_1 = 5\%$, $\delta_2 = 7\%$
- $T = 1$
- $\sigma_1 = 0.25$, $\sigma_2 = 0.35$ and $\rho_{12} = -0.63$
- Payoff: $\max\{0.58S_1 + 0.42S_2 - K, 0\}$
Coordinate Transformation + Stretching + Sparse Grids

- Payoff: \( \max \left( n_1 s_1 + n_2 s_2 - K, 0 \right) \)
  (left: no transformation, no stretching, right: stretching and transformation).

![Payoff graph](image1)

![Payoff graph](image2)
Coordinate Transformation + Sparse Grids

- Solution $u$ at $t = 0$:
  (left: no transformation, no stretching, right: stretching and transformation).

Talk at Numerical Analysis Day, 29.4.2005 01.02.2005 /nr. 31
## Numerical Results

### 2D Black-Scholes

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Basket Option: 3D

Three-asset European basket call option (from Tavella’s book).

- $K = 100$
- $r = 4.5\%$
- $\delta_1 = 5\%$, $\delta_2 = 7\%$, $\delta_3 = 4\%$
- $T = 1$
- $\sigma_1 = 0.25$, $\sigma_2 = 0.35$, $\sigma_3 = 0.20$
- $\rho_{12} = -0.63$, $\rho_{13} = 0.25$, $\rho_{23} = 0.5$
- Payoff: $\max\{0.38S_1 + 0.22S_2 + 0.4S_3 - K, 0\}$
Numerical Results

Full grid computations: $30 \times 30 \times 30 \times 15$

<table>
<thead>
<tr>
<th></th>
<th>Eq Second</th>
<th>Eq Fourth</th>
<th>Stretch Second</th>
<th>Stretch Fourth</th>
<th>Tavella</th>
</tr>
</thead>
<tbody>
<tr>
<td>3D</td>
<td>5.21</td>
<td>5.46</td>
<td>5.47</td>
<td>5.43</td>
<td>5.47</td>
</tr>
</tbody>
</table>

Sparse grid with transformation, stretching and 4th order discretization:

<table>
<thead>
<tr>
<th></th>
<th>8</th>
<th>16</th>
<th>32</th>
<th>64</th>
<th>128</th>
<th>256</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tavella</td>
<td>5.526</td>
<td>5.492</td>
<td>5.493</td>
<td>5.482</td>
<td>5.476</td>
<td>5.476</td>
</tr>
</tbody>
</table>
Conclusions

- **Options on Single Asset**: Accurate option values with grid stretching in space and 4th order discretization in space and time
- **The sparse grid method** (recombination technique) is an interesting choice if the problem dimension increases
- **Basket options**: Accurate option values with coordinate transformation, grid stretching, 4th order discretization in space and time and the sparse grid method.