Pricing Options with Discrete Dividends by High Order Finite Differences and Grid Stretching

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Outline

- Discretization for Black-Scholes equation
  - use only a few grid points
- Discrete dividend
- American-style options

$\Rightarrow$ "PDE on a grid" is straightforward, some modeling questions remain

Black-Scholes option pricing

Point of Departure (here)

- The asset price follows the lognormal random walk.
- Interest rate r and volatility σ_c are known functions of t.
- Transaction costs for hedging are not included in the model.
- There are no arbitrage possibilities.

$\Rightarrow$ Black-Scholes partial differential equation:
(for a European option)

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma_c^2 S^2 \frac{\partial^2 u}{\partial S^2} + rS\frac{\partial u}{\partial S} - ru = 0$$

- The Black-Scholes equation is a parabolic partial differential equation

Options on dividend-paying equities

- At the time that a dividend is paid there will be a drop in the value of the stock.
- The price of an option on an dividend-paying asset is affected by these payments.
- Different structures are possible for the dividend payment (deterministic or stochastic with payments continuously or at discrete times)
- We consider discrete deterministic dividends, whose amount and timing are known.
- Arbitrage arguments require:

$$u(S, t_d^-) = u(S - D, t_d^+)$$

Final/Boundary conditions

- European Call option: Right to buy assets at maturity $t = T$ for exercise price K.
- Final condition: $u(S,T) = \max(S-K,0)$
- Boundary conditions $S = 0$: $u(0,t) = 0$,

  for $S \to \infty$: $u(S_{max},t) = S_{max} - Ke^{-r(T-t)} - De^{-r(t_d-t)}$ or $u_{ss} = 0$.
- The strategy to solve the Black-Scholes equation numerically is as follows
  - Start solving from $t = T$ to $t = t_d$ with the usual pay-off.
  - Apply an interpolation to calculate the new asset and option price on the grid discounted with D.
  - Restart the numerical process with the PDE from the interpolated price as final condition from t_d to $t = 0$.

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Discretization

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma_c^2 S^2 \frac{\partial^2 u}{\partial S^2} + rS\frac{\partial u}{\partial S} - ru = 0$$

- Grid in space and time with N and M points; mesh width $h = 1/N, k = 1/M$
- Finite differences, based on Taylor's expansion
- $O(h^2 + k^2)$ is easily achieved by central differencing and Crank-Nicolson discretization
- Our aim: High accuracy with only a few grid points

$\Rightarrow$ Grid stretching in space and 4th order discretizations in space and in time

Grid stretching

- A coordinate transformation that clusters points in the region of interest.
- Boundary at infinity: truncate the domain at a safe place (option value is not influenced) according to a well-known formula
- An equidistant grid discretization can be used after the analytic transformation
- Consider a general parabolic PDE with non-constant coefficients

$$\begin{aligned} \frac{\partial v}{\partial t} &= \alpha(s)\frac{\partial^2 v}{\partial s^2} + \beta(s)\frac{\partial v}{\partial s} + \gamma(s)v(s,t) \\ v(a,t) &= L(t), \;\; v(b,t) = R(t), \;\; v(s,0) = \phi(s). \end{aligned}$$

Grid stretching

- Consider a coordinate transformation $y = \psi(s)$ (one-to-one), inverse $s = \varphi(y) = \psi^{-1}(y)$ and let $\hat{v}(y,t) := v(s,t)$.
- Chain rule, the first and second derivative:

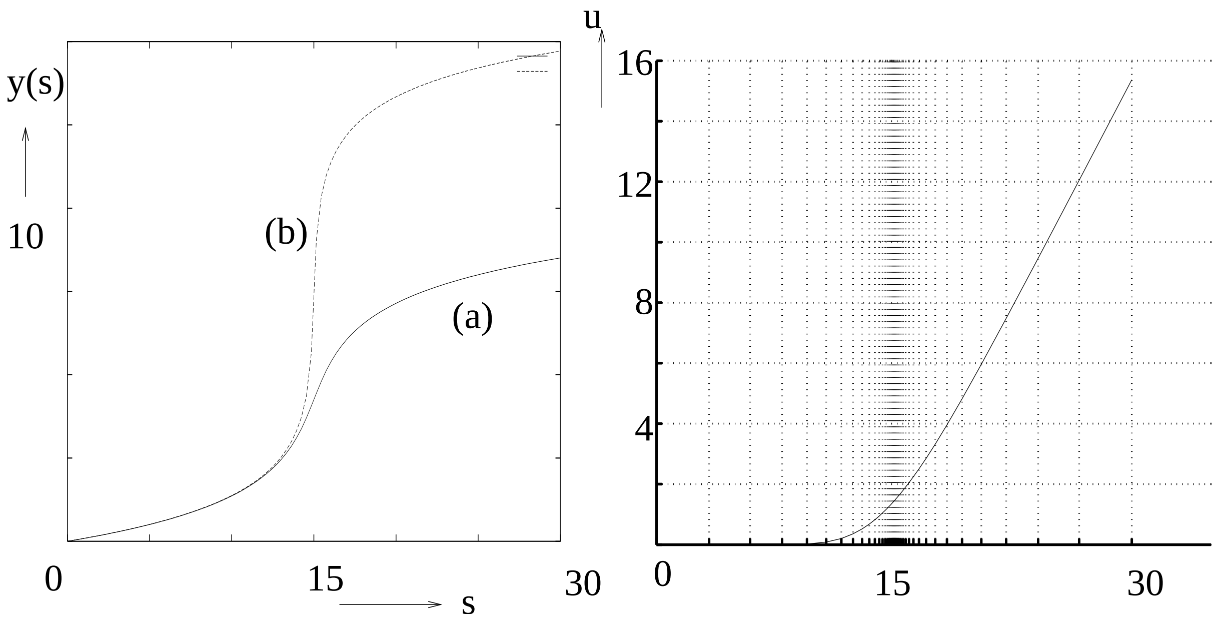
$$\frac{\partial v}{\partial s} = \frac{1}{\varphi'(y)}\frac{\partial \hat{v}}{\partial y}, \tag{1}$$

$$\frac{\partial^2 v}{\partial s^2} = \frac{1}{(\varphi'(y))^2}\frac{\partial^2 \hat{v}}{\partial y^2} - \frac{\varphi''(y)}{(\varphi'(y))^3}\frac{\partial \hat{v}}{\partial y}. \tag{2}$$

Application changes the factors α, β and γ into:

$$\widehat{\alpha}(y) = \frac{\alpha(\varphi(y))}{(\varphi'(y))^2}, \quad \widehat{\beta}(y) = \frac{\beta(\varphi(y))}{\varphi'(y)} - \alpha(\varphi(y))\frac{\varphi''(y)}{(\varphi'(y))^3}, \quad \widehat{\gamma}(y) = \gamma(\varphi(y)). \tag{3}$$

Grid stretching



- Spatial transformation used for Black-Scholes [Clarke-Parrott, Tavella-Randall]:

$$y = \psi(s) = \sinh^{-1}(\mu(s-K)) + \sinh^{-1}(\mu K) \, . \tag{4}$$

- The grid is refined around $s = K$, i.e. the nondifferentiability in the final condition.
- Parameter μ determines the rate of stretching; keep μK constant
- Stretching is possible at several places: grid is defined numerically

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Discretization

- Fourth order in space (long stencils):

$$\frac{\partial \hat{v}_i}{\partial t} = \frac{1}{12h^2}\widehat{\alpha}_i \left(-\hat{v}_{i+2} + 16\hat{v}_{i+1} - 30\hat{v}_i + 16\hat{v}_{i-1} - \hat{v}_{i-2}\right) + \\ + \frac{1}{12h}\widehat{\beta}_i \left(-\hat{v}_{i+2} + 8\hat{v}_{i+1} - 8\hat{v}_{i-1} + \hat{v}_{i-2}\right) + \widehat{\gamma}_i\hat{v}_i + O(h^4), \quad 2 \leq i \leq N-2. \tag{5}$$

- Fourth order in time: BDF4 scheme (preceded by CN, BDF3). BDF4 reads

$$\left(\frac{25}{12}I - kL\right)u^{j+1} = 4u^j - 3u^{j-1} + \frac{4}{3}u^{j-2} + \frac{1}{4}u^{j-3}, \tag{6}$$

- No stability complications observed
- Well-suited for linear complementarity problems (for American options)

Accuracy

European option pricing experiment, no dividend

- Error in u_h and hedge parameters Δ_h, Γ_h
- $K = 15,\ s_0 = K,\ \sigma_c = 0.3,\ r = 0.05,\ D = 0.03,\ T = 0.5.$

Scheme	Grid	$\|u - u_{ex}\|_\infty$	c_∞	$\|\Delta - \Delta_{ex}\|_\infty$	c_∞	$\|\Gamma - \Gamma_{ex}\|_\infty$	c_∞
	10×10	1.1×10^{-2}		2.4×10^{-2}		6.3×10^{-3}	
$\mathbf{O}(h^4 + k^4)$	20×20	1.1×10^{-3}	10.1	3.1×10^{-3}	7.6	1.3×10^{-3}	4.8
	40×40	9.4×10^{-5}	11.2	2.9×10^{-4}	10.8	9.7×10^{-5}	13.6

Scheme	Grid	$\|u - u_{ex}\|_\infty$	c_∞	$\|\Delta - \Delta_{ex}\|_\infty$	c_∞	$\|\Gamma - \Gamma_{ex}\|_\infty$	c_∞
	10×10	2.7×10^{-1}		1.7×10^{-1}		4.2×10^{-2}	
$\mu = 12$	20×20	1.5×10^{-2}	18.1	1.5×10^{-2}	11.5	4.2×10^{-3}	9.9
stretching	40×40	9.1×10^{-4}	16.5	1.7×10^{-3}	8.6	5.3×10^{-4}	8.0
$\mathbf{O}(h^4 + k^4)$	80×80	5.7×10^{-5}	16.0	1.5×10^{-4}	11.6	4.2×10^{-5}	12.7

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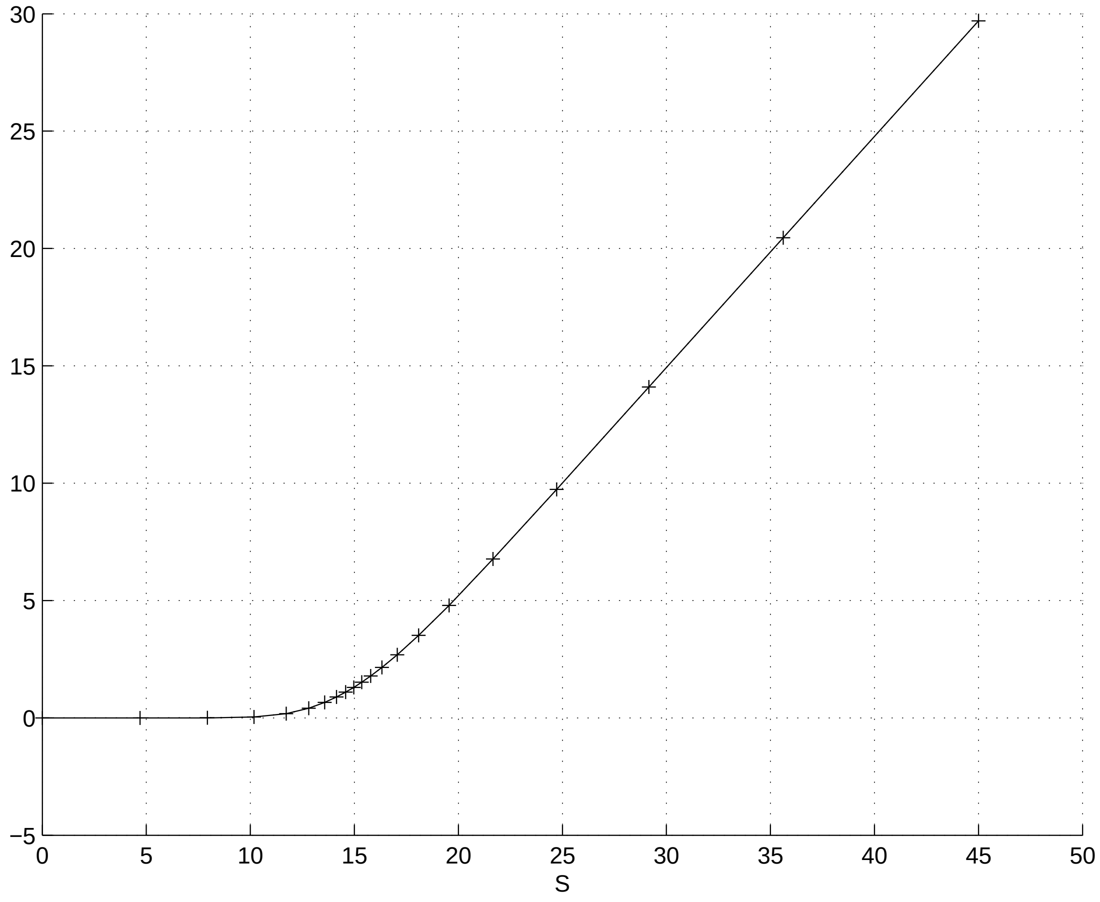
Accuracy

European option pricing experiment



Example European option

Multiple discrete dividends

- Multiple discrete dividends: analytic solution not available
- Parameters: $s_0 = K = 100,\ r = 0.06, \sigma_c = 0.25$, multiple dividends of 4 (ex-dividend date is each half year), $T = 1, 2, 3, 4, 5, 6$. Grid: $s_{max} = RK(3 \leq R \leq 7), \mu = 0.15$

Grid	$T=1$	Grid	$T=2$	Grid	$T=3$
20×20	10.660	20×40	15.202	20×80	18.607
40×40	10.661	40×80	15.201	40×160	18.600
Lewis (Wilmott Mag. 2003)	10.661		15.199		18.598
Grid	$T=4$	Grid	$T=5$	Grid	$T=6$
20×80	21.370	20×100	23.697	20×120	25.710
40×160	21.362	40×200	23.691	40×240	25.698
Lewis	21.364		23.697		25.710

Example European option

Zero interest rate

- Case $r = 0$, the ex-dividend date t_d should not matter for the option price.
- Black-Scholes does not satisfy this market principle.
- Correction of volatility in Black-Scholes:

$$dS = \mu S dt + (S - De^{-rt_d})dW \qquad t \in [0, t_d]$$

$$\sigma(S,t,D) = \begin{cases} \sigma_c \frac{S - D\exp(-rt_d)}{S} & t \in [0, t_d] \\ \sigma_c & t \in [t_d, T] \end{cases}$$

	t_d =0	t_d =3 months	t_d =6 months	t_d =9 months	t_d =12 months
Black-Scholes	8.3386	8.5522	8.7590	8.9587	9.1511
Vol. correction	8.3386	8.3386	8.3386	8.3386	8.3386

European Call, $K = 100$, $D = 7$, $r = 0$, $T = 1$, $\sigma = 0.3$

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American Options

Linear Complementarity

- American options are contracts that may be exercised early. This right to exercise is valuable: The American option cannot be worth less than the equivalent European.
- The problem we need to solve for an American call option contract reads:

$$
\begin{aligned}
\mathcal{A}u := \frac{\partial u}{\partial t} \; &+ \; \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 u}{\partial S^2} + rS\frac{\partial u}{\partial S} - ru \leq 0 \\
u(S,T) \; &= \; \max\left(S-K,0\right), \quad u(S,t_d) = \max\left\{S - Ke^{-r(T-t_d)} - D, S-K\right\}, \\
u(S,t) \; &\geq \; \text{final condition} \\
\frac{\partial u}{\partial S} & \quad \text{continuous} \\
u(S_{max},t) \; &= \; \max\{S_{max} - Ke^{-r(T-t)} - De^{-r(t_d-t)}, S_{max} - Ke^{-r(t_d-t)}\}, \quad t < t_d
\end{aligned}
$$

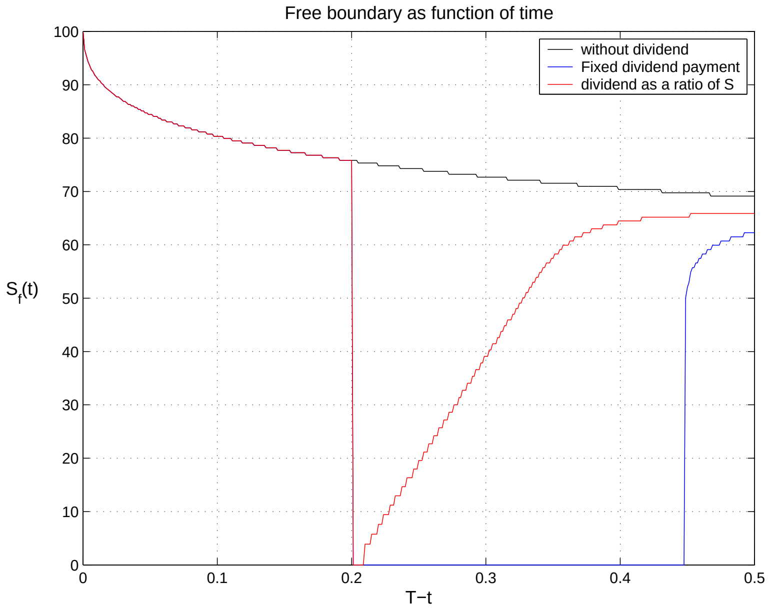
- Early exercise valuable only if $D > K(1-e^{-r(T-t_d)}$, just before the asset goes ex-dividend [Kwok].
- Reformulation of the obstacle problem into a linear complementarity problem:

American Put with one Discrete Dividend

- $K = 100, T = 0.5, d = 2.0, t_d = 0.3, \sigma_c = 0.4, r = 0.08,\ \mu = 0.15, s_{max} = 3K$

Grid	$u_h(80, t=0)$	$u_h(100, t=0)$	$u_h(120, t=0)$
20×20	0.223	0.105	0.043
40×40	0.223	0.105	0.043
Meyer (J. C. Fin. 2001):	0.223	0.105	0.043

- $d = 0$ (black line) , $d = 2$ (blue line) vs. $d = 0.98S$ (red line)



American Call with one Dividends

- Problem parameters: $K = 100, \sigma_c = 0.3, r = 0.05, t_d = 51$ weeks, $D_1 = 4, T = 50$ weeks versus T=1 year

	Vorst	Haug	Black Scholes	Vol. correction	Eur. corr.
$t_d = 51$ weeks, T=50 weeks	13.88	13.92	13.92	13.92	13.49
$t_d = 51$ weeks, T=52 weeks	13.63	13.64	14.08	13.65	

- American price is lower than European
- One should maybe (but this will not happen in practice !) adapt the European price to avoid this contradiction with the volatility correction

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Conclusions

- Accurate option values with grid stretching in space and 4th order discretization in space and time
- Option price and hedge parameters are accurate with 20 -40 points
- Multiple discrete dividend payment can be included in a straightforward way
- American style early exercise does not pose any problems

⇒ Discrete dividends lead to interesting modeling issues.