Efficient Portfolio Valuation Incorporating Liquidity Risk

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Abstract

According to the theory proposed by Acerbi & Scandolo (2008), the value of a portfolio is defined in terms of public market data and idiosyncratic portfolio constraints imposed by an investor holding the portfolio. Depending on the constraints, one and the same portfolio could have different values for different investors. As it turns out, within the Acerbi-Scandolo theory, portfolio valuation can be framed as a convex optimization problem. We provide useful MSDC models and show that portfolio valuation can be solved with remarkable accuracy and efficiency.
Keywords: liquidity risk, portfolio valuation, ladder MSDC, liquidation sequence, exponential MSDC, approximation.

1 Introduction

According to the theory developed by Acerbi & Scandolo (2008) the value of a portfolio is determined by market data and a set of portfolio constraints. The market data is assumed to be publicly available and is the same for all investors. The market data of interest consists of price quotes corresponding to different trading volumes. These quotes for an asset are represented in terms of a mathematical function referred to as a Marginal Supply-Demand Curve (MSDC). See Section 2.1.

The portfolio constraints may vary across different players. These idiosyncratic constraints—collectively referred to as a *liquidity policy*—refer to restrictions that any portfolio held by the investor should be prepared to satisfy. Examples of such portfolio constraints are

- minimum cash amounts to meet shorter term liquidity needs;
- risk limits such as VaR or credit limits;
- capital limits.

To illustrate the use of the first type of constraint, while holding identical portfolios, an investor interested in relatively long term stable returns may have less strict cash requirements compared to a fund aiming for high short term redemptions. In contrast with the former, the investment fund might want to be able to liquidate all or part of its positions very quickly in order to meet short term liquidity demands. This would naturally translate into a tighter cash constraint. As will be expected, tighter cash constraints would generally yield a lower portfolio value, which is indeed what the Acerbi-Scandolo theory predicts. See Section 2.3.

To value her portfolio, the investor will mark all the positions she could possibly unwind to satisfy the constraints to the best price she is able to quote from the market. The cash amount she could maximally get from unwinding her positions accordingly will mark as the value of that portfolio. As it turns out, within Acerbi and Scandolo's theory, the valuation of a portfolio of assets can be framed as a convex optimization problem. The associated constraint set is represented by a liquidity policy. Although this was already pointed out by Acerbi and Scandolo themselves, the practical implications of this point have as yet not been investigated. Such is the aim of the present paper.

We present the fundamental concepts of Acerbi-Scandolo theory in Section 2. Thereafter, the portfolio valuation function will be studied more extensively, assuming different forms of the market data function (i.e., the MSDC). We first consider a very general setting where the MSDC is shaped as a non-increasing step function (referred to as a *ladder MSDC*) in Section 3. This corresponds to normal market situations for relatively actively traded products. We will present an algorithm for portfolio valuation assuming ladder MSDCs and a cash portfolio constraint. In Section 4, we will look at MSDCs which are shaped as decreasing exponential functions and see how the exponential functions can be used as approximations of ladder MSDCs.

All numerical results are collected in Section 5. We will find that in a wide range of cases, the approximation of ladder MSDCs by exponential MSDCs appears to be accurate, suggesting that not all market price information represented in ladder MSDCs is necessary for accurate portfolio valuation. We present our conclusions in Section 6.

2 The Portfolio Theory

In this section we present the main concepts and relevant results from Acerbi-Scandolo portfolio theory. For further details and discussion we refer to (Acerbi & Scandolo 2008).

2.1 Asset

An asset is an object traded in a market. Examples of types of assets are securities, derivatives or commodities. We make the assumption that an asset can be traded in terms of units of some standardized amount.

An asset is generally not quoted by a single price, but by a series of bid and ask prices. Each bid and each ask price is associated with a maximum trading volume. What we buy beyond the maximum trading volume is quoted for a lower price, which is also linked with a maximum trading volume, etc. On the other hand, what we sell beyond the maximum trading volume is quoted for a higher price, which is also linked to a maximum trading volume, etc. Finally, we stipulate that bid prices are always lower than ask prices. This is basically a no-arbitrage assumption.

In Acerbi and Scandolo's theory, all available market price information is represented in terms of a mathematical function referred to as a *Marginal Supply-Demand Curve (MSDC)*. Let a real-valued variable s denote the trading volume. Whenever $s > 0$ we think of this as a sale of s units of an asset; whenever $s < 0$, we think of this as a purchase of $|s|$ units of the asset. We have exclude a value for $s = 0$, as we will not be able to quote a price for trading nothing. An MSDC records the last price hit in a trade of volume s.

This leads to the following definition.

Definition 2.1. An **asset** is an object traded in a market and which is characterized by a Marginal Supply-Demand Curve (MSDC). This is defined as a map $m : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ satisfying the following two conditions:

1. $m(s)$ is non-increasing, i.e., $m(s_1) \geq m(s_2)$ if $s_1 < s_2$;

2. $m(s)$ is càdlàg (i.e., right-continuous with left limits) for $s < 0$ and làdcàg (i.e., left-continuous with right limits) for $s > 0$.

Condition 1 represents the no-arbitrage assumption mentioned above. Condition 2 ensures that MSDCs have elegant mathematical properties. In contrast with Condition 1, we will not heavily use this condition and we only mention it for the sake of completeness of exposition. Instead, what we need most of the time is that an MSDC is (Riemann) integrable on its domain.

(a) A list of prices for a stock

	shares	price (in euro)
Asks	1500	2.8710
	1000	2.8700
	2800	2.8690
	2070	2.8680
	2070	2.8660
Bids	1170	2.8600
	2070	2.8590
	900	2.8580
	500	2.8570
	3521	2.8560

(b) The MSDC of the stock

	$s \in$	$m(s)$
Asks	[-9440, -7940)	2.8710
	[-7940, -6940)	2.8700
	[-6940, -4140)	2.8690
	[-4140, -2070)	2.8680
	[-2070,0)	2.8660
Bids	(0, 1170]	2.8600
	(1170, 3240]	2.8590
	(3240, 4140]	2.8580
	(4140,4640]	2.8570
	(4640,8161]	2.8560

Table 1: A list of prices and the MSDC for a stock

Table 1(a) shows a real-time chart of the order book for a certain stock at some given time. The chart summarizes the lowest five ask and the highest five bid prices together with their maximum trading volumes. Table 1(b) represents this price information in terms of an MSDC. Note that this MSDC is a piecewise constant function. We will refer to piecewise constant MSDCs as *ladder MSDCs*. They are discussed in Section 3.

We call the limit $m^+ := \lim_{h \downarrow 0} m(h)$ the *best bid* and $m^- := \lim_{h \uparrow 0} m(h)$ the *best ask*. The *bid-ask spread*, denoted by δm, is the difference between the best ask and the best bid, i.e., $\delta m := m^- - m^+$.

An important example of an asset is the *cash* asset.

Definition 2.2. Cash is the asset representing the currency paid or received when trading any asset. It is characterized by a constant MSDC, $m_0(s) = 1$ (i.e., one unit) for every $s \in \mathbb{R} \setminus \{0\}$.

Cash is referred to as a *perfectly liquid* asset based on the following definition.

Definition 2.3. An asset is called **perfectly liquid** if the associated MSDC is constant.

We call a *security* any asset whose MSDC is a positive function (e.g., a stock, a bond, a commodity) and a *swap* any asset whose MSDC can take both positive and negative values (e.g., an interest rate swap, a

CDS, a repo transaction). A negative MSDC can be converted into a security by defining a new MSDC as $m^*(s) := -m(-s)$.

We presuppose one currency as the cash asset. For example, if we choose the euro as the cash asset, relative to the euro, the US dollar will be considered as an illiquid asset. Dependent on trading volumes, the US dollar can then be bought or sold at different bid or ask prices and hence its associated MSDC is not constant. If we choose the US dollar as our cash asset, then the opposite holds true.

2.2 Portfolio

A portfolio is characterized by listing the holding volumes of different assets in the portfolio. Each portfolio always holds a cash component.

Definition 2.4. Given are $N+1$ assets labeled $0,1,\ldots,N$. We let asset 0 denote the cash asset. A **portfolio** is a vector of real numbers, $\mathbf{p} = (p_0, p_1, \ldots, p_N) \in \mathbb{R}^{N+1}$, where p_i represents the holding volume of asset i. In particular, p_0 denotes the amount of cash in the portfolio.

When we specifically want to highlight the portfolio cash we tend to write a portfolio as $\mathbf{p} = (p_0, \overrightarrow{p})$. We henceforth presuppose a set of portfolios referred to as the *portfolio space* $\mathscr{P}$. We will assume that $\mathscr{P}$ is a vector space so that it becomes meaningful to add portfolios together and to multiply portfolios by scalar numbers. Let $\mathbf{p} = (p_0, \overrightarrow{p}) \in \mathscr{P}$ and suppose we have an additional amount a of cash. We write $\mathbf{p} + a = (p_0 + a, \overrightarrow{p})$. Note that this overloads the notation for addition. The context will usually make clear what is meant.

We sometimes refer to a holding volume p_i as the *position* in asset i. $p_i > 0$, $p_i < 0$ or $p_i = 0$ implies that we have a *long*, *short* or *zero position* in asset i respectively. Whenever we bring our long or short position in asset i to the zero position we will say that we *liquidate* our position in asset i.

An important stepping stone towards Acerbi and Scandolo's general definition of portfolio value is formed by the *liquidation Mark-to-Market value* and the *uppermost Mark-to-Market value* of a portfolio.

Definition 2.5. The **liquidation Mark-to-Market value** $L(\mathbf{p})$ of a portfolio $\mathbf{p}$ is defined as:

$$L(\mathbf{p}) := \sum_{i=0}^{N} \int_0^{p_i} m_i(x)\mathrm{d}x = p_0 + \sum_{i=1}^{N} \int_0^{p_i} m_i(x)\mathrm{d}x. \tag{1}$$

The liquidation MtM value is the total cash an investor receives from the liquidation of all her positions. The liquidation MtM value of a portfolio **p** can be viewed as the value of **p** for an investor who should be able to liquidate all her positions in exchange for cash.

The opposite case is to keep the portfolio as it is and to mark all illiquid (i.e., non-cash) assets to the best bid price or to the best ask price, depending on whether a long or short position was taken. This leads to the following definition.

Definition 2.6. The **uppermost Mark-to-Market(MtM) value** $U(\mathbf{p})$ of **p** is given by

$$U(\mathbf{p}) := \sum_{i=0}^{N}(m_i^+ \cdot \max(p_i,0) + m_i^- \cdot \min(p_i,0)) = p_0 + \sum_{i=1}^{N}(m_i^+ \cdot \max(p_i,0) + m_i^- \cdot \min(p_i,0)). \qquad (2)$$

where m_i^+ and m_i^- are the best bid and the best ask for asset i, respectively.

The uppermost MtM value can be viewed as the value of a portfolio for an investor who has no cash demands. In this sense, the portfolio is unconstrained.

Note that, as MSDCs are non-increasing, $U(\mathbf{p}) \geq L(\mathbf{p})$. The difference between $U(\mathbf{p})$ and $L(\mathbf{p})$ is termed the *uppermost liquidation cost* and is defined as $C(\mathbf{p}) := U(\mathbf{p}) - L(\mathbf{p})$.

2.3 Liquidity policy

The definitions of the liquidation MtM value $L(\mathbf{p})$ and the uppermost MtM value $U(\mathbf{p})$ suggest that the value of a portfolio **p** is subject to constraints, which represent certain cash constraints an investor should be able to meet by wholly or partly liquidating positions she has taken. There could be other types of constraints besides. For example, an investor might want to impose market risk VaR limits on her positions, or credit limits, or capital constraints. All the constraints that an investor imposes can be represented as a subset of the underlying portfolio space $\mathscr{P}$. These constraints are collectively referred to as a liquidity policy. For completeness we quote the definition from (Acerbi & Scandolo 2008).

Definition 2.7. A **liquidity policy** $\mathscr{L}$ is a closed and convex subset of $\mathscr{P}$ satisfying the following conditions:

1. if $\mathbf{p} = (p_0, \vec{p}) \in \mathscr{L}$ and $a \geq 0$, then $\mathbf{p} + a = (p_0 + a, \vec{p}) \in \mathscr{L}$;
2. if $\mathbf{p} \in \mathscr{L}$, then $(p_0, \vec{0}) \in \mathscr{L}$.

For the purpose of the present paper, the exact mathematical definition of a liquidity policy and the type of constraints constituting it are not very important. The only properties that we do care about is that any liquidity policy is a *closed and convex* subset of the underlying portfolio space. As said in the introduction, portfolio valuation can be framed as a convex optimization problem. In view of this, to demand that a liquidity policy is closed and convex ensures that each portfolio *has* a value and that this value is *unique* (see Section 2.4 for further discussion).

Example 2.1 (Liquidating-nothing policy)**.** The uppermost MtM value operator U corresponds to the *liquidating-nothing policy*

$$\mathscr{L}^{\mathrm{U}} := \mathscr{P}. \tag{3}$$

The liquidating-nothing policy effectively imposes no constraint on a portfolio. It can be viewed as a requirement imposed by an investor who has no cash demands her portfolio should be prepared to satisfy.

Example 2.2 (Liquidating-all policy)**.** The liquidation MtM value operator L corresponds to the *liquidating-all policy*

$$\mathscr{L}^{\mathrm{L}} := \{\mathbf{p} = (p_0, \vec{p}) \in \mathscr{P} | \vec{p} = \vec{0}\}. \tag{4}$$

In a sense, the liquidating all policy imposes a very strict constraint on a portfolio. It can be viewed as a portfolio requirement from an investor who should be prepared to liquidate all her positions in return for cash.

Example 2.3 (α-liquidation policy)**.** Let $\alpha = (\alpha_1, \ldots, \alpha_N)$, with $\alpha_i \in [0,1]$; $i = 1, \ldots, N$. The following liquidity policy, the *α-liquidation policy*, specifies to liquidate part of a given portfolio $\mathbf{p} = (p_0, \vec{p})$:

$$\mathscr{L}^{\alpha} := \{\mathbf{q} = (q_0, \vec{q}) \in \mathscr{P} | q_0 \geq p_0 + L(\alpha \cdot \vec{p})\}. \tag{5}$$

In this definition, $\cdot$ denotes the termwise product: $\alpha \cdot \vec{p} = (\alpha_1 p_1, \ldots, \alpha_N p_N)$. This policy indicates that an investor needs to be able to liquidate α_i parts of position p_i in return for cash.

Example 2.4 (Cash liquidity policy)**.** A liquidity policy setting a minimum cash requirement, c, is a *cash liquidity policy*:

$$\mathscr{L}(c) := \{\mathbf{p} \in \mathscr{P} | p_0 \geq c \geq 0\}. \tag{6}$$

An investor endorsing a cash liquidity policy should be prepared to liquidate her positions to such an extent that minimum cash level c is obtained. We will extensively use cash liquidity policies in Sections 3 and 4. We refer to (Acerbi 2008) for additional examples of liquidity policies.

Note that a portfolio is not supposed to satisfy a liquidity policy all the time. The meaning of the policy is that the portfolio will be *prepared* to satisfy that policy instantaneously if needed, which will be clarified in the next section.

2.4 Portfolio value

In this section, we present Acerbi and Scandolo's definition of the portfolio value function. We first need the following definition.

Definition 2.8. Let $\mathbf{p}, \mathbf{q} \in \mathscr{P}$ be portfolios. We say that $\mathbf{q}$ is **attainable** from $\mathbf{p}$ if $\mathbf{q} = \mathbf{p} - \mathbf{r} + L(\mathbf{r})$ for some $\mathbf{r} \in \mathscr{P}$. The set of all portfolios attainable from $\mathbf{p}$ is written as $\mathsf{Att}(\mathbf{p})$.

It means that a portfolio $\mathbf{q}$ is attainable from $\mathbf{p}$ if, starting from $\mathbf{p}$, liquidating $\mathbf{r}$ in return for an amount $L(\mathbf{r})$ of cash, yields $\mathbf{q}$.

The following definition is key:

Definition 2.9. The **Mark-to-Market (MtM) value** (or the **value**, for short) of a portfolio $\mathbf{p}$ subject to a liquidity policy $\mathscr{L}$ is the value of the function $V^{\mathscr{L}} : \mathscr{P} \to \mathbb{R} \cup \{-\infty\}$ defined by

$$V^{\mathscr{L}}(\mathbf{p}) := \sup\{U(\mathbf{q}) | \mathbf{q} \in \mathsf{Att}(\mathbf{p}) \cap \mathscr{L}\}. \tag{7}$$

If $\mathsf{Att}(\mathbf{p}) \cap \mathscr{L} = \varnothing$, meaning that no portfolio attainable from $\mathbf{p}$ satisfies $\mathscr{L}$, then we stipulate the portfolio value to be $-\infty$.

Proposition 2.1 (Acerbi & Scandolo (2008))**.** *The portfolio value function $V^{\mathscr{L}}$ from Definition 2.9 can be alternatively defined as*

$$V^{\mathscr{L}}(\boldsymbol{p}) = \sup\{U(\boldsymbol{p} - \boldsymbol{r}) + L(\boldsymbol{r}) | \boldsymbol{r} \in \mathscr{P}, \boldsymbol{p} - \boldsymbol{r} + L(\boldsymbol{r}) \in \mathscr{L}\}. \tag{8}$$

To prove this is not very difficult; see (Acerbi & Scandolo 2008). The proposition above allows us to frame the determination of the value of a portfolio as an optimization problem with explicit constraints, namely:

$$\begin{cases} \text{maximize} \quad U(\mathbf{p} - \mathbf{r}) + L(\mathbf{r}); \\ \text{subject to:} \quad \mathbf{p} - \mathbf{r} + L(\mathbf{r}) \in \mathscr{L}; \\ \qquad\qquad\quad \mathbf{r} \in \mathscr{P}. \end{cases} \tag{9}$$

(We ignore the case $V^{\mathscr{L}}(\mathbf{p}) = -\infty$.) This optimization problem is convex as $\mathscr{L}$ is a convex set. Since $\mathscr{L}$ is also closed, this problem has a unique optimal value (which could be $-\infty$).

Proposition 2.2. *The previous maximization problem (9) has the same optimal solution as the following* minimization *problem*

$$\begin{cases} \textit{minimize} \quad C(\boldsymbol{r}); \\ \textit{subject to:} \quad \boldsymbol{p} - \boldsymbol{r} + L(\boldsymbol{r}) \in \mathscr{L}; \\ \qquad\qquad\quad \boldsymbol{r} \in \mathscr{P}. \end{cases} \tag{10}$$

Proof. Note that $U(\mathbf{p}-\mathbf{r}) = U(\mathbf{p}) - U(\mathbf{r})$ by the definition of uppermost MtM value. It follows that the objective function of problem (9) can be rewritten as

$$U(\mathbf{p}) - U(\mathbf{r}) + L(\mathbf{r}).$$

Since, given $\mathbf{p}$, we can always determine $U(\mathbf{p})$, maximizing this function under the given constraints will yield the same optimal solution $\mathbf{r}^*$ as maximizing the following function under the same constraints:

$$-U(\mathbf{r}) + L(\mathbf{r}).$$

Obviously, minimizing

$$U(\mathbf{r}) - L(\mathbf{r})$$

again yields the same optimal solution $\mathbf{r}^*$. Noting that $C(\mathbf{r}) = U(\mathbf{r}) - L(\mathbf{r})$ proves the result. □

Informally, this result implies that to determine the value of a portfolio is to determine a portfolio $\mathbf{r}^*$ such that liquidating $\mathbf{r}^*$ in exchange for cash minimizes the uppermost liquidation costs $C(\mathbf{r}^*)$. This result will prove useful at a later stage.

3 Portfolio Valuation Using Ladder MSDCs

In the present section we will provide an algorithm providing an exact global solution for problem (9) under the assumption that the MSDC for the illiquid assets are piecewise constant, as we will name them *ladder MSDCs*.

Within the Acerbi-Scandolo theory, ladder MSDCs will play a key role to model the liquidity of the assets. Equipped with the fast and accurate algorithm discussed in this section, one could solve the convex optimization problem incurred in portfolio valuation more efficiently than using some conventional optimization techniques.

3.1 The optimization problem

Ladder MSDCs can represent a market wherein we can quote a price for each volume we wish to trade, i.e., a market of "unlimited depth". In a real-world market context, we will typically only be able to trade volumes within certain bounds. We could say that an MSDC represents a market of limited depth if its domain is a closed interval of reals. The upper and the lower bound of this domain represent the market depth: the upper bound represents the maximum volume we will be able to sell against prices we can quote from the market and the lower bound represents the maximum we will be able to buy against prices we will be able to quote from the market. In what follows, we assume MSDCs representing markets of limited depth.

Reconsider problem (9). Using a cash liquidity policy $\mathscr{L}(c)$ this becomes

$$\begin{cases} \text{maximize} \quad U(\mathbf{p}-\mathbf{r})+L(\mathbf{r}); \\ \text{subject to:} \quad p_0 - r_0 + L(\mathbf{r}) \geq c; \\ \qquad\qquad\quad \mathbf{r} \in \mathscr{P}. \end{cases} \tag{11}$$

The inequality constraint can be replaced by the equality constraint $p_0 - r_0 + L(\mathbf{r}) = c$ without affecting the optimal value of the original problem. Furthermore, we may assume that the cash component r_0 equals 0 as it does not play a role in the optimization problem. To find the optimal solution we hence might as well solve

$$\begin{cases} \text{maximize} \quad U(\mathbf{p}-\mathbf{r})+L(\mathbf{r}); \\ \text{subject to:} \quad L(\mathbf{r}) = c - p_0; \\ \qquad\qquad\quad \mathbf{r} \in \mathscr{P}. \end{cases} \tag{12}$$

Note that without loss of generality we may assume that $p_0 = 0$; otherwise use the cash liquidity policy $\mathscr{L}(c - p_0)$.

3.2 A calculation scheme for portfolio valuation with ladder MSDCs

In case of portfolio valuation based on ladder MSDCs we can solve the associated optimization problem (12) numerically, for example, by an interior point algorithm (see (Boyd & Vandenberghe 2004)). However, this might give us only local optima as we often start from an arbitrarily chosen initial solution. In addition, the algorithm could be computationally inefficient in the sense that an interior point algorithm approximates any (local) solution and that several iterations might be required to bring this approximation within reasonable bounds. Hence, the aim of this section is to provide an algorithm for problem (12) yielding an exact global optimal solution $\mathbf{r}^*$. Unless otherwise noted, throughout the remainder of this section we assume (i) that an investor holds a portfolio $\mathbf{p}$ consisting of *long* positions only and (ii) that she uses a cash liquidity policy $\mathscr{L}(c)$ for some $c > 0$.

Given that all assets are assumed to be characterized by ladder MSDCs, we can conveniently break up each and every position into a finite number of volumes. To each of these volume there corresponds a definite market quote as represented by the MSDC. The idea of the algorithm is to consider all of these portfolio bits together and to liquidate them in a systematic and orderly manner, starting with the portions which will be liquidated with the smallest cost relative to the best bid, and subsequently to the ones that can be liquidated with second smallest cost, and so on, until the cash constraint is met.

If the minimum cash requirement that the portfolio should be prepared to satisfy exceeds the liquidation MtM value of the entire portfolio, then we will never be able to meet the cash constraint; by definition, we set the portfolio value to be $-\infty$.

Alternatively, suppose we sell off a fraction of each position against the best bid price and that the total cash we subsequently receive in return exceeds the cash constraint. Then the value of the portfolio equals the uppermost MtM value and there exists infinitely many optimal solutions.

We will now make this formal, starting with the following definition.

Definition 3.1. Given is an asset i, characterized by MSDC m_i. The **liquidity deviation** of a volume s of asset i is defined as:

$$S_i(s) := \frac{m_i^+ - m_i(s)}{m_i^+}. \tag{13}$$

The liquidity deviation is the relative difference between the best bid price and the last market quote $m_i(s)$ hit for a volume s. In this sense, it measures the liquidity of asset i at s_i units traded relative to the best bid. Given any asset, the liquidity deviation is a non-decreasing function, as the MSDC corresponding to that asset is non-increasing. For a security, the values of the liquidity deviation are in $[0,1]$, as the lower bound of the corresponding MSDC is 0. For a swap, the values are in $[0,+\infty)$. Since the MSDC of an asset is assumed to be piecewise constant, each value of liquidity deviation corresponds to a maximum bid size.

Using the previously defined liquidity deviation, positions are liquidated in a definite order, as follows. Given a portfolio $\mathbf{r} = (r_0, r_1, \ldots, r_N)$, assume that we want to liquidate all the r_i, $i > 0$. Each non-cash position r_i can be written as a sum

$$r_i = \sum_{j=1}^{J_i} r_{ij}, \quad i = 1, \ldots, N.$$

where r_{ij} is called a *liquidation size*.

To define the liquidation size r_{ij}, consider the bid part of a ladder MSDC m_i, which is constructed by a finite number of bid prices with maximum bid sizes. For each r_i in asset i, we can identify a finite number J_i of bid prices m_{ij} with liquidation sizes r_{ij}, $j = 1, \ldots, J_i$. For the first $J_i - 1$ liquidation sizes r_{ij} ($j = 1, \ldots, J_i - 1$), they are equal to the first $J_i - 1$ maximum bid sizes recognized from the market; for the J_i-th liquidation size r_{ij}, it is less than or equal to the J_i-th maximum bid size. Moreover, each liquidation size r_{ij} corresponds to each bid price m_{ij}, particularly with r_{i1}, the first liquidation size of each asset, corresponding to the best bid $m_i^+ = m_{i1}$. Afterwards, the liquidity deviation for each liquidation size can be written as $S_{ij} = \frac{m_i^+ - m_{ij}}{m_i^+} = \frac{m_{i1} - m_{ij}}{m_{i1}}$.

Now we put the liquidity deviations S_{ij} in ascending order indexed by k, and we generically refer to any term of this sequence as $S_k(\mathbf{r})$ (the addition of $\mathbf{r}$ as an extra parameter will prove convenient later on). Note that the length of the liquidation sequence equals $K = J_1 + \cdots + J_N$.

In addition, we observe that there exists a natural one-one correspondence between the sequence $(S_k(\mathbf{r}))_k$, the sequence of liquidation size $(r_{ij})_{(i,j)}$ and the sequence of bid prices $(m_{ij})_{(i,j)}$. Hence, while preserving these one-one correspondences, we relabel the sequences $(r_{ij})_{(i,j)}$ and $(m_{ij})_{(i,j)}$ as $(r_k)_k$ and $(m_k)_k$, respectively. So we call the sorted index k the *liquidation sequence*, which is a permutation of the index (i, j). We also note that the first N terms of the sequence $(m_k)_k$ are the best bids m_i^+, $i = 1, \ldots, N$.

(a) Asset 1

Maximum Bid Size	Bid Price
200	11.65
200	11.55
200	11.45

(b) Asset 2

Maximum Bid Size	Bid Price
200	19.58
600	19.5
200	19.2

Table 2: Bid price information of assets 1 and 2

To illustrate the above concepts, consider an example as follows. Given two illiquid assets, the bid part of which can be read from the market are shown in Table 2. Assume that we hold a portfolio which contains 600 units in asset 1 and 900 in asset 2. Then the liquidation sizes for the two assets are shown in Table 3 and the sorted liquidity deviations as well as the liquidation sequence are presented in Table 4.

(a) Asset 1

Liquidation Size		Bid Price	
r_{11}	200	m_{11}	11.65
r_{12}	200	m_{12}	11.55
r_{13}	200	m_{13}	11.45

(b) Asset 2

Liquidation Size		Liquidation Size	
r_{21}	200	m_{21}	19.58
r_{22}	600	m_{22}	19.5
r_{23}	100	m_{23}	19.2

Table 3: Liquidation size of our portfolio $\mathbf{r} = (0, 600, 900)$

Liquidation Sequence	Asset	Liquidation Size	Bid Price	Liquidity Deviation
(1, 1)	1	200	11.65	0
(2, 1)	2	200	19.58	0
(2, 2)	2	600	19.5	0.004085802
(1, 2)	1	200	11.55	0.008583691
(1, 3)	1	200	11.45	0.017167382
(2, 3)	2	100	19.2	0.019407559

Table 4: Liquidity deviation and liquidation sequence

To meet the cash constraint embodied in the cash liquidity policy we start liquidating the portfolio from $S_1(\mathbf{r})$, then $S_2(\mathbf{r})$, and so on, until we have met the cash requirement. The liquidation sequence effectively directs the search process throughout the constraint set towards the global solution, and exactly so. This is summarized in the following theorem, which we will prove subsequently. Note that the theorem assumes any liquidity policy, not specifically a cash liquidity policy.

Proposition 3.1. *Given is a portfolio* $\mathbf{p}$ *such that each asset is characterized by a ladder MSDC. Assume any liquidity policy* $\mathscr{L}$*. Then optimization problem (9) has the same optimal solution as the following (using the same notations as above):*

$$\begin{cases} \textit{minimize} \quad \sum_{k=1}^{K} S_k(\boldsymbol{r}); \\ \textit{subject to:} \quad \boldsymbol{p} - \boldsymbol{r} + L(\boldsymbol{r}) \in \mathscr{L}; \\ \qquad\qquad\quad \boldsymbol{r} \in \mathscr{P}. \end{cases} \tag{14}$$

Loosely put, the optimal solution is the one yielding the minimum total sum of liquidity deviation.

Proof. Let a portfolio $\mathbf{p} = (p_0, p_1, \ldots, p_N)$ be given and suppose we liquidate a portfolio $\mathbf{r} = (r_0, r_1, \ldots, r_N)$ to meet a liquidity policy $\mathscr{L}$. Asset i has a corresponding MSDC m_i, $i = 0, 1, \ldots, N$. From Proposition 2.2, the optimal solution of (9) minimizes the uppermost liquidation cost. Using that all assets are characterized by ladder MSDCs, the objective function $C(\mathbf{r})$ can be rewritten as follows:

$$\begin{aligned} C(\mathbf{r}) &= U(\mathbf{r}) - L(\mathbf{r}) \\ &= \sum_{i=1}^{N} \sum_{j=1}^{J_i} \left(m_i^+ r_{ij} - m_{ij} r_{ij} \right) \end{aligned}$$

Here p_0 and r_0 are set to be 0 as above for simplicity.

Note that for each asset i, $m_i^+ \geq m_{ij}$ for all j. It follows that the minimum of the sum of the absolute differences between the $m_i^+ r_{ij}$ and $m_{ij} r_{ij}$ is the same as the minimum of the sum of the relative differences. Hence, to find the optimal solution we might as well minimize

$$\begin{aligned} \sum_{i=1}^{N} \sum_{j=1}^{J_i} \frac{m_i^+ r_{ij} - m_{ij} r_{ij}}{m_i^+ r_{ij}} &= \sum_{i=1}^{N} \sum_{j=1}^{J_i} \frac{m_i^+ - m_{ij}}{m_i^+} \\ &= \sum_{k=1}^{K} S_k(\mathbf{r}). \end{aligned}$$

On the last line, $K = J_1 + \cdots + J_N$. □

Based on this result, we now state the algorithm for portfolio valuation assuming only ladder MSDCs and a cash liquidity policy $\mathscr{L}(c)$. For the sake of clarity we recall that the optimal solution $\mathbf{r}^*$ of problem (12) should satisfy $L(\mathbf{r}^*) = c - p_0$. Also, we assume that $p_0 = 0$ and $r_0 = 0$. (Otherwise, we can set the cash requirement $c = c - p_0$.) We continue using the same notations as above. The pseudocode is summarized in Algorithm 3.1.

The optimal solution $\mathbf{r}^*$ can be found by recording the liquidation parts of corresponding assets in the above calculation procedure of Algorithm 3.1.

The piecewise constant MSDCs in the convex optimization problem generally increase the difficulty of the search for the global optimal solution with standard software. With the aforementioned calculation scheme listed in Algorithm 3.1, instead, we can solve the optimization problem efficiently via a liquidation sequence.

4 Portfolio valuation using continuous MSDCs

There is typically no analytic solution to the convex optimization problem (12). However, it can be shown that if we model the MSDC as a continuous function simple analytic solutions result from the Lagrange multiplier method. In Section 4.1 we will first look at continuous MSDCs without imposing any specific form for them. We will then look at MSDCs shaped as exponential functions in Section 4.2. We then propose to use exponential MSDCs to approximate ladder MSDCs in order to improve the efficiency of portfolio valuation in Section 4.3. We will assume the cash liquidity policy in this section.

Algorithm 3.1 Algorithm for portfolio valuation assuming ladder MSDCs and a cash liquidity policy $\mathscr{L}(c)$

Calculate:
$U(\mathbf{p}) = \sum_{i=1}^{N} m_i^+ \cdot p_i$;
$L(\mathbf{p}) = \sum_{i=1}^{N} \sum_{j=1}^{J_i} m_{ij} \cdot p_{ij}$;
$V_1(\mathbf{p}) = \sum_{i=1}^{N} m_i^+ \cdot p_{i1}$;
$S_{ij} = \frac{m_{i1} - m_{ij}}{m_{i1}}$;
Sort the S_{ij} as an ascending sequence with index variable k. // *With k running from* 1 *to* $J_1 + \cdots + J_N$
if $c > L(\mathbf{p})$ **then**
  return $V^{\mathscr{L}(c)}(\mathbf{p}) = -\infty$; // *There is no optimal solution satisfying the cash constraint.*
else
  if $c \leq V_1(\mathbf{p})$ **then** // *Liquidating the* p_{i1} *to the respective best bids meets the cash constraint.*
    return $V^{\mathscr{L}(c)}(\mathbf{p}) = U(\mathbf{p})$; // *There are infinitely many optimal solutions.*
  else
    $U(\mathbf{r}) = V_1(\mathbf{p})$;
    $c = c - V_1(\mathbf{p})$;
    $k = N + 1$; // *Start loop from the first part with non-zero liquidity deviation until c is zero.*
    while $c > 0$ **do**
      if $\frac{c}{m_k} > p_k$ **then**
        $U(\mathbf{r}) = U(\mathbf{r}) + m_k^+ \cdot p_k$;
        $c = c - m_k \cdot p_k$;
        $k = k + 1$;
      else
        $U(\mathbf{r}) = U(\mathbf{r}) + m_k^+ \cdot \frac{c}{m_k}$;
        $c = 0$;
      end if
    end while
    return $V^{\mathscr{L}(c)}(\mathbf{p}) = U(\mathbf{p}) - U(\mathbf{r}) + c$ //*Here we have* $L(\mathbf{r}) = c$.
  end if
end if

4.1 The general case

We first assume N illiquid assets labeled $1, \ldots, N$. The corresponding MSDCs m_i are supposed to be continuous on $\mathbb{R}$ (i.e., $m_i(0)$ is supposed to exist at first, but we will exclude the point $m_i(0)$ later in this section); furthermore, the m_i are assumed to be strictly decreasing. Adopting the cash liquidity policy, valuing a portfolio consisting of positions in these assets comes down to solving the optimization problem (12). The solution to this optimization problem can be analytically derived, as is shown by the following proposition.

Proposition 4.1 (Acerbi & Scandolo (2008))**.** *Assuming continuous strictly decreasing MSDCs and the cash liquidity policy* $\mathscr{L}(c)$*, the optimal solution* $\mathbf{r}^* = (0, \vec{r}^*)$ *to optimization problem (12) is unique and given by*

$$r_i^* = \begin{cases} m_i^{-1}\left(\frac{m_i(0)}{1+\lambda}\right), & \text{if } \; p_0 < c, \\ 0, & \text{if } \; p_0 \geq c, \end{cases} \tag{15}$$

where m_i^{-1} *denotes the inverse of the MSDC function* m_i*, and the Lagrange multiplier* λ*, representing the* *marginal liquidation cost, can be determined from the equation* $L(\boldsymbol{r}^*) = c - p_0$.

Note that we can extend the above to the case where the MSDCs are not continuous at the point 0, i.e., the case where there is a positive bid-ask spread. We only have to change the definition of the value at $m_i(0)$ to the limit m_i^+ in the case of long positions or to m_i^- in the case of short positions.

Obviously, by using the Lagrange multiplier method, we can generalize the case to any liquidity policy giving rise to equality constraints. When using a general liquidity policy which results in inequality constraints, we can solve the optimization problem (9) by checking the Karush-Kuhn-Tucker (KKT) conditions. In addition, the Lagrange dual method may be useful as well.

4.2 Exponential MSDCs

We continue the discussion by looking at a particular example of a continuous MSDC proposed by Acerbi & Scandolo (2008), i.e., the exponential MSDC. As it turns out, the exponential MSDCs form an effective model to characterize a security-type asset and to determine the portfolio value by convex optimization. We will discuss this in Section 4.3.

Suppose that there are N illiquid assets $1, 2, \ldots, N$ characterized by exponential MSDCs

$$m_i(s) = M_i e^{-k_i s}, \tag{16}$$

with $M_i, k_i > 0$ for all $i = 1, \ldots, N$. We call M_i the *market risk factor* and k_i the *liquidity risk factor* for the corresponding asset i ($i = 1, \ldots, N$). Note that the range of an exponential MSDC is bounded from below by 0. Hence exponential MSDCs serve to characterize security-type assets.

We find for the uppermost MtM value

$$U(\mathbf{p}) = p_0 + \sum_{i=1}^{N} m_i(0) p_i = p_0 + \sum_{i=1}^{N} M_i p_i, \tag{17}$$

and for the liquidation MtM value

$$L(\mathbf{p}) = p_0 + \sum_{i=1}^{N} \int_0^{p_i} m_i(x) \mathrm{d}x = p_0 + \sum_{i=1}^{N} \frac{M_i}{k_i} (1 - e^{-k_i p_i}). \tag{18}$$

The value of a portfolio under the cash liquidity policy $\mathscr{L}(c)$ with $p_0 < c$ follows from Proposition 4.1:

$$r_i^* = \frac{\log(1+\lambda)}{k_i}, \quad i = 1, \ldots, N,$$
$$\text{with} \quad \lambda = \frac{c - p_0}{\sum_{i=1}^{N} \frac{M_i}{k_i} - c + p_0}. \tag{19}$$

Hence,

$$V^{\mathscr{L}(c)}(\mathbf{p}) = U(\mathbf{p} - \mathbf{r}^*) + L(\mathbf{r}^*) = \sum_{i=1}^{N} M_i(p_i - \frac{\log(1+\lambda)}{k_i}) + c. \tag{20}$$

The use of exponential MSDCs for modeling gives rise to very efficient computations. For actively traded stocks, the liquidity risk factor is estimated to vary from 10^{-9} to 10^{-7}. This was confirmed by experiments with real market data using the method of least squares (see Section 4.3).

For bid prices of a security-type asset, an approximation by an exponential MSDC appears sufficiently accurate. For ask prices, however, exponential MSDCs may be less appropriate, as it gives rise to a steep slope for ask prices without an upper bound.

4.3 Approximating ladder MSDCs by exponential MSDCs

In Section 3, we have defined a fast calculation scheme for portfolio valuation with ladder MSDCs. In the real world, however, we may face a situation that to collect the price information to form a ladder MSDC is too costly, or that the information is incomplete or non-transparent, e.g., in an over-the-counter (OTC) market.

One could model ladder MSDCs as the modeling of order book dynamics which appear to have a common basis with ladder MSDCs. For example, in (Bouchaud, Mézard & Potters 2002), the trading volume at each bid (or ask) price in the stock order book follows a Gamma distribution. In (Cont, Stoikov & Talreja 2010), a continuous Markov chain is used to model the evolution of the order book dynamics.

In our paper, we use the basic continuous MSDC models to approximate the ladder MSDC directly, as we can then apply the Lagrange multiplier method and other convex optimization techniques to obtain analytic solutions and thus improve the efficiency. For actively traded security-type assets, a portfolio valuation based on exponential MSDCs, with their analytic solutions, is significantly faster than with ladder MSDCs. For OTC traded assets, lacking price information, exponential MSDCs with high liquidity risk factors can be a first modeling attempt.

Generally, when using exponential MSDC models, we need to estimate or model the parameters M_i and k_i. The dynamics of the market risk factors M_i can be estimated by the best bid prices for long positions, or modeled by asset price models (e.g., geometric Brownian motion). If we assume that k_i is independent of M_i, we can employ time series or stochastic processes to model k_i. If k_i is assumed to be correlated with M_i, we also need to model the correlation. Furthermore, for security-type assets traded in an OTC market, we may use the mere price information of the asset to estimate market risk and liquidity risk factors in the MSDC models. In particular, the liquidity risk factor may be set at a high level (e.g., 10^{-3}) to represent the illiquidity of the asset.

For the approximation of a ladder MSDC by an exponential MSDC, we assume that the portfolio consists of long positions in N illiquid security-type assets. The best bid price of asset i, m_i^+, is set here as the market risk factor, M_i, in the exponential function. If we assume additionally that the liquidity risk factor of asset i, k_i, is independent of market risk factor M_i, then parameter k_i can be estimated from the ladder MSDC of asset i by the method of least squares as follows. We transform the exponential function as $-\log(\frac{m_i(s)}{M_i}) = k_i s$, and estimate k_i by n discrete pairs $(s_n, -\log(\frac{m_i(s_n)}{M_i}))$ provided M_i has already been determined, minimizing the merit function:

$$\sum_{j=1}^{n}(-\log(\frac{m_i(s_j)}{M_i}) - k_i s_j)^2.$$

The least squares estimate of parameter k_i then reads

$$\widehat{k_i} = \frac{-\sum_{j=1}^{n} s_j \log(\frac{m_i(s_j)}{M_i})}{\sum_{j=1}^{n} s_j^2} \tag{21}$$

The validity of the model requires insight in modeling errors, and in parameter regimes for which the exponential MSDC may become inaccurate.

The relative difference between the ladder MSDC and its approximation may serve as an indication for the error in the modeling. For different cases this difference in the portfolio is of different shape. For certain parameters, the relative difference increases monotonically.

A factor related to the validity of the exponential function is the occurrence of significant jumps in the ladder MSDCs when liquidating an asset with different bid prices. For asset i, we define a *jump indicator*, $I_i(s_i)$, to measure the size of a jump and the related modeling error as

$$I_i(s_i) := S_i(s_i^+) - S_i(s_i^-) = \frac{m_i(s_i^-) - m_i(s_i^+)}{m_i^+} \tag{22}$$

S_i denotes the liquidity deviation, as defined in Section 3. The jump indicator is non-negative and less than 1. With $I_i(s_i) = 0$, the ladder MSDC is continuous at point s_i and there is no jump in the MSDC at the trading volume s_i. When $I_i(s_i) > 0$, the ladder MSDC is discontinuous at s_i.

As the jump indicator is defined as a relative value, we can compare the impact of jumps occurring in different ladder MSDCs. This jump indicator can also give insight, to some extent, in the shape of the relative error (see, for example, Figures 3(b) and 5(b) in Section 5) in the portfolio values. After the calculation of each asset's liquidity deviation, the difference between two adjacent liquidity deviations is computed for each asset. This is represented by the jump indicator at the margin of one ladder of an MSDC. By sorting the non-zero liquidity deviations in an ascending order (i.e., the liquidation sequence, see Section 3) in combination with the non-zero jump indicators, the impact of a modeling error on the portfolio valuation for different liquidation requirements can be estimated.

For modeling purposes, we may set a tolerance level for jump indicators. A jump indicator exceeding this level is an indication for a significant modeling error.

5 Numerical Results

In this section we give examples for the various concepts discussed in this paper. In particular, we explain the calculation scheme for efficient portfolio valuation by means of an example.

5.1 Portfolio with four illiquid assets

The example here is based on the case of four illiquid security-type assets. We deal with a portfolio $\mathbf{p} = (0, 3400, 2400, 3200, 2800)$. The bid prices with liquidation sizes for the portfolio are chosen at a given time as presented in Table 5.

(a) Asset 1

Liquidation Size	Bid Price
200	11.65
200	11.55
200	11.45
200	11.1
200	11.05
200	11
200	10.3
500	9.3
500	6.5
1000	6.46

(b) Asset 2

Liquidation Size	Bid Price
200	19.58
600	19.5
200	19.2
200	19.15
200	19.1
200	18.6
200	18.5
200	16.85
200	16.1
200	16.05

(c) Asset 3

Liquidation Size	Bid Price
400	29.3
200	29.16
400	29.15
400	28.9
200	28
600	27.8
200	27.15
200	27
400	26
200	22

(d) Asset 4

Liquidation Size	Bid Price
200	43.1
400	42.65
200	41.9
400	41
200	40.86
200	40.4
200	39
400	37
400	36
200	35.1

Table 5: Bids of assets 1-4

It is easy to calculate the uppermost MtM value $U(\mathbf{p})$ and the liquidation MtM value $L(\mathbf{p})$ from the tables, that is, $U(\mathbf{p}) = 3.01042 \times 10^5$ and $L(\mathbf{p}) = 2.73720 \times 10^5$. Hence, the uppermost liquidation cost equals

$C(\mathbf{p}) = 0.27322 \times 10^5$. If the true portfolio value is equal to the liquidation MtM value, but if we would use however the uppermost MtM value instead, it would overestimate the portfolio value by as much as 10%.

For different cash requirements, we use the sorted liquidity deviations (see Table 6) to find the liquidation sequence and then calculate the portfolio values (see Figure 1). From the last row of Table 6, we can see that the liquidity deviation can be as large as 44.5% for the most illiquid part of the MSDC for asset 1, which indicates a high level of liquidity risk.

From Figure 1, we infer that the portfolio value decreases at a faster rate as we have to liquidate positions of an increasing number of illiquid assets to meet the cash requirements, which will definitely cause significant losses during liquidation.

Liquidation Sequence	Asset	Liquidation Size	Bid Price	Best Bid	Liquidity Deviation
(1, 1)	1	200	11.65	11.65	0
(2, 1)	2	200	19.58	19.58	0
(3, 1)	3	400	29.3	29.3	0
(4, 1)	4	200	43.1	43.1	0
(2, 2)	2	600	19.5	19.58	0.004085802
(3, 2)	3	200	29.16	29.3	0.004778157
(3, 3)	3	400	29.15	29.3	0.005119454
(1, 2)	1	200	11.55	11.65	0.008583691
(4, 2)	4	400	42.65	43.1	0.010440835
(3, 4)	3	400	28.9	29.3	0.013651877
(1, 3)	1	200	11.45	11.65	0.017167382
(2, 3)	2	200	19.2	19.58	0.019407559
(2, 4)	2	200	19.15	19.58	0.021961185
(2, 5)	2	200	19.1	19.58	0.024514811
(4, 3)	4	200	41.9	43.1	0.027842227
(3, 5)	3	200	28	29.3	0.044368601
(1, 4)	1	200	11.1	11.65	0.0472103
(4, 4)	4	400	41	43.1	0.048723898
(2, 6)	2	200	18.6	19.58	0.050051073
(3, 6)	3	600	27.8	29.3	0.051194539
(1, 5)	1	200	11.05	11.65	0.051502146
(4, 5)	4	200	40.86	43.1	0.051972158
(2, 7)	2	200	18.5	19.58	0.055158325
(1, 6)	1	200	11	11.65	0.055793991
(4, 6)	4	200	40.4	43.1	0.062645012
(3, 7)	3	200	27.15	29.3	0.07337884
(3, 8)	3	200	27	29.3	0.078498294
(4, 7)	4	200	39	43.1	0.09512761
(3, 9)	3	400	26	29.3	0.112627986
(1, 7)	1	200	10.3	11.65	0.115879828
(2, 8)	2	200	16.85	19.58	0.139427988
(4, 8)	4	400	37	43.1	0.141531323
(4, 9)	4	400	36	43.1	0.164733179
(2, 9)	2	200	16.1	19.58	0.17773238
(2, 10)	2	200	16.05	19.58	0.180286006
(4, 10)	4	200	35.1	43.1	0.185614849
(1, 8)	1	500	9.3	11.65	0.201716738
(3, 10)	3	200	22	29.3	0.249146758
(1, 9)	1	500	6.5	11.65	0.442060086
(1, 10)	1	1000	6.46	11.65	0.445493562

Table 6: Liquidity deviation and liquidation sequence

The calculation scheme in Algorithm 3.1 provides an efficient search direction to the optimal value guided by the liquidation sequence. For this four-asset example with the cash liquidity policy, we compare our calculation scheme with the *fmincon* function with an interior point algorithm in MATLAB. The optimization is repeated for 2.5×10^5 different cash requirements and the total computation time is recorded.[1] The av-

[1]The computer used for all experiments has an Intel Core2 Duo CPU, E8600 @3.33GHz with 3.49 GB of RAM and the code is

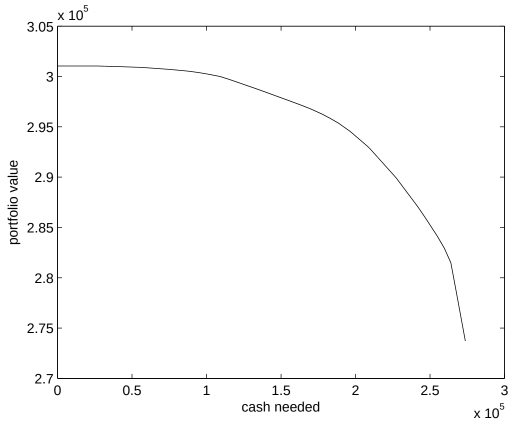


Figure 1: Portfolio value with different cash requirements

eraged time for each cash liquidity policy equals 0.568 millisecond for our scheme, whereas *fmincon* takes 202.7 milliseconds, which implies that the time difference is a factor of 300. More importantly, we reach an accurate optimal value.

Since the ascending sequence of liquidity deviations shows the illiquidity of different parts of the corresponding asset, liquidating the portfolio along the liquidation sequence will cause minimum loss of values compared to the other kinds of liquidation.

An interior point method, on the other hand, may reach different local optimal values from different starting points. In addition, the non-smoothness of the ladder MSDCs increases the difficulty of implementing conventional convex optimization algorithms.[2]

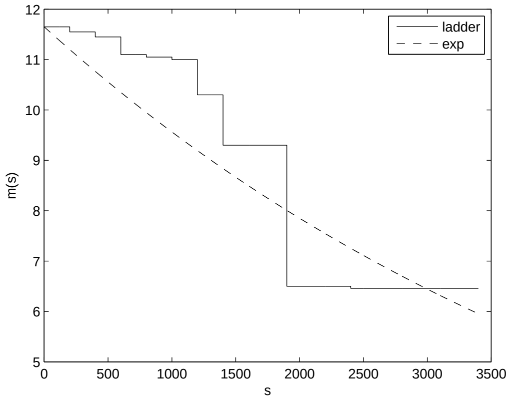
5.2 Using exponential MSDCs to approximate ladder MSDCs

For the four-asset example with the ladder MSDCs from Section 5.1, Figure 2 illustrates the ladder MSDCs and the corresponding exponential approximating MSDCs. The latter MSDCs are estimated by least squares. The liquidity risk factors in the exponential MSDCs are found as $k_1 = 1.9738 \times 10^{-4}$, $k_2 = 6.1091 \times 10^{-5}$, $k_3 = 4.3015 \times 10^{-5}$ and $k_4 = 6.8139 \times 10^{-5}$. Hence, asset 1 is most illiquid and asset 3 is the most liquid in general.

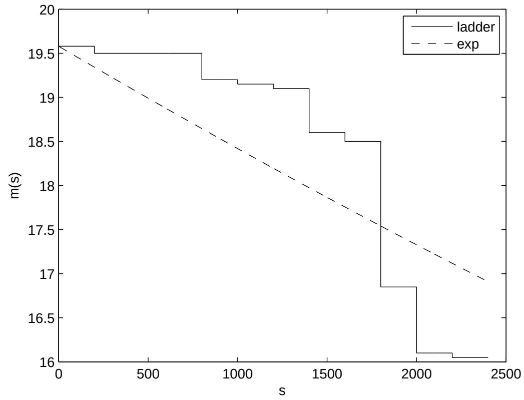
In Figure 3(a), we compare the portfolio values obtained by using the exponential MSDCs with the reference portfolio values by the ladder MSDCs under different cash requirements. The relative difference in the

written in MATLAB R2009b.

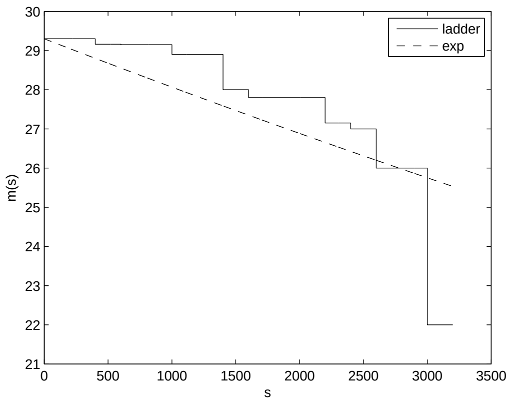
[2]For example, the optimality conditions in the interior point algorithm will not apply at non-smooth points of the ladder MSDC. See (Boyd & Vandenberghe 2004) for more information.



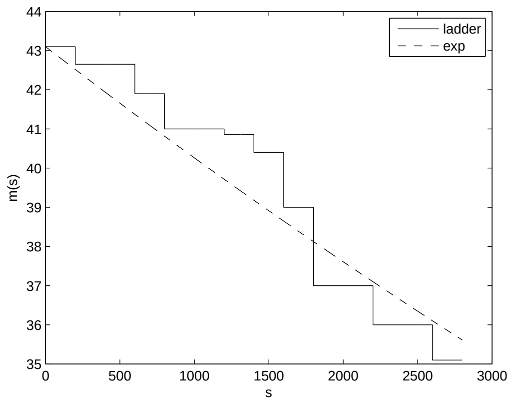
(a) Asset 1



(b) Asset 2



(c) Asset 3



(d) Asset 4

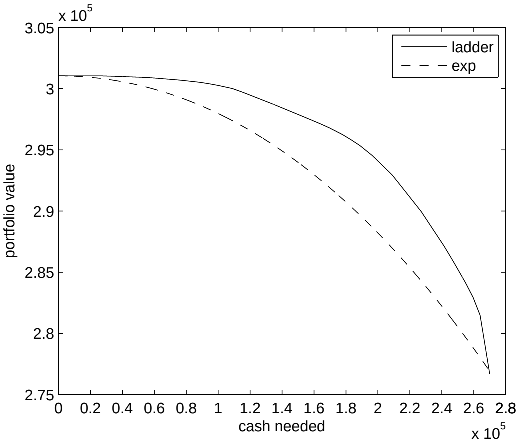
Figure 2: Exponential MSDCs versus ladder MSDCs for the bid prices of assets 1-4

portfolio values is presented in Figure 3(b). The relative difference found is at most 2.04%, so that in this example the exponential MSDCs are accurate approximations.

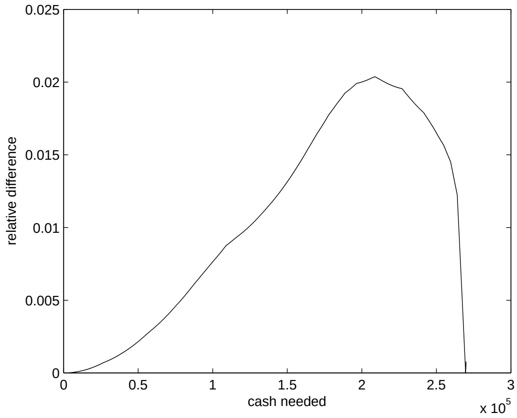
5.3 Modeling error

To test the exponential MSDC model, we construct an artificial extreme case based on the four-asset example which is used in Section 5.1. We just modify the last part of the bid prices for all four assets to be 0.46, 0.05, 0.1 and 0.1, respectively, with other data being unchanged. The new estimated exponential MSDCs are shown in Figure 4. We calculate portfolio values under different cash liquidity policies based on exponential and ladder MSDCs (see Figure 5(a)), and find that the modeling error is significant when increasing the cash requirement (see Figure 5(b)). It is clear that the exponential MSDC model fails in this case.

Since the jump indicator is able to measure jump sizes and the related modeling errors, we use it for the

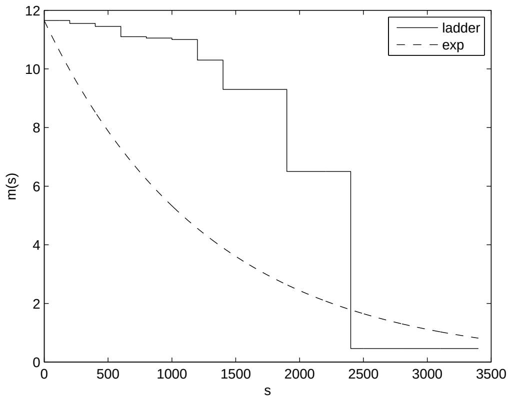


(a) Comparison of portfolio values

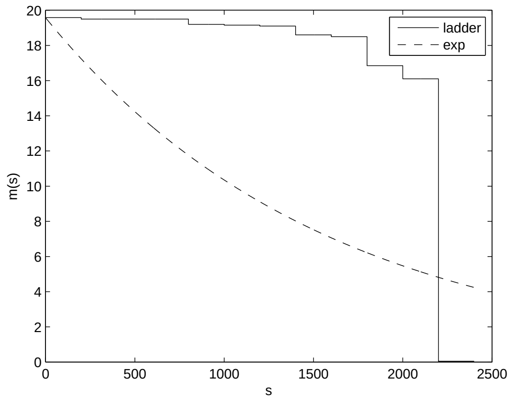


(b) Relative difference in portfolio values

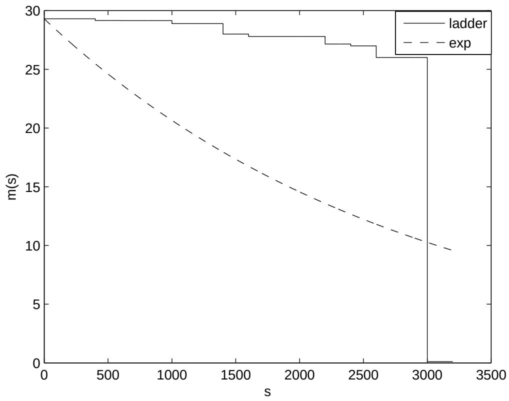
Figure 3: Modeling ladder MSDCs by exponential MSDCs



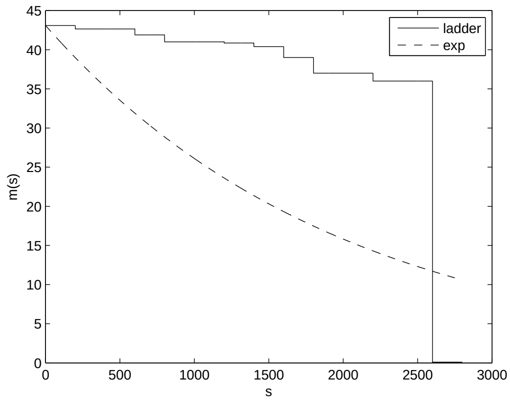
(a) Asset 1



(b) Asset 2

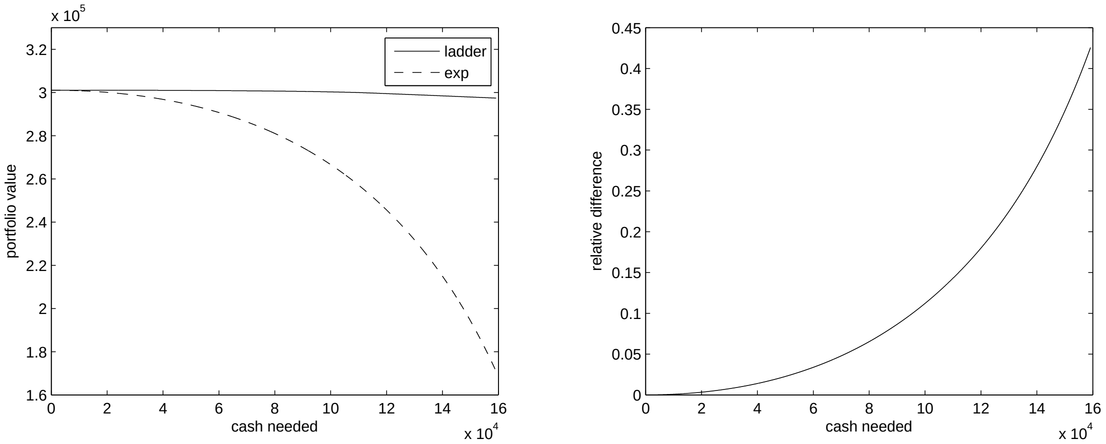


(c) Asset 3



(d) Asset 4

Figure 4: Exponential MSDCs versus ladder MSDCs for bid prices (extreme example)

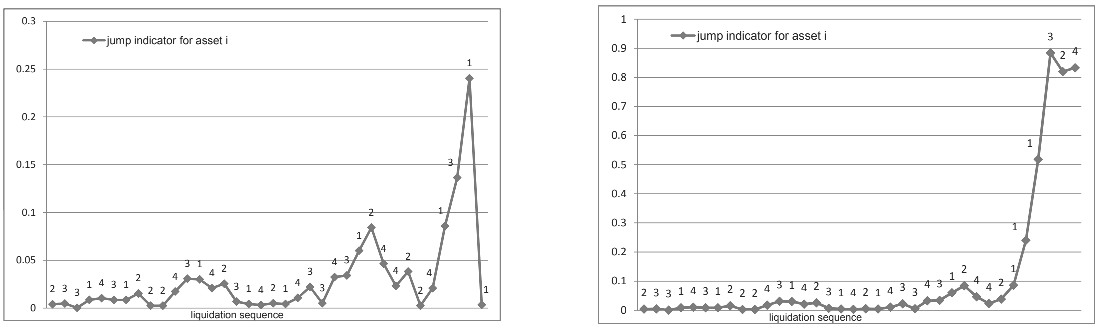


(a) Comparison of portfolio values (extreme example)

(b) Relative difference (extreme example)

Figure 5: Modeling of ladder MSDCs (extreme example)

extreme four-asset example here. When using exponential MSDCs to model ladder MSDCs we find that the liquidity risk factor for asset 3 is the smallest of the four assets. However, from the calculation of jump indicators for assets 1 to 4, we find that the most significant jumps occur at the final part of the ladder MSDC for asset 3. Hence, using the exponential MSDC to model the ladder MSDC for asset 3 will thus give rise to a significant modeling error.



(a) Example in Section 5.2

(b) Extreme example in this section

Figure 6: Impact of jump indicators on modeling portfolio values

The jump indicator may also be helpful for understanding the shapes of the modeling errors, for example, in the Figures 3(b) and 5(b). In Figure 6(a) a trend is visible in the jump indicators. At the right-hand side of the graph the jump indicator drops to a relatively small value, which can partly explain why there is some decrease of the modeling error in Figure 3(b). The increasing values in the jump indicator in Figure 6(b) are related to increasing errors in Figure 5(b) for the portfolio valuation in the extreme example.

6 Conclusions and Discussions

Within the theory proposed by Acerbi & Scandolo (2008) the valuation of a portfolio can be framed as a convex optimization problem. We have proposed a useful and efficient algorithm using a specific form of the market data function, i.e., all price information is represented in terms of a ladder MSDC. We have also considered approximations of ladder MSDCs by exponential functions.

As an outlook, one may improve the modeling techniques either by means of improved methods to estimate the liquidity risk and market risk factors in the exponential function, or by sophisticated models replacing the exponential functions.

Whereas in regulated markets such as stock exchanges price information is relatively easily available, bid and ask prices for assets traded in the over-the-counter (OTC) markets may not be easily obtained at any given time. Hence, it may be nontrivial to employ this portfolio theory to these markets. Extracting all relevant price information from OTC markets is however a challenge for all researchers.

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