Option Pricing with The Fourier Transform Method, Based on The Stochastic Volatility Model

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Dedicated to

My Parents

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Abstract

This thesis applies the Fourier transform method to evaluate Bermudan and American options, based on Heston's stochastic volatility model. Because accuracy relies heavily on the step size of the grid where the Fourier transform method computes the option prices, a finer grid seems necessary to give sastisfactory results. However, this may be expensive since more grid points are involved. A new extrapolation scheme is applied to facililate the approximation with the results obtained with the Fourier transform method in the coarse grid. This extrapolation scheme is compared with the Gaussian quadrature scheme in performance. It shows a significant gain in the computational time and results in enough accuracy as well. Options with various parameters and in the single and multi-asset case are evaluated with this extrapolation scheme. Besides the Fourier transform method, the multigrid technique and the Green function method are applied as alternative ways to evaluate options in Heston's model.

Declaration

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Chapter 1

Introduction

1.1 Basic Concepts

An option is a financial instrument that gives one the right to make a specified transaction at (or by) a specified date at a specified price. In this definition, people who buy the options are called the buyers or holders of the options and those who issue the options, the writers or sellers. The asset on which the transaction may take place is known as the underlying. The prescribed date is called the maturity time or expiry. And the price at which the buyers may purchase or sell the underlying is called the exercise price or strike price.

The history of options dates back hundreds of years ago. Wim Schoutens [7] mentioned the stories of Thales in Greece who used the ancient type of options to secure a low price for olives in advance of the harvest. In The Netherlands, trading in tulip derivatives blossomed during the early 1600s. In 1973, the Chicago Board Options Exchange (CBOE) started trading in call options on some stocks. Trading in put options began later in 1977. Nowadays the option market has grown to be so huge, as shown by John C. Hull [3], that the size of the OTC[1] market has reached $220.1 trillion by June 2004.

[1] OTC: over-the-counter

Usually options have two primary usages, speculation and hedging. Speculative investors trade the options to make profit according to their judgments on the trends of the asset prices. For example, such an investor will choose to buy a call option if he thinks the underlying price is going to increase in the following months. If his forecast is correct, he makes money, otherwise he loses money. By contrast to buying the stock share, options can be a cheap way for investment when the stock price is much higher than the cost of the option. Hedging happens when investors want to minimize the risk that they may be subjected to because of the unpredictable events in the financial market, i.e. due to the random walk of the underlying price. In the market, asset holders can choose to buy put options if they think that the asset price is going to fall. If they are correct, the profit they make from the options will reduce the loss due to holding the asset. Therefore this hedging strategy becomes an insurance against the adverse movements in the underlying.

In the financial market, options are classified into different categories according to the elementary concepts which make up those options. If the option gives the buyer the right to buy the underlying, it is a call option. By contrast, it is a put option if the buyer has the right to sell the underlying at (or by) the expiry. American options enable the buyers to exercise them before the expiry, and European options can only be exercised at the expiry. Call and put options are known as the plain vanilla options because they are basic. There are also some other complicated types of options. Asian options' strike prices are prescribed as some form of the average of the underlying prices over a period. Lookback options depend on the maximum or minimum price. Barrier options can either come into existence or become worthless if the underlying asset reaches some prescribed value before expiry. Such complicated options as Asian, lookback and barrier options are usually called exotic options.

Because they offers their buyers some privileges, options have some value. An option's value should be equal to its price when traded in the market, otherwise the arbitrage opportunity will appear. A correct mathematical valuation of the option value is thus one of the targets in the research on options. Roughly speaking, an option's value is influenced by many factors. For example, a European call option will have a positive value if the underlying asset's price is higher than the strike price, otherwise it may be profitless. Time to expiry plays also a role. An option with an expiry of 6 months has more value than one which will expire tomorrow, because the underlying price may have more potential to change over a longer period. There are also many other factors, such as the option's type, the bank interest rate, dividends and so on, which may influence an option's value. The relationship between those factors and the option value is described differently in various models.

In the theory of option pricing, there are some assumptions for modelling. There is no market friction, no default risk and no arbitrage. This means: no transaction costs, no bid/ask spread, perfect liquid markets, no taxes, no margin requirements, no restrictions on short[2] sales, no transaction delays, market participants act as price takers, market participants prefer more to less. The bank interest rate is assumed to be fixed in short-term and does not change. The basic theory says, if the interest rate is r and an amount of money B is deposited at time 0, its value will be Be^{rt} at time $t > 0$. Equivalently, if we borrow Be^{-rt} currency units, we will have to pay back B currency units at time t later.

Many assets, such as equities, pay out dividends. The dividends can be seen as the payments to the share holders out of the profits made by the company. The likely dividend stream of a company in the future is reflected by its current share price. When pricing options, we should count the effect of dividends.

This thesis is focused on 2 different problems in quantitative finance. The first one is how to apply the extrapolation scheme when computing option prices with the Fourier transform method based on Heston's stochastic volatility model. The second

[2]Short position: selling assets that one does not own. Its opposite is a long position.

problem is to explore the alternative numerical methods to compute option prices. Chapter 2 introduces the mathematical models and methods frequently applied in option pricing. In chapter 3, the Fourier transform method and the extrapolation scheme are applied to evaluate options. In chapter 4, the implementation details and the data results are presented.

1.2 Notations and Remarks

Before moving on to the following sections, we introduce the notation used in this report. The option value is denoted by V. To make the distinction between call and put options' values, C and P are used. To show that the option value is a function of the current value of the underlying asset price S and time, t, V is sometimes denoted as $V(S,t)$. The option value depends on the following parameters:

- σ, the volatility of the underlying asset;
- K, the exercise price;
- T, the expiry date;
- r, the interest rate;
- D, the dividend.

Chapter 2

Mathematical Models

2.1 Basics

2.1.1 Random Walk of Asset Prices

In the research on option pricing, the dynamics of the asset price is usually represented by its relative change, dS/S, called return. The most common model, geometric Brownian motion model (GBM), says that the return of the asset price is made up of two parts as [7]

$$\frac{dS}{S} = \mu dt + \sigma dX, \tag{2.1}$$

where μ, known as the drift, marks the average rate of growth, and σ is called volatility that keeps the information of the standard deviation of the return. The first part μdt reflects a predictable, deterministic and anticipated return which is similar to the return of the investment in banks. The second part σdX simulates the random change in the asset price in response to external effects, such as uncertain events. The quantity dX contains the information of the randomness of the asset price and is known as Wiener process or Brownian motion. It is a random variable which follows a normal distribution, with mean zero and variance dt. This means that dX can be written as $dX = \phi\sqrt{dt}$. Here ϕ is a random variable with a standardized normal distribution. Its probability density function is given by

$$f(\phi) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2} d\phi,$$

for $-\infty < \phi < \infty$. With the definition of expectation operator $\mathcal{E}$ by

$$\mathcal{E}[F(.)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\phi) e^{-\frac{1}{2}\phi^2} d\phi,$$

for any function F, we have

$$\mathcal{E}[\phi] = 0, \qquad \mathcal{E}[\phi^2] = 1.$$

2.1.2 Arbitrage and Martingales

In the theory of option pricing, one fundamental and essential concept is arbitrage. Formally speaking, it states that there is never any opportunity to make an instantaneous risk-free profit. More correctly, such opportunities cannot exit for a significant length of time before prices move to eliminate them. Almost all financial theories assume the existence of risk-free investments that give a guaranteed return with no chance of default, e.g. a government bond or a deposit in a sound bank. The greatest risk-free return that one can make on a portfolio of assets is the same as the return if the equivalent amount of cash were placed in a bank. In the definition of arbitrage, the key words are "instantaneous" and "risk-free". This means, by investing in equities, one can probably beat the bank, but this cannot be certain. If one wants a greater return, one must accept a greater risk. This is explained clearly in [6]. In the binomial model, if r is the spot rate and the stock price process can be represented as

$$\begin{cases} S_0 = s, \\ S_1 = s \cdot Z, \end{cases}$$

where Z is a stochastic variable defined as

$$\begin{cases} Z = u, \text{with probability } p_u, \\ Z = d, \text{with probability } p_d, \end{cases}$$

where $u > d$ and $p_u + p_d = 1$, then "free of arbitrage" results in

$$d \leq (1 + r) \leq u.$$

The arbitrage theory leads to the definition of the risk-neutral measure, or martingale measure: a probability measure $\mathbb{Q}$ is called martingale if the following condition holds

$$S_0 = \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}[S_1].$$

The risk-neutral measure is the basis of the valuation in this thesis.

2.1.3 Payoff Function

The value of an option at its expiry is usually called the payoff function. For a European call option with a strike price K, the payoff is [7]

$$C(S_T, T) = \begin{cases} S_T - K & \text{if } S_T > K, \\ 0 & \text{otherwise.} \end{cases}$$

This can also be written more concisely as $max(S_T - K, 0)$, or $(S_T - K)^+$. In the case of $S_T > K$, the option is called "in the money". It is said to be "out of the money" if $S_T < K$. If $S_T = K$, it is "at the money". Similarly, the payoff function is $(K - S_T)^+$ for a European put option. American options will be covered in the following sections.

2.1.4 Itô's lemma

In practice, stock prices are discrete values at discrete time points. Changes can be observed only when the exchange is open. Nevertheless, the continuous-variable, continuous-time processes prove to be useful models for many purposes. To value an option, it is necessary to set up the mathematical models in the continuous time limit $dt \to 0$. And it is more efficient to solve the resulting differential equations, rather than to simulate the random walk on a practical time scale. Therefore, it is needed to handle the dX term in equation (2.1) as $dt \to 0$. In [7], Itô's lemma provides such a kind of machinery as

$$df = \sigma S \frac{\partial f}{\partial S} dX + (\mu S \frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t})dt. \tag{2.2}$$

Detailed derivation refers to Appendix (A.1). Because logarithmic asset prices are widely used, the differentiation of $f(S) = \log(S)$ gives

$$\frac{df}{dS} = \frac{1}{S} \qquad and \qquad \frac{d^2 f}{dS^2} = -\frac{1}{S^2},$$

and it leads to

$$df = \sigma dX + (\mu - \frac{1}{2}\sigma^2)dt. \tag{2.3}$$

This is a constant coefficient stochastic differential equation, which says that the difference df is normally distributed. Consider f itself: it is the sum of the jumps df (in the limit, the sum becomes to be an integral). Since a sum of normal variables is also normal, $f - f_0$ has a normal distribution with mean $(\mu - \frac{1}{2}\sigma^2)t$ and variance $\sigma^2 t$, where t is the time elapsed between f and f_0, and $f_0 = \log(S_0)$ is the initial value of f. The probability density function of $f(S)$ is known as

$$\frac{1}{\sigma\sqrt{2\pi t}} e^{-(f - f_0 - (\mu - \frac{1}{2}\sigma^2)t)^2/2\sigma^2 t} \tag{2.4}$$

for $-\infty < f < \infty$. Thereafter, it is not difficult to know that the probability density function of S is

$$\frac{1}{\sigma S\sqrt{2\pi t}} e^{-(\log(S/S_0) - (\mu - \frac{1}{2}\sigma^2)t)^2/2\sigma^2 t} \tag{2.5}$$

for $0 < S < \infty$.

2.2 The Black-Scholes Model

The most famous model in option pricing is the Black-Scholes model. It is based on the GBM (geometric Brownian motion) model of asset prices: $dS/S = \mu dt + \sigma dX$ where μ and σ are fixed values during the lifetime of the option. According to Itô's lemma and the arbitrage theory, a partial differential equation can be obtained by means of setting a portfolio and eliminating the random items by hedging. The result of the derivation is the Black-Scholes equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rS\frac{\partial V}{\partial S} - rV = 0. \tag{2.6}$$

If the logarithmic price is used as $x(t) = logS$ and $t = T - t^*$, the equation will change to be

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial x^2} - (r - \frac{1}{2}\sigma^2)\frac{\partial V}{\partial x} - rV = 0. \tag{2.7}$$

Detailed derivation refers to Appendix (A.2). With its extensions and variants, it plays a major role in option pricing problems. Usually the boundary conditions of the Black-Scholes equation are defined as

$$V(S,t) = V_a(t) \qquad on \qquad S = a,$$

and

$$V(S,t) = V_b(t) \qquad on \qquad S = b.$$

Because the equation (2.7) is of the backward type, there is a final condition

$$V(S,t) = V_T(S) \qquad on \qquad t = T$$

where V_T is a known function, usually the payoff function.

Explicit solutions to the Black-Scholes equation are available for European call and put options. For a European call option, the explicit solution is

$$C(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2), \tag{2.8}$$

where $N(\cdot)$ is the cumulative distribution function for a standardized normal random variable, given by

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2} dy,$$

and for d_1 and d_2, it says

$$d_1 = \frac{\log\left(\frac{S}{K}\right) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$

and

$$d_2 = \frac{\log\left(\frac{S}{K}\right) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

For a put, the solution is

$$P(S,t) = Ke^{-r(T-t)}N(-d_2) - SN(-d_1). \tag{2.9}$$

Figures (2.1) and (2.2) shows the explicit solutions. Since $N(d_1)$ and $N(d_2)$ can be interpreted as the "adjusted" probabilities, these explicit solutions also prove Feynman-Kač's proposition mentioned in [6] and reveal the simple phenomenon that the European option value at time t is just equal to the discounted expectation of the option value at expiry T:

$$V(t) = e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}[V(T)], \tag{2.10}$$

where $\mathbb{Q}$ denotes the risk-neutral measure.

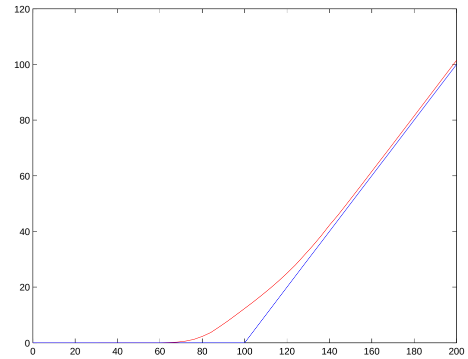

Figure 2.1: European Call and Payoff Diagram

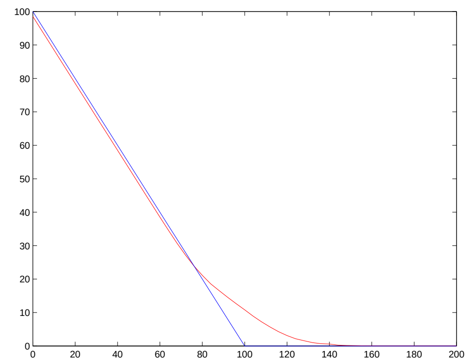

Figure 2.2: European Put and Payoff Diagram

2.3 American Options

American options differ from their European counterparts because of the possibility of early exercise. This difference makes it more difficult to model and solve the option pricing problems related to them.

Take an European put for example, if S lies in the range where S is much smaller than the exercise price and $P(S,t) < (K-S)^+$, there is an obvious arbitrage opportunity in the American case: one can buy the asset in the market at price of S and the put option of P, one can get the immediate riskless profit if one exercises the option by selling for K, because here is $K - P - S > 0$. So the conclusion is, when the early exercise is permitted, the following constraint must be imposed

$$V(S,t) \geq (K - S)^+.$$

In the examples mentioned above, there exist some values of S for which it is optimal from the holder's point of view to exercise the American option. Typically at each time t there is a particular value of S which marks the boundary between two regions: to one side one should hold the option, and to the other side one should exercise it. Therefore the option pricing problem in the American case is one kind of free boundary problem, and the boundary value, which generally varies with time, is referred to as optimal exercise price, denoted by $S_f(t)$. In the free boundary problem, it is unknown where is the free boundary or where to apply the boundary conditions in advance.

The simple arbitrage argument used for the European option no longer leads to a unique value for the return on the portfolio, only to an inequality. In summary, an American put option is written as a free boundary problem [7] as follows. For each time t, the S axis must be divided into two distinct regions. The first one is $0 \leq S < S_f(t)$, where early exercise is optimal and

$$P = K - S, \qquad \frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP < 0. \tag{2.11}$$

In the other region, $S_f(t) < S < \infty$, early exercise is not optimal and

$$P > K - S, \qquad \frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP = 0. \tag{2.12}$$

The boundary conditions at $S = S_f(t)$ are: P and its slope (delta) should be continuous:

$$P(S_f(t), t) = (K - S_f(t), 0)^+, \qquad \frac{\partial P}{\partial S}(S_f(t), t) = -1. \tag{2.13}$$

The mathematical analysis of American options is more complicated than that of European options. And it is impossible to find an exact solution to a given free boundary problem to analytically decide the value of an American option. Therefore the primary objective is to solve the problem with efficient and robust numerical methods or other powerful methods. This needs a theoretical framework where the free boundary problem may be treated in fairly general terms. The linear complementary problem is one of such frameworks with the advantage that the free boundary need not to be treated explicitly. In this thesis, American options are evaluated by a Fourier transform method.

2.4 Alternative Asset Price Models: Stochastic Volatility and Jumps

Based on the Black-Scholes model, explicit analytical solutions are available in [7] when we evaluate European call or put options. For American options, numerical methods have been developed for decades to solve the equation. These solutions actually match the real data very well. However, some important properties of the market have been ignored by the Black-Scholes model. The first flaw is in the assumption of the normal distribution adopted in the random walk of the underlying price. In equation (2.1), the underlying price is supposed to follow mainly within the 6-sigma range centered by μdt according to the statistical theory about the normal distribution. And it says that the underlying prices outside the 6-sigma range are of very small probabilities, say, "impossible". However, the real data in the market shows that this is not true. Stock prices, for example, are more widely dispersed than expected. In statistical terms this is known as "fat tails". Secondly, the volatility in equation (2.1) is set to be invariant with respect to time and asset space, but observation shows that this is actually not the case at all. For equity and foreign exchange, the implied volatilities display a strong dependence with respect to the exercise price as well. Thirdly, the Black-Scholes model is built up on the basis of a continuous theory. However, discontinuities in the financial market show up very frequently, which means that the continuous model does not fit the market very well. In a word, in GBM, whereas in the financial market, the asset price process has fat tails, skewness, discontinuities and is typically non-symmetrical. Therefore alternative models, such as stochastic volatility models and jump diffusion models, have been built up to import on the flaws mentioned above.

2.4.1 Heston's Model

Heston's model [9] allows the volatility to be stochastic. It is assumed that the volatility also follows a stochastic process, which is described by a stochastic differential equation:

$$\begin{aligned} dx(t) &= (\mu - \frac{1}{2}v(t))dt + \sqrt{v(t)}dW_1(t), \\ dv(t) &= -\lambda(v(t) - \bar{v})dt + \eta\sqrt{v(t)}dW_2(t). \end{aligned} \tag{2.14}$$

Here $x(t)$ is the logarithm of the underlying price, $v(t)$ is the volatility, λ is the speed of mean reversion, $\bar{v}$ is the mean level of variance and η is the volatility of volatility. The Wiener processes $W_1(t)$ and $W_2(t)$ correlate with the correlation coefficient ρ. The highlight of the Heston's model is the variable volatility of the underlying price. Therefore it is possible to simulate the real case more accurately.

Similar to the derivation that leads to the Black-Scholes equation, i.e. applying Itô's lemma and the arbitrage theory leads to the Heston's equation,

$$\frac{\partial V}{\partial \tau} = \frac{1}{2}v\frac{\partial^2 V}{\partial x^2} + \rho\eta v\frac{\partial^2 V}{\partial x \partial v} + \frac{1}{2}v\eta^2\frac{\partial^2 V}{\partial v^2} + (r - \frac{1}{2}v)\frac{\partial V}{\partial x} - (\lambda(v - \bar{v}) - \theta v)\frac{\partial V}{\partial v} - rV, \tag{2.15}$$

where θ is known as the market price of risk. With respect to volatility and asset space, Heston's model is a two dimensional model.

2.4.2 Models with Jump Processes

Jump processes are included to model the discontinuities in the underlying dynamics. When a jump term is added to the GBM dynamics of the logarithmic stock price, we have

$$\begin{aligned} dx(t) &= (\mu(t, x(t)) - \frac{1}{2}\sigma^2(t, x(t)))dt + \sigma(t, x(t))dW(t) + \phi_{N(t)}dN(t), \\ x(t_0) &= x_0, \end{aligned}$$

where $x(t) = \log S(t)$, $\phi_{N(t)} = \log J_{N(t)}$, $J_{N(t)}$ is the jump size and $N(t)$ is a Poisson process. This leads to the pricing equation as

$$(\frac{\partial V}{\partial t} - (r + \lambda)V + (\mu - \frac{1}{2}\sigma^2)\frac{\partial V}{\partial x}) + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2} + \lambda\mathbb{E}_{\phi_{N(t)}}[V(t, x(t) + \phi_{N(t)})] = 0,$$

which is the partial integro-differential equation for option price V.

In Merton's model [14], the jump sizes are assumed to be independent of each other and log-normally distributed. The probability density function of the jump size reads

$$f(J) = \frac{e^{-\frac{\log(J)^2 - \mu_J}{2\sigma_J^2}}}{\sqrt{2\pi}\sigma_J}.$$

This results in the PIDE of the related option's value:

$$\frac{\partial V}{\partial t} + (r - \kappa\lambda)S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - (r + \lambda)V + \lambda \int_0^\infty f(J)V(t, JS(t))dJ = 0,$$

where σ, λ, μ_J and σ_J can be fit to the market and $\kappa = \mathbb{E}[J - 1]$ If $\lambda = 0$, then the Black-Scholes equation will be obtained.

Kou's model [15] is similar to Merton's model, but the difference is that the choice for the distribution of the jump sizes is a non-symmetrically double exponentially distribution. Thus the probability density function of the jump size reads

$$f(\phi) = p\alpha_1 e^{\alpha_1 \phi} I_{\{\phi \geq 0\}} + q\alpha_2 e^{\alpha_2 \phi} I_{\{\phi < 0\}}, \alpha_1 > 1, \alpha_2 > 0,$$

where p, q are positive real numbers and $p + q = 1$. α_1 and α_2 are parameters that can be fitted to the market. This distribution can be seen as a weighted average between an exponential distribution on the positive real line and a mirrored exponential distribution on the negative part. The pricing equation is the same as that in Merton's model, except for the density function of the jump size density.

Another model for modeling asset prices is the so-called Bates model [10]. It takes jumps into account and uses a Poisson process to model the jumps,

$$\begin{aligned} dx(t) &= (\mu - \frac{1}{2}v(t))dt + \sqrt{v(t)}dW_1(t) + j_{N(t)}dN(t), \\ dv(t) &= -\lambda(v(t) - \bar{v})dt + \eta\sqrt{v(t)}dW_2(t), \end{aligned}$$

where $N(t)$ is a Poisson process with intensity ξ, and $j_{N(t)}$ is the jump size with the normal distribution, $j_{N(t)} \sim N(\mu_j, \sigma_j^2)$. The Poisson process $N(t)$ is independent of the Brownian motions and the jump distribution. With the jump consideration, the underlying price can be simulated better.

Chapter 3

The Numerical Valuation Methods

European call and put options under GBM have the explicit known solutions from the Black-Scholes equation. But for other options, we usually have to solve the pricing equations differently. When we have the pricing equations which are partial integro-differential equations, numerical methods are available to approximate the solutions. The directions time, stock price and even volatility can be discretized on a grid. Differentials in the pricing equations will be substituted by finite differences. Integrals can be replaced by their numerical approximations. By doing this, the relationship between the approximation values at grids will be built up in the form of a linear system of equations (LSE). Efficient algorithms like iteration schemes and the multigrid technique can solve the LSE quickly to get the numerical solutions.

The binomial method, on the other hand, assumes that the asset price changes only at discrete times and its value is also discrete. The asset price at a next step takes only the two neighbor values which are closest to the value at the current step, with known probabilities into account. This method works in the risk-neutral world, which means that the return from the underlying is the risk-free interest rate. It firstly builds a tree of possible values of asset prices and their probabilities, given an initial asset price, then uses this tree to determine the possible asset prices at expiry and the probabilities of these asset prices being realized. The possible values of the security at expiry can then be calculated, and by working back down the tree, the option can be valued. One useful consequence is that we can quite easily deal with the possibility of early exercise such as in American options and options with dividends.

Monte Carlo simulation is also applicable to option pricing problems. It simulates many paths of the underlying's random walk. Then a payoff can be obtained from each simulation. The price of the option is the discounted average of the simulated payoffs. Obviously this can be very time-consuming. However this method is applicable to many derivative pricing problems, such as path-dependent and multi-asset options, which is the main advantage of this method.

3.1 The Characteristic Function

Here we start with the description of a very efficient pricing technique which will result in the so-called CONV (convolution-based) method [8]. According to Feynman-Kač's proposition [6], the solution of the Black-Scholes equation (2.7) has a solution of the form

$$V(t,S) = e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}[V(T,S)], \tag{3.1}$$

where T is the maturity time and $\mathbb{Q}$ is the risk-neutral measure. This equation can be written as an integral:

$$V(x,T) = e^{-r(T-t)}\int_{-\infty}^{\infty} g(x_T)f(x_T|x)dx_T, \tag{3.2}$$

where x is the logarithmic asset price, $g(x_T) = (\phi(e^{x_T} - K))^+$, is the payoff function ($\phi = 1$ (call) or $\phi = -1$ (put)) at maturity and $f(x_T|x)$ is the transition probability density of reaching $x(T)$ from $x(t)$.

The transition probability density function is usually difficult to be found analytically, whereas its Fourier transform, called the characteristic function, is comparatively easy to be obtained, by means of the moment generating function. The characteristic function reads

$$\hat{f} = \int_{-\infty}^{\infty} e^{iws} \cdot f(s)ds. \tag{3.3}$$

Therefore it is convenient to switch the computation to the frequency domain with the help of the characteristic function to solve the option pricing problems.

The characteristic function of the logarithmic asset price in Black-Scholes model [7] is

$$\phi(w) = e^{iw\bar{\mu} - \frac{1}{2}\bar{\sigma}^2 w^2}, \tag{3.4}$$

where $\bar{\mu} = (r - \frac{1}{2}\sigma^2)\triangle t$ and $\bar{\sigma} = \sigma\sqrt{\triangle t}$.

Because the volatility is stochastic in Heston's model, the option price is not only conditional on the initial asset price, but also conditional on the initial volatility. Therefore equation (3.2) should be changed:

$$V(x, v, T) = e^{-r(T-t)} \int_{-\infty}^{\infty} g(x_T) f(x_T|x, v) dx_T.$$

For Heston's model (2.14), the characteristic function of the transition probability density of the logarithmic asset price $f(x_T|x_0, v_0)$ [9] reads

$$\phi_T(w) = e^{iwrT + \frac{v_0}{\eta^2}\left(\frac{1-e^{-DT}}{1-Ge^{-DT}}\right)(\lambda - \rho\eta iw - D) + \frac{\lambda\bar{v}}{\eta^2}\left(T(\lambda - \rho\eta iw - D) - 2\log\left(\frac{1-Ge^{-DT}}{1-G}\right)\right)}, \tag{3.5}$$

where

$$\begin{aligned} D &= \sqrt{(\lambda - \rho\eta iw)^2 + (w^2 + iw)\eta^2}, \\ G &= \frac{\lambda - \rho\eta iw - D}{\lambda - \rho\eta iw + D}, \end{aligned}$$

here v_0 and v_T are the volatility at initial time and maturity, respectively.

The characteristic function in Bates model [10] is

$$\begin{aligned} \phi_T(w) \;&=\; e^{iw(r-\kappa\xi)T + \frac{v_0}{\eta^2}\left(\frac{1-e^{-DT}}{1-Ge^{-DT}}\right)(\lambda - \rho\eta iw - D) + \frac{\lambda\bar{v}}{\eta^2}\left(T(\lambda - \rho\eta iw - D) - 2\log\left(\frac{1-Ge^{-DT}}{1-G}\right)\right)} \\ &\quad \times e^{\xi T\left(e^{iw\mu_j - \frac{1}{2}\sigma_j^2 w^2 - 1}\right)}, \end{aligned} \tag{3.6}$$

where $\kappa = \mathbb{E}[e^{j_{N(t)}} - 1]$.

3.2 The Fourier Transform Method

Equation (3.2) shows a general form of a representation of option prices. Once we know the characteristic function, we can transform the computation from the asset price domain to the frequency domain. The reason why we do this way is that characteristic functions are easier to be obtained than the density functions themselves.

Bermudan options allow the buyers to exercise the options at some prescribed times before the expiry. The price of a Bermudan option can be computed as follows. The transform method is applied between every two neighbor exercisable times, backward from the expiry to the initial time. After the option price is obtained at each exercisable time, it is compared with the payoff. The maximum values are selected as the new final condition to start the transform in the next time interval backward. Because an American option enables the buyers to exercise the option any time during its lifetime, it is regarded as the limit of a Bermudan option when the number of exercise dates goes to infinity. This is one way to approximate an American option if we set the number of Bermudan option exercise times big enough. Another way to compute American options is to apply the extrapolation technique.

3.2.1 Valuation Under The Black-Scholes Model

For the one dimensional exponential Levy model, we have $f(x_T|x) = f(z)$ with $z = x_T - x$. As a result, we have

$$V(x,T) = e^{-r(T-t)} \int_{-\infty}^{\infty} g(T, x_T) f(x_T - x) dx_T.$$

Taking $z = x_T - x$ leads to

$$V(x,T) = e^{-r(T-t)} \int_{-\infty}^{\infty} g(T, z + x) f(z) dz.$$

Then we apply Fourier transform on $V(t,x)$, with the damping factor $e^{\alpha x}$ to ensure the existence of the Fourier transform,

$$\begin{aligned}
& e^{-r(T-t)}\mathcal{F}\{e^{\alpha x}V(t,x)\} \\
= & e^{-r(T-t)}\int_{-\infty}^{\infty} e^{\imath\omega x}e^{\alpha x}V(t,x)dx \\
= & \int_{-\infty}^{\infty} e^{\imath\omega x}\left[\int_{-\infty}^{\infty} e^{\alpha x}g(t,x+z)f(z)dz\right]dx \\
= & \int_{-\infty}^{\infty} e^{\imath\omega x+\alpha x}\int_{-\infty}^{\infty} g(t,x+z)f(z)dzdx \\
= & \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} e^{\imath(\omega-i\alpha)x}g(t,x+z)dx\right]f(z)dz \\
= & \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} e^{\imath(\omega-i\alpha)(y-z)}g(t,y)dy\right]f(z)dz \\
= & \int_{-\infty}^{\infty} e^{\imath(\omega-i\alpha)y}g(t,y)dy\int_{-\infty}^{\infty} e^{\imath(-\omega+i\alpha)z}f(z)dz \\
= & \hat{g}(w-i\alpha)\hat{f}(-(w-i\alpha)).
\end{aligned}$$

This gives

$$e^{-r(T-t)}\hat{V}(w-i\alpha)=\hat{g}(w-i\alpha)\hat{f}(-(w-i\alpha)), \tag{3.7}$$

where α is the damping factor, $\hat{f}(\cdot)$ is the characteristic function, $\hat{V}(\cdot)$ is the Fourier transform of $V(x_T)$ and $\hat{g}(\cdot)$ is the Fourier transform of the payoff function. With the transformation, we can compute the right hand side of equation (3.7) and then get the option's value by performing the inverse Fourier transform.

3.2.2 Valuation Under Heston's Model

Heston's model is a two dimensional model with the logarithmic asset price and the volatility directions. Based on Heston's model, the valuation of European options can be calculated [17] by the following equation

$$V(x_0,v_0,t)=e^{-r(T-t)}\int_{-\infty}^{\infty} g(x_T)f(x_T|x_0,v_T|v_0)dx_T, \tag{3.8}$$

where x_0 and v_0 represent the logarithmic asset price and the volatility at the start time, and x_T is the logarithmic asset price at maturity.

For Bermudan options, the payoff function at each exercise time depends on the volatility at that time [17]. That is, the expectation of the payoff function gives

$$\mathbb{E}_{t_0}[g(x_T, v_T)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_T, v_T)p(x_0 \to x_T, v_0 \to v_T)dx_T dv_T, \tag{3.9}$$

where $p(x_0 \to x_T, v_0 \to v_T)$ is the transition probability density from x_0 and v_0, the logarithmic asset price and the volatility at the initial time, to x_T and v_T, their counterparts at the maturity time.

In [17], it states that the transition probability density in equation (3.9) depends on the logarithmic asset price and the volatility at maturity, given the initial volatility:

$$p(x_0 \to x_T, v_0 \to v_T) = p(z_x, v_T|v_0), \tag{3.10}$$

where $z_x = x_T - x_0$. Then the expectation of the payoff function $g(x_T, v_T)$ gives

$$\mathbb{E}_{t_0}[g(x_T, v_T)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_T, v_T)p(z_x, v_T|v_0)dz_x dv_T, \tag{3.11}$$

which equals

$$\mathbb{E}_{t_0}[g(x_T, v_T)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_T, v_T)p(z_x|v_T, v_0)p_v(v_T|v_0)dz_x dv_T. \tag{3.12}$$

Here $p_v(v_T|v_0)$ is the transition density of the volatility, which has the analytical form given by Feller (1951):

$$p_v(v_T|v_0) = ce^{-b-x}\left(\frac{x}{b}\right)^{\frac{(a-1)}{2}} I_{a-1}\left(2\sqrt{bx}\right), \tag{3.13}$$

with $c = 2\lambda/((1-e^{-\lambda T}\eta^2))$, $b = cv_0e^{-\lambda T}$, $x = cv_T$ and $a = 2\lambda\bar{v}/\eta^2$, and $I_a(x)$ being the modified Bessel function of the first kind in this formula.

By means of the Fourier transform in [17], we have

$$\hat{F}(w - i\alpha, v_0) = \int_{-\infty}^{\infty} \hat{g}(w - i\alpha, v_T)\cdot\hat{p}(-(w - i\alpha))\cdot p_v(v_T|v_0)dv_T, \tag{3.14}$$

where α is the damping factor, $\hat{g}(\cdot,\cdot)$ is the Fourier transform of the payoff function, and $\hat{p}(w)$ is the characteristic function of the logarithmic asset price given v_0 and v_T which is known analytically,

$$\hat{p}(w) = e^{iw[r(T-t_0)+\frac{\rho}{\eta}(v_T-v_0-\lambda\bar{v}(T-t_0))]}\phi\left(w\left(\frac{\lambda\rho}{\eta}-\frac{1}{2}\right)+\frac{1}{2}iw^2(1-\rho^2)\right), \tag{3.15}$$

with $\phi(a)$ being the characteristic function of $\int_{-t_0}^{T} v(s)ds$ given v_0 and v_T. In Broadie and Kaya (2004), it gives

$$\begin{aligned}\phi(a) = \quad & \frac{\gamma(a)e^{-\frac{1}{2}(\gamma(a)-\lambda)(T-t_0)}(1-e^{-\lambda(T-t_0)})}{\lambda(1-e^{-\gamma(a)(T-t_0)})} \\ \times \quad & e^{\frac{v_0+v_T}{\eta^2}\left[\frac{\lambda(1+e^{-\lambda(T-t_0)})}{1-e^{-\lambda(T-t_0)}}-\frac{\gamma(a)(1+e^{-\gamma(a)(T-t_0)})}{1-e^{\gamma(a)(T-t_0)}}\right]} \\ \times \quad & \frac{I_{\frac{1}{2}d-1}\left[\sqrt{v_0 v_T}\frac{4\gamma(a)e^{-\frac{1}{2}\gamma(a)(T-t_0)}}{\eta^2(1-e^{-\gamma(a)(T-t_0)})}\right]}{I_{\frac{1}{2}d-1}\left[\sqrt{v_0 v_T}\frac{4\lambda e^{-\frac{1}{2}\lambda(T-t_0)}}{\eta^2(1-e^{-\lambda(T-t_0)})}\right]},\end{aligned} \quad (3.16)$$

where $\gamma(a) = \sqrt{\lambda^2 - 2\eta^2 ia}$, $d = 4\bar{v}/\eta^2$ and $I_v(x)$ is the modified Bessel function of the first kind.

3.3 Error Estimation

The numerical methods applied in the thesis result in computational errors, such as the truncation error, the integral error, and other kinds of errors.

The truncation error shows up when we approximate the partial derivatives with value at grid points. A well known example is that if we the central difference scheme $\frac{V_{i-1}-2V_i+V_{i+1}}{h^2}$ (h is the step size) to approximate $\frac{\partial^2 V}{\partial S^2}$ in the asset price axis, it gives rise to the truncation error of $\mathcal{O}(h^2)$ for sufficiently smooth functions. Moreover, the axes of the asset price and the volatility are usually truncated to be finite domains. This also leads to the computational error.

In this thesis, there are many integral computations, for example, performing the Fourier transform. When numerical schemes are adopted to approximate the integral, they incur the integral error. We see that the Simpson's rule results in $(O)(h^4)$ error as

$$\int_a^b f(x)dx = \frac{h}{3}(f(a) + 4f(\frac{a+b}{2} + f(b))) - \frac{h^5}{90}f^{(4)}(\xi),$$

where $h = \frac{b-a}{2}$, and $\xi in(a,b)$.

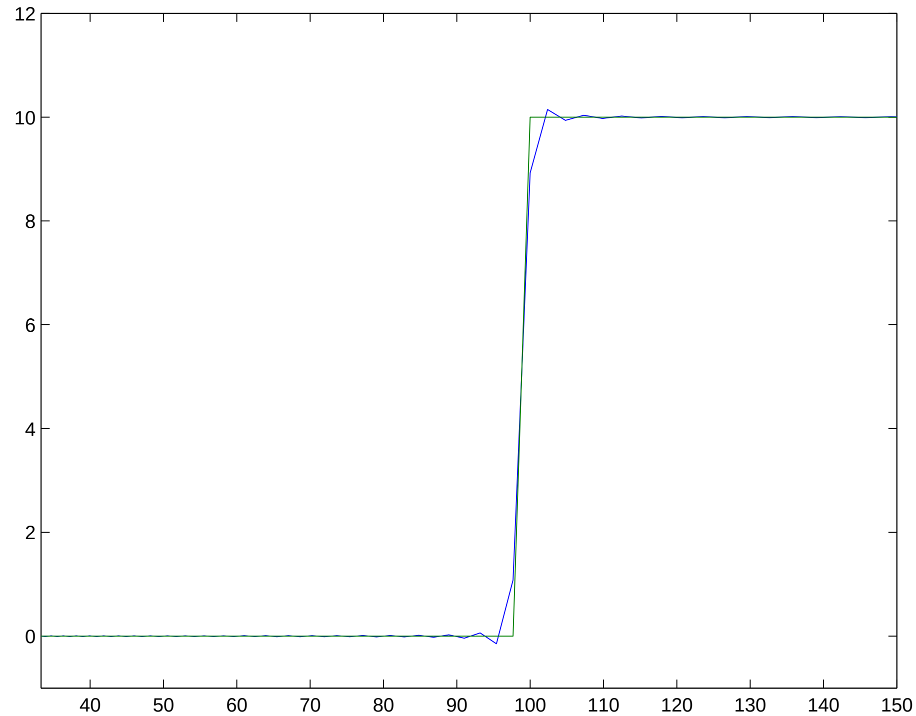

Figure 3.1: Gibbs phenomenon when using the Fourier transform method, dt=0.001

The computational error can be shown by the following example. When we treat the binary option, due to the discontinuity in the payoff function at $S = K$, it is expected that the Gibbs phenomenon will show up when Fourier transform is used to compute the numerical solution. This is actually the case. A typical example is shown in Figure 3.1, where the oscillations occur when the time step is chosen to be extremely small, the grid size in the asset price axis is big, and the Fourier transform is performed only one step backward ($dt = 0.001, dy = 0.02353, y = \log(S/K)$).

3.4 The Extrapolation Scheme

As mentioned before, American options can be approximated by evaluating a Bermudan options whose exercisable times are sufficiently many. However it is very time-consuming when the computation is done based on Heston's model and by means of Fourier transform method. The reason is that the complexity in this case is

$\mathcal{O}(N \log(N) M J^2)$, where N, M and J are the numbers of grids in the directions of asset price, time and volatility. Therefore we turn to the Richardson extrapolation method to find a way which can approximate the accurate solution, in expectation of paying a smaller price in computation.

Equation (3.14) can be regarded as a simple form,

$$\hat{F}(w - i\alpha, v_0) = \int_{-\infty}^{\infty} k(v_T) dv_T, \tag{3.17}$$

which shows that the right hand side of equation (3.17) is actually a function with respect to the volatility. This means that the valuation of $\hat{F}(w - i\alpha, v_0)$ depends on the numerical computation of the integral on the right hand side of equation (3.17), where the Richardson extrapolation technique can be applied.

If h is the mesh size in the volatility direction, a function $K(h)$ which will be the option price in our case, can be written as

$$N(h) = M + K_1 h + K_2 h^2 + K_3 h^3 + \cdots + a_n h^n + O(h^{n+1}), \tag{3.18}$$

where K_i $(i = 1, 2, 3, \cdots)$ are unknown coefficients, N is the approximation value and M is the theoretical value. In terms of the extrapolation algorithm, $K(h)$ can be approximated by

$$\begin{aligned} A_{i,0} &= K(h_i) \\ A_{i,m} &= A_{i+1,m_1} + \frac{A_{i+1,m-1} - A_{i,m-1}}{h_i / h_{i+m} - 1} \end{aligned}$$

This is described by the following process:

h_i	$A_{i,0}$	$A_{i,1}$	$A_{i,2}$	$A_{i,3}$	$\cdots$
h_1	$A_{1,0}$	$A_{1,1}$	$A_{1,2}$	$A_{1,3}$	
h_2	$A_{2,0}$	$A_{2,1}$	$A_{2,2}$		
h_3	$A_{3,0}$	$A_{3,1}$			
h_4	$A_{4,0}$				
$\cdots$					

With this algorithm, the extrapolation technique can be applied to the valuation based on equation (3.14).

The extrapolation scheme discussed above is known as Richardson extrapolation. When applied to price American options based on the Black-Scholes model, it proves to be efficient and cheap. However, when we turn to compute one Bermudan option's price under Heston's model, we are faced with the problem of how to apply it in the directions of volatility, stock and time. Generally these directions are discretized on grids and the numerical approximation of the related Bermudan option prices can be computed by means of the Fourier transform method. But some computational results are rather inaccurate. For example, when the time direction has 32 points (a Bermudan option with 32 exercisable dates), a considerable computational error will show up if the numbers of points in stock and volatility are small (for example smaller than 24). This means that we will fail to get an accurate result from Richardson extrapolation. One can see this clearly with the computational results which are obtained by discretizing the volatility direction with 2, 4, 8 and 16 points. This will completely make the results with Richardson extrapolation inaccurate. On the other hand, if we start from 32, 64, 128 and 256 grids, we will see only a small gain in the computational time in contrast with the reference value, which may be computed from 512 points.

One reason why Richardson extrapolation fails is because the computational results from few points have big errors. To avoid this, we focus firstly on the principal idea which leads to Richardson extrapolation, equation (3.18), to figure out new schemes. Theoretically speaking, when we use a number of grids to compute one Bermudan option's price, the result we get is an approximation to the accurate value. Suppose that we compute 4 times with different numbers of points (or different mesh sizes),

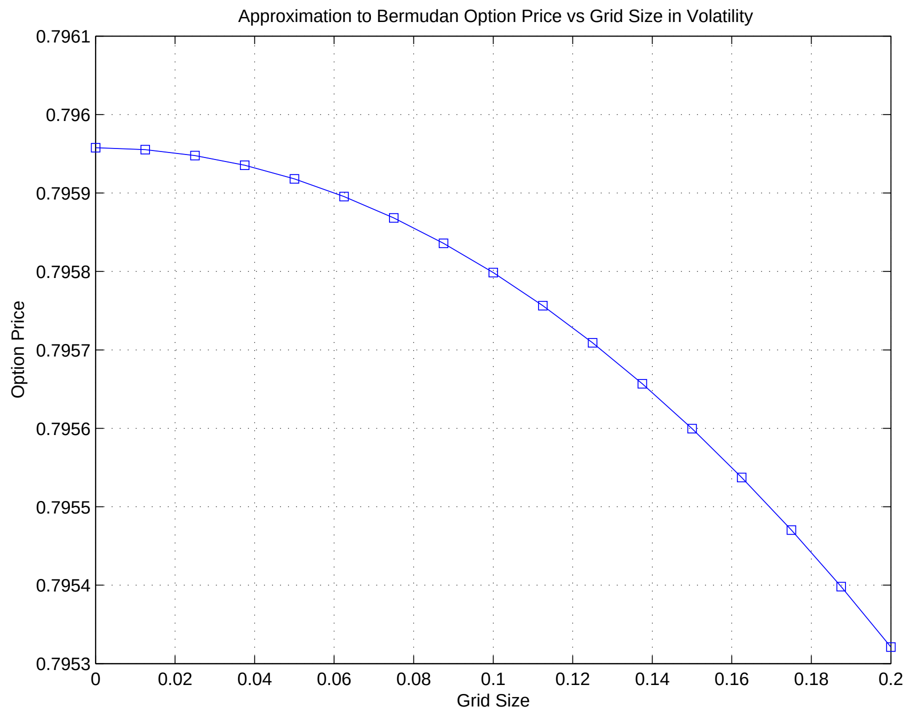

Figure 3.2: Option price converges when the grid size in volatility and stock goes to 0

we will have 4 results of the approximated value, as the equations below show:

$$
\begin{aligned}
N1 &= M + K_1(a_1h) + K_2(a_1h)^2 + K_3(a_1h)^3 + \cdots, \\
N2 &= M + K_1(a_2h) + K_2(a_2h)^2 + K_3(a_2h)^3 + \cdots, \\
N3 &= M + K_1(a_3h) + K_2(a_3h)^2 + K_3(a_3h)^3 + \cdots, \\
N4 &= M + K_1(a_4h) + K_2(a_4h)^2 + K_3(a_4h)^3 + \cdots,
\end{aligned}
$$

where M is the theoretical value, N_i $(i = 1, 2, 3, 4)$ are the computation results (approximation to the theoretical value), a_ih $(i = 1, 2, 3, 4)$ are the step sizes used in each computation and K_j $(j = 1, 2, 3, \cdots)$ are the coefficients in the Taylor expansion (3.18). In Richardson extrapolation, we have $a_{j+1} = a_j/2$ which does not give satisfactory results when applied in the volatility direction under Heston's model. An alternative technique of how to apply the extrapolation scheme is following: (1) to ensure the accuracy, we choose to start from quite a few points, (2) to save computation time, we choose small differences in the numbers of points. For example,

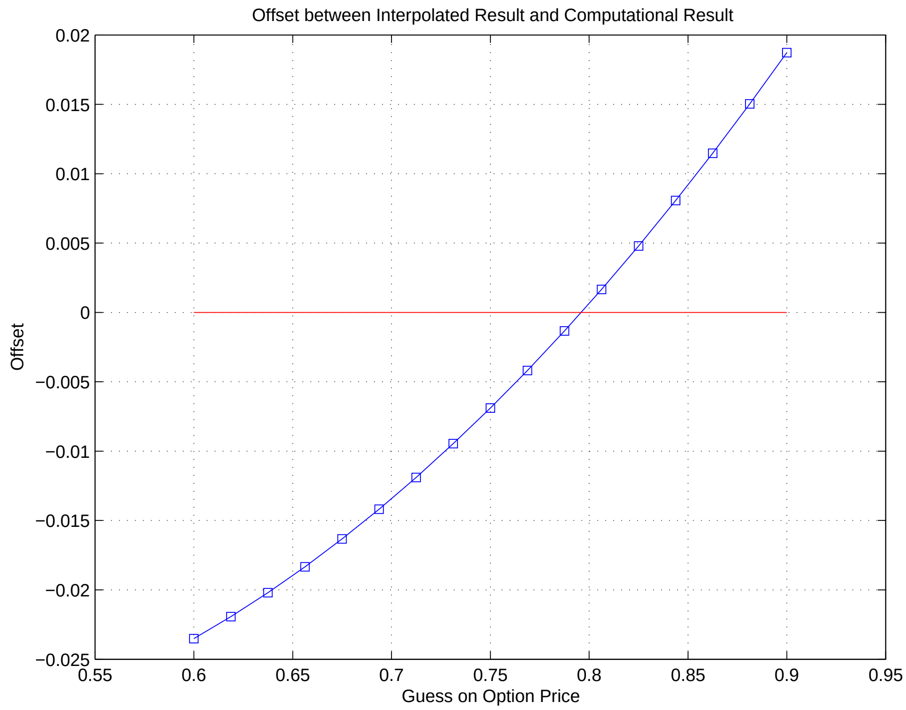

Figure 3.3: Estimate the theoretical value and apply interpolation to check this.

we compute the value with 20, 52, 84, 116 grids, say N_1, N_2, N_3 and N_4. All the computational results include the information of the theoretical value. If we check equations (3.19), we see that N_i (the computational results) and $a_i h$ (the grid size like $L/20$, $L/52$, $L/84$ and $L/116$) are known, L is the range in volatility or stock price and the unknowns are M and K_j ($j = 1, 2, 3$). We can eliminate K_j to get the value of M.

Another technique is based on the assumption that all the computational results fit a smooth curve as Figure (3.2) shows that converges to the theoretical value when the grid size goes to 0. According to Weierstrass approximation theorem[1], the curve

[1]Weierstrass Approximation Theorem: Suppose that f is defined and continuous on $[a, b]$. For each $\epsilon > 0$, there exists a polynomial $P(x)$, with the property that $|f(x) - P(x)| < \epsilon$, for all x in $[a, b]$

is approximated by the Lagrange polynomial in our new method,

$$\begin{aligned} P(x) &= f(x_0)L_{n,0}(x) + f(x_1)L_{n,1}(x) + \cdots + f(x_n)L_{n,n}(x) \\ &= \sum_{k=0}^{n} f(x_k)L_{n,k}(x), \end{aligned}$$

where

$$\begin{aligned} L_{n,k}(x) &= \frac{(x-x_0)(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)} \\ &= \prod_{i=0, i\neq k}^{n} \frac{(x-x_i)}{x_k-x_i}. \end{aligned}$$

This means that the computational results (N_1, N_2, N_3, N_4) that we obtain with 20, 52, 84, 116 points (grid sizes: $L/20$, $L/52$, $L/84$, $L/116$, where L is the length of the interval in volatility) and the theoretical value can be fitted to the curve of a Lagrange polynomial. With a next computation result, i.e. N_5 from 148 points, it should also lie on this curve. If we estimate a value for the asymptotic value and apply the interpolation scheme to compute the value N_5' at $L/148$, we should know how accurate the initial estimate was. If the difference $|N_5 - N_5'|$ is zero, we assume that the estimate is a good approximation to the theoretical value. The procedure to find the solution is a root-finding subroutine in our program.

The extrapolation can save the computational time greatly. However, the computational error converges in an unusual way as Figure (3.4) shows. The reason why the error shows an irregular behavior comes from the fact that we use a Lagrange polynomial to approximate Bermudan option prices with respect to the grid size in volatility and stock. According to the error estimation formula given in [1], when we use the function value at x_0, x_1, $\cdots$, x_n, to interpolate the value at x and suppose f is the unknown function, it says that a number $\xi(x)$ exists with

$$f(x) = P(x) + \frac{f^{n+1}(\xi(x))}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n), \tag{3.19}$$

where $P(x)$ is the interpolating polynomial. One example is shown in Figure (3.5). In this example, 5 points are used to interpolate the function $y = \sqrt{x}$. The difference

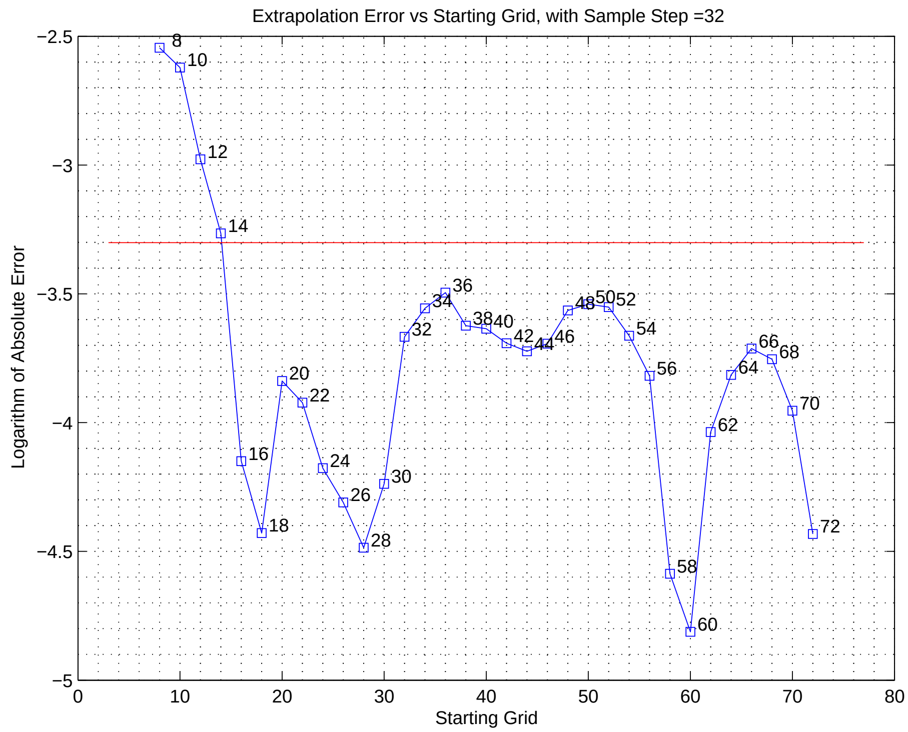

Figure 3.4: Computational error of extrapolation where Lagrange polynomial is used

between $P(x)$ and $\sqrt{x}$ can be seen in the figure. In our extrapolation, when we use the computational results with grid sizes $L/20$, $L/52$, $L/84$, $L/116$ and estimate a price at $L/1000$ to interpolate the price at $L/148$, the computational error depends partially on the difference of the grid sizes which are the term $(x-x_0)(x-x_1)\cdots(x-x_n)$ in the previous estimation formula. When we think that our initial guess is correct, it includes actually the error which responds to the interpolation error.

Here the error of the extrapolation method employed is explored. Suppose we have the computational results $f_1(x=x_1)$, $f_2(x=x_2)$, $f_3(x=x_3)$, $f_4(x=x_4)$ and the value to extrapolate is $v_0(x=x_0)$. If the guess is v_0+e where e is the error and we

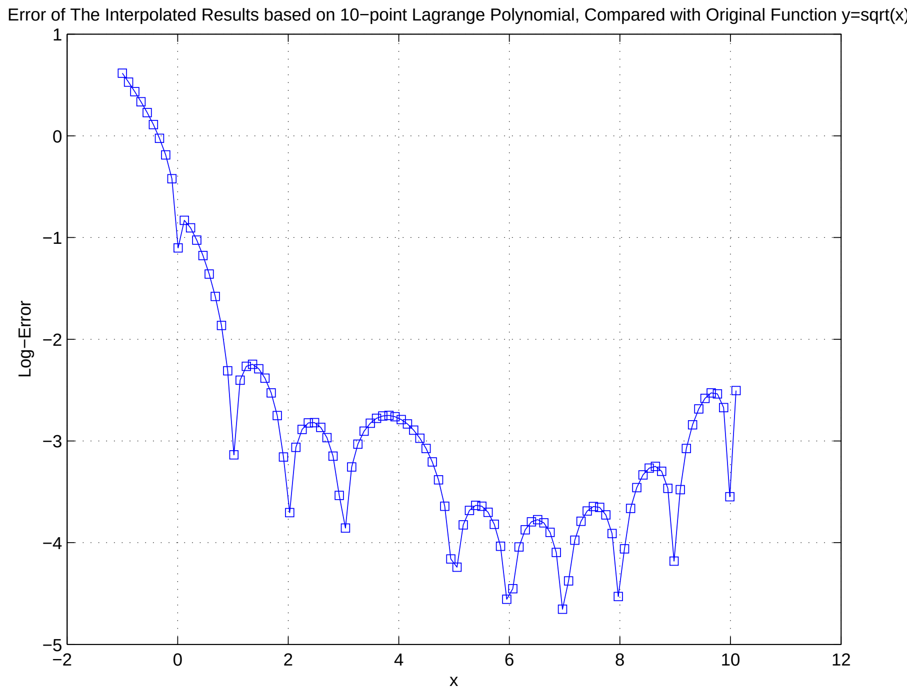

Figure 3.5: Error of the interpolated results based on 10-point Lagrange Polynomial

start to interpolate the value at $x = x_5$ with its function value f_5, we have

$$\begin{aligned}
f_5 &= (v_0 + e)\frac{(x_5 - x_1)(x_5 - x_2)(x_5 - x_3)(x_5 - x_4)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)(x_0 - x_4)} \\
&+ f_1\frac{(x_5 - x_0)(x_5 - x_2)(x_5 - x_3)(x_5 - x_4)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)} + f_2\frac{(x_5 - x_0)(x_5 - x_1)(x_5 - x_3)(x_5 - x_4)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)(x_2 - x_4)} \\
&+ f_3\frac{(x_5 - x_0)(x_5 - x_1)(x_5 - x_2)(x_5 - x_4)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)(x_3 - x_4)} + f_4\frac{(x_5 - x_0)(x_5 - x_1)(x_5 - x_2)(x_5 - x_3)}{(x_4 - x_0)(x_4 - x_1)(x_4 - x_2)(x_4 - x_3)} \\
&+ \frac{f^6(\xi(x))}{6!}(x_5 - x_0)(x_5 - x_1)(x_5 - x_2)(x_5 - x_3)(x_5 - x_4).
\end{aligned}$$

Because x_i and f_i $(i = 1, 2, 3, 4, 5)$ are known, all terms except the first one on the right hand side are known once we have the computational results. If the points we use to discretize in volatility and stock are g_0, $g_0 + d$, $g_0 + 2d$, $g_0 + 3d$ and $g_0 + 3d$, we have $x_1 = L/g_0$, $x_2 = L/(g_0 + d)$, $x_3 = L/(g_0 + 2d)$, $x_4 = L/(g_0 + 3d)$ and $x_5 = L/(g_0 + 4d)$, where L represents the range of volatility and stock price, g_0 is the starting grid and d is the step size. Substitution x_i into equation (3.20), it says that the error of the new extrapolation depends on the starting grid g_0, the grid difference d and the unknown value of the option v_0,

$$e = e(v_0, g_0, d). \tag{3.20}$$

This interprets why the computational error decreases if the start grid g_0 is a big number and the step size d is large.

3.5 Multi-Asset Option Pricing

3.5.1 Asset Prices' Random Walk and Pricing Equation

In the case of two-asset options, if the dynamics of the underlying is modeled by GBM, we have

$$dS_1/S_1 = \mu_1 dt + \sigma_1 dW_1,$$
$$dS_2/S_2 = \mu_2 dt + \sigma_2 dW_2,$$

where $\mu_i (i = 1, 2)$ are drift parameters, $\sigma_i (i = 1, 2)$ are volatility parameters and $W_i (i = 1, 2)$ are Brownian motions. We have

$$\mathbb{E}[dW_i] = 0, var[dW_i] = dt, (i = 1, 2)$$

and dW_1 and dW_2 are correlated as

$$\mathbb{E}[dW_1 dW_2] = \rho dt.$$

If $V(S_1, S_2, t)$ denotes the option value, the differential of V can be obtained according to the 2-dimensional Itô's lemma:

$$dV = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \rho\sigma_1^2 S_1^2 \sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_1^2} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2})dt + \frac{\partial V}{\partial S_1} dS_1 + \frac{\partial V}{\partial S_2} dS_2.$$

We construct a portfolio including one long option position and two short positions in some quantities of underlying assets. Then we compute these quantities to eliminate risk as before. Finally we arrive at obtaining the pricing equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \rho\sigma_1^2 S_1^2 \sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_1^2} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + rS_1 \frac{\partial V}{\partial S_1} + rS_2 \frac{\partial V}{\partial S_2} - rV = 0.$$

If continuous dividends are paid, it changes to be

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma_1^2S_1^2\frac{\partial^2 V}{\partial S_1^2}+\rho\sigma_1^2S_1^2\sigma_2^2S_2^2\frac{\partial^2 V}{\partial S_1^2}+\frac{1}{2}\sigma_1^2S_1^2\frac{\partial^2 V}{\partial S_1^2}+(r-q_1)S_1\frac{\partial V}{\partial S_1}+(r-q_2)S_2\frac{\partial V}{\partial S_2}-rV=0,$$

where $q_i(i=1,2)$ are the dividend yields. The final condition for the pricing equation is

$$V(S_1,S_2,T)=f(S_1,S_2),$$

where $f(S_1,S_2)$ is the payoff function. Some well-known payoffs are

- "Maximum call": $f(S_1,S_2)=(max(S_1,S_2)-K)^+$.
- "Maximum put": $f(S_1,S_2)=(K-max(S_1,S_2))^+$.
- "Minimum call": $f(S_1,S_2)=(min(S_1,S_2)-K)^+$.
- "Minimum put": $f(S_1,S_2)=(K-min(S_1,S_2))^+$.
- "Spread option": $(S_1-S_2-K)^+$.

For more general case of n assets, it gives

$$dS_i=\mu_iS_idt+\sigma_iS_idW_i. \tag{3.21}$$

where μ_i, σ_i and $W_i(i=1,2,\cdots,n)$ have the meaning as mentioned before. dW_i and dW_j are correlated:

$$\mathbb{E}[dW_idW_j]=\rho_{ij}dt,(i,j=1,2,\cdots,n,i\neq j).$$

Here ρ_{ij} is the correlation coefficient. The correlation matrix is made up of these correlation coefficients and it is symmetric and positive definite. By means of the multi-dimensional Itô's lemma, the differential of function $V(S_1,S_2,\cdots,S_n,t)$ reads

$$dV=(\frac{\partial V}{\partial t}+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_i\sigma_j\rho_{ij}S_iS_j\frac{\partial^2 V}{\partial S_i\partial S_j})dt+\sum_{i=1}^{n}\frac{\partial V}{\partial S_i}dS_i.$$

This leads to the pricing equation for basket options.

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_i\sigma_j\rho_{ij}S_iS_j\frac{\partial^2 V}{\partial S_i\partial S_j}+\sum_{i=1}^{n}(r-q_i)S_i\frac{\partial V}{\partial S_i}-rV=0. \tag{3.22}$$

The final condition is

$$V(S_1,S_2,\cdots,S_n,T)=f(S_1,S_2,\cdots,S_n).$$

3.5.2 Multi-dimensional Fourier Transform and CONV Method

The multi-dimensional Fourier transform is defined as

$$\mathcal{F}^m\{F(\vec{x})\} = \int_{\mathbb{R}^m} e^{i\vec{w}\cdot\vec{x}} F(\vec{x}) d\vec{x} = \hat{F}(\vec{w}),$$

where "$\cdot$" denotes the inner product of vectors and m, the number of dimensions. The inverse Fourier transform in m-dimensions reads

$$\mathcal{F}^{-m}\{F(\vec{w})\} = \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} e^{-i\vec{x}\cdot\vec{w}} \hat{F}(\vec{w}) d\vec{w}.$$

In [6], Björk points out that the solution to the multi-dimensional Black-Scholes PDE (3.22) has the risk-neutral valuation form:

$$V(\vec{S}, t) = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}_{t,\vec{S}}[V(\vec{S}, T)].$$

Here $\mathbb{Q}$ means the martingale measure and $V(\vec{S}, T)$ is the payoff function. In order to use the CONV method to compute option prices in the multi-dimensional case, we make the assumption that the multi-variate conditional probability density function, $f(y_1, y_2, \cdots, y_n | x_1, x_2, \cdots, x_n)$, of the vector of state variables $\{y_1, y_2, \cdots, y_n\}$ conditioned on $\{x_1, x_2, \cdots, x_n\}$ is equal to the transition probability density $f(y_1 - x_1, y_2 - x_2, \cdots, y_n - x_n)$:

$$f(y_1, y_2, \cdots, y_n | x_1, x_2, \cdots, x_n) = f(y_1 - x_1, y_2 - x_2, \cdots, y_n - x_n). \tag{3.23}$$

Then from the risk-neutral expression, we obtain

$$V(\vec{x}, t) = e^{-r(T-t)} \int_{\mathbb{R}^m} V(\vec{y}, T) \cdot f(\vec{y}|\vec{x}) d\vec{y},$$

where $\vec{y}$ and $\vec{x}$ denote vectors of state variables at time T and $t (t < T)$, respectively. With the assumption (3.23), it gives

$$e^{r(T-t)} V(\vec{x}, t) = \int_{\mathbb{R}^m} V(\vec{y}, T) \cdot f(\vec{y} - \vec{x}) d\vec{y} = \int_{\mathbb{R}^m} V(\vec{x} + \vec{z}, T) \cdot f(\vec{z}) d\vec{z} \tag{3.24}$$

by changing variables with $\vec{z} = \vec{y} - \vec{x}$. Damping $V(\vec{x}, t)$ with $e^{\vec{\beta}\cdot\vec{x}}$ then taking Fourier transform on both sides of equation (3.24) leads to

$$\begin{aligned}
e^{r(T-t)}\mathcal{F}^m\{e^{\vec{\beta}\cdot\vec{x}}V(\vec{x},t)\} &= \int_{\mathbb{R}^m} e^{i\vec{w}\cdot\vec{x}}e^{\vec{\beta}\cdot\vec{x}}V(\vec{x},t)\}d\vec{x} \\
&= \int_{\mathbb{R}^m} e^{i\vec{w}\cdot\vec{x}}\Big[\int_{\mathbb{R}^m} e^{\vec{\beta}\cdot\vec{x}}V(\vec{x}+\vec{z},T)f(\vec{z})d\vec{z}\Big]d\vec{x} \\
&= \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} e^{i\vec{w}\cdot\vec{x}+\vec{\beta}\cdot\vec{x}}V(\vec{x}+\vec{z},T)f(\vec{z})d\vec{z}d\vec{x} \\
&= \int_{\mathbb{R}^m}\Big[\int_{\mathbb{R}^m} e^{i(\vec{w}-i\vec{\beta})\cdot\vec{x}}V(\vec{x}+\vec{z},T)d\vec{x}\Big]f(\vec{z})d\vec{z} \\
&= \int_{\mathbb{R}^m}\Big[\int_{\mathbb{R}^m} e^{i(\vec{w}-i\vec{\beta})\cdot(\vec{y}-\vec{z})}V(\vec{y},T)d\vec{y}\Big]f(\vec{z})d\vec{z} \\
&= \int_{\mathbb{R}^m} e^{i(\vec{w}-i\vec{\beta})\cdot\vec{y}}V(\vec{y},T)d\vec{y}\int_{\mathbb{R}^m} e^{i(-\vec{w}+i\vec{\beta})\cdot\vec{z}}f(\vec{z})d\vec{z} \\
&= \hat{V}(\vec{w}-i\vec{\beta})\phi(-\vec{w}+i\vec{\beta}),
\end{aligned}$$

where

$$\phi(\vec{w}) = \int_{\mathbb{R}^m} e^{i\vec{w}\cdot\vec{x}}f(\vec{x})d\vec{x}$$

is the characteristic function of the multi-variate probability density function $f(\vec{x})$. The option price can be recovered by the inverse Fourier transform and un-damping as

$$e^{r(T-t)}V(\vec{x},t) = e^{-\vec{\beta}\cdot\vec{x}}\mathcal{F}^{-m}\{\hat{V}(\vec{w}-i\vec{\beta})\phi(-\vec{w}+i\vec{\beta})\}.$$

3.5.3 Multi-Variate Characteristic Function

When the underlying assets $\vec{S}$ follows the random walks as (3.21), $\vec{S}$ is normally distributed. As [2] (page 17) shows, the joint characteristic function of multi-variate normal variables reads

$$\begin{aligned}
\phi(\vec{w}) &= \mathbb{E}[e^{i\vec{w}\cdot\vec{S}}] \\
&= exp\Big[i\sum_{i=1}^{m} w_i\mu_i - \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} w_iw_jCov(dW_i,dW_j)\Big].
\end{aligned}$$

For two-asset options, the characteristic function is

$$\phi(w_1,w_2) = exp\Big[i(w_1\mu_1+w_2\mu_2) - \frac{1}{2}(w_1^2\sigma_1^2+2\rho w_1w_2\sigma_1\sigma_2+w_1^2\sigma_1^2)\Big].$$

Therefore we can calculate the option prices with multi-asset once the characteristic function is available.

Chapter 4

Implementation and Results

When using the Fourier transform method to evaluate option prices, the continuous functions in the pricing equations should be replaced by their discrete versions. In order to do this, discretization is performed in the axis of time, the logarithmic asset price and the volatility. Similarly some other problems like computing the integrals over an infinite domain must be considered as well.

In this chapter, the implementation of the option pricing methods is covered. The results of applying the extrapolation scheme with the Fourier transform in Heston's model are also presented. The results of applying extrapolation scheme in the multi-asset case are included as well. Two sections after this shows the results obtained with the multigrid method and the Green function method.

4.1 Implementation Details

In this section, we give some details about the implementation of the extrapolation scheme and the Fourier transform to calculate option prices. First of all, the axes of time, the logarithmic asset price and the volatility are discretized and the numbers of points are denoted by N, M and J, respectively.

The domains of the integrals are truncated to be finite intervals as follows. The range in the logarithmic asset price is chosen to be $2L$ times the volatility (standard deviation) of the logarithmic asset price, which proves to be enough to ensure the accuracy with $L = 10$. On the volatility axis, the integral range is chosen as $[0, 2]$. After the discretization and the truncation, integrals over discrete grid points within finite intervals are approximated by the Trapezoidal rule.

The characteristic function in Heston's model takes both the initial volatility and the final volatility as input arguments. The complexity of the method $\mathcal{O}(N \log(N) M J^2)$ has the square of J inside.

The damping factor α is introduced to ensure that the Fourier transform exists. Experiments show that the impact of the damping factor is not significant. Therefore we usually choose $\alpha = 0$.

The reference data is taken from the literature if available. Otherwise we use a computation on finer grids, for example, with $N = 512, M = 10, J = 512$ in Heston's model for a 10-time exercisable Bermudan option.

The grid sizes in the logarithmic asset price and in the frequency domain follow the reciprocity relation $dx * dw = 1/N$. Therefore the grid size in the frequency domain is determined after we discretized the axis of the logarithmic asset price. The grid points in the logarithmic asset price direction are centered at $\log(S_0/K)$ and the grid points in the frequency domain are centered at $w = 0$.

The extrapolation scheme usually is based on 5 numerical results calculated by the Fourier transform method. The number of the points with which the numerical results have been generated are usually chosen $G_0 + k * G_s(k = 0, 1, 2, 3, 4)$ to be used in the extrapolation scheme, together with the option prices evaluated by the CONV method. To explain the extrapolation results easily, G_0 is the "start point" and G_s, the "step size".

4.2 Performance of The Extrapolation Scheme

The price of a Bermudan option is calculated to explore the characteristics of the extrapolation scheme. The reference data is obtained in a very fine grid $N = 512, M = 4, J = 512$. The numbers of points in the logarithmic asset price and the volatility direction are set to be equal $N = J$ when the CONV method is applied to compute the pre-extrapolation results. The parameters in Heston's model which describes the dynamics of the option's underlying are listed in Table (4.14):

Table 4.1: Parameters with the option and Heston's Model

$S_0 = 10$	$K = 10$	$T = 0.25$	$r = 0.1$	$\lambda = 5.0$
$\eta = 0.9$	$\bar{v} = 0.16$	$v_0 = 0.25$	$\rho = 0.1$	

The experimental results show that the accuracy of the extrapolation scheme is influenced by the choice of the start point and the step size, as shown in the Tables (4.2)(4.3) and Figures (4.1)(4.2).

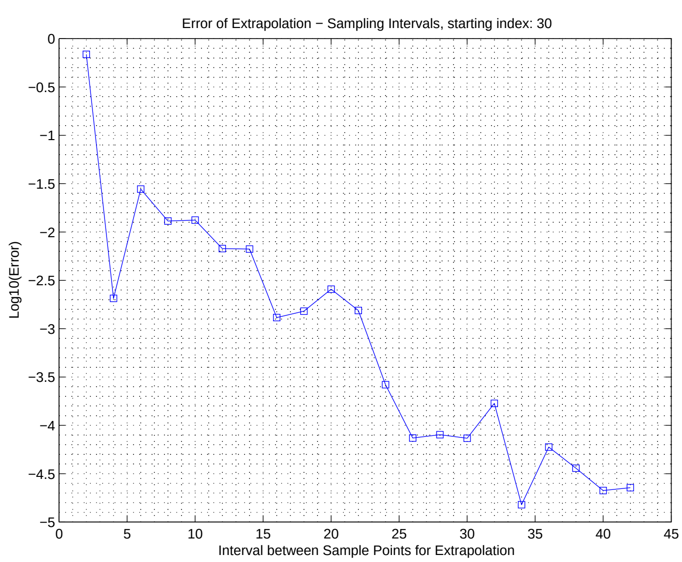

Figure 4.1: The accuracy of the extrapolation scheme is influenced by the number of start points.

Table 4.2: The accuracy of the extrapolation scheme is influenced by the start point. The step size is fixed at 30.

Start Number of Points	Absolute Error
12	2.3814e-04
16	3.1331e-05
20	7.5692e-05
24	1.0209e-04
28	2.2178e-06
32	2.5172e-04
36	2.7581e-04
40	8.3468e-05
44	8.9020e-05
48	1.0838e-04
52	3.7273e-05
56	1.0016e-04
60	7.7828e-05
64	2.5583e-04
68	1.4077e-04
72	3.7797e-05
76	7.2500e-06
80	4.9491e-05

Table 4.3: The accuracy of the extrapolation scheme is influenced by the grid step. The first grid in the extrapolation scheme contains 30 points.

Step Size	Absolute Error
2	6.8824e-01
4	2.0565e-03
6	2.7766e-02
8	1.2986e-02
10	1.3289e-02
12	6.7349e-03
14	6.6606e-03
16	1.3032e-03
18	1.5178e-03
20	2.5625e-03
22	1.5409e-03
24	2.6357e-04
26	7.3880e-05
28	8.0015e-05
30	7.3625e-05
32	1.6907e-04
34	1.5113e-05
36	5.9652e-05
38	3.6078e-05
40	2.1193e-05
42	2.2665e-05

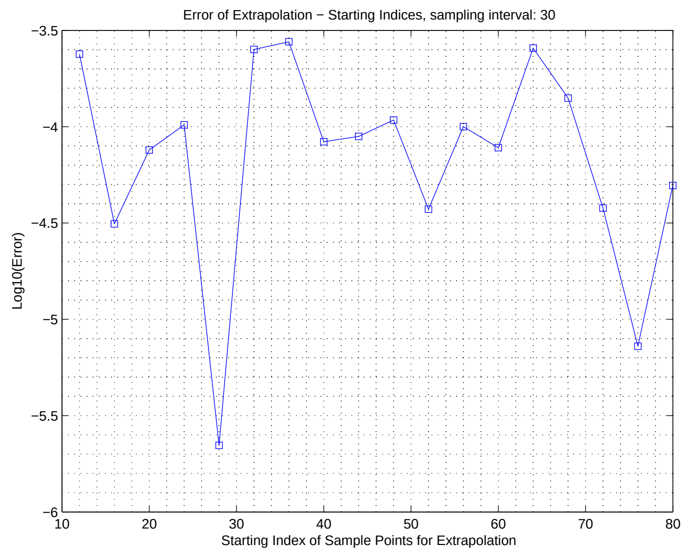

Figure 4.2: The accuracy of the extrapolation scheme is influenced by the step size.

4.3 Comparison with Gaussian Quadrature

Gaussian quadrature is a powerful scheme to approximate integrals. In the method where the extrapolation scheme is adopted, Gaussian quadrature is not applied. The results generated by these 2 schemes are compared to see the difference in their performance. The experiment used to do the comparison is the same Bermudan option as in the previous section with the same reference data.

Two sets of grid points are used in the experiments with the extrapolation scheme. The first calculates the result with grid points 20, 30, 40, 50 and 60. The second uses 20, 36, 52, 68 and 84 points.

The pre-extrapolation data is generated in two ways. In the first test, the number of points in the logarithmic asset price N and the volatility J are not equal. N is set to be fixed at $N = 512$ and J is varying. In the second test, we let both of them be varying and satisfy $N = J$. With the Gaussian quadrature, N is always equal to J.

The comparison between the results given by Gaussian quadrature and those evaluated by the CONV-extrapolation method with four choices the various setting of N, J and the choices of step size is listed in Tables (4.4), (4.5), (4.6) and (4.7).

Table 4.4: Computational error with 20, 30, 40, 50, 60 points; N fixed, J variable

Number of Points	Gaussian Quadrature	Extrapolation
100	2.0066e-03	2.6453e-03
120	1.3502e-03	1.3846e-03
140	9.5873e-04	4.4404e-04
160	7.0540e-04	2.9668e-04
180	5.3412e-04	9.0006e-04
200	4.0932e-04	1.4032e-03

Table 4.5: Computational error with 20, 36, 52, 68, 84 points; N fixed, J variable

Number of Points	Gaussian Quadrature	Extrapolation
100	2.0066e-03	2.6889e-03
120	1.3502e-03	1.4974e-03
140	9.5873e-04	6.4609e-04
160	7.0540e-04	5.5993e-07
180	5.3412e-04	5.1212e-04
200	4.0932e-04	9.2875e-04

From the comparison, we recognize that when the pre-extrapolation results are calculated numerically with $N = J$ and the extrapolation scheme chooses the number of start points and step size carefully, we can get the same accuracy as the Gaussian quadrature in Table (4.6). Table (4.8) shows the main benefit of the extrapolation scheme, which is the less computational time.

Table 4.6: Computational error with 20, 30, 40, 50, 60 points; $N = J$, both variable

Number of Points	Gaussian Quadrature	Extrapolation
100	2.0066e-03	8.7841e-04
120	1.3502e-03	6.2254e-04
140	9.5873e-04	4.6682e-04
160	7.0540e-04	3.6525e-04
180	5.3412e-04	2.9546e-04
200	4.0932e-04	2.4551e-04

Table 4.7: Computational error with 20, 36, 52, 68, 84 points; $N = J$, both variable

Number of Points	Gaussian Quadrature	Extrapolation
100	2.0066e-03	1.1640e-03
120	1.3502e-03	1.1377e-03
140	9.5873e-04	1.2047e-03
160	7.0540e-04	1.3076e-03
180	5.3412e-04	1.4213e-03
200	4.0932e-04	1.5349e-03

Table 4.8: Comparison between Gaussian quadrature and the extrapolation scheme.

Method	Option Price	Absolute Error	Computational Time (sec)
Gaussian Quadrature	7.882735e-01	5.871015e-05	4297.3
Extrapolation	7.882526e-01	7.957564e-05	303.3

In the next experiment, Gaussian quadrature is used with $N = 512$, $M = 4$, $J = 512$. Extrapolation is compared with $N = J = 20, 52, 84, 116, 148$, and $M = 4$. The reference data is obtained with $N = J = 1000$ and $M = 4$. We can see a significant benefit of the extrapolation scheme.

4.4 Extrapolation for Various Options

In this section the extrapolation technique is evaluated for various options. The parameters used in this section are listed in Table (4.9), where CP denotes the type of option. With $CP = 1$, it is a call option, otherwise it is a put option.

Table 4.9: Parameters for different options to test the extrapolation scheme.

	Bermudan I	Bermudan II	Bermudan III	American
S_0	10	10	100	10
CP	1	-1	1	-1
K	10	10	100	10
T	0.25	1.25	1	0.25
q	0	0	0	0
r	0.1	0.1	0.06	0.1
$\bar{v}$	0.144	0.144	0.232	0.16
v_0	0.029	0.029	0.083	0.25
λ	0.136	2.232	1.136	5
η	0.190	0.690	0.690	0.9
ρ	-0.834	-0.334	0.375	0.1
M	4	4	10	∞

The experimental results give the table are presented in the tables (4.10)(4.11)(4.12)(4.13) below. The computational time is in seconds.

Table 4.10: Bermudan option I (Reference: $2.5698e-01$).

	Gaussian Quadrature	Extrapolation
Comp. Result	2.5686e-01	2.5606e-01
Absolute Error	1.1468e-04	9.2056e-04
Comp. Time (sec)	3781.5	598.8

Table 4.11: Bermudan option II (Reference: $9.0940e-01$).

	Gaussian Quadrature	Extrapolation
Comp. Result	9.0937e-01	9.0941e-01
Absolute Error	2.6032e-05	1.4219e-05
Comp. Time	3562.6	996.7

Table 4.12: Bermudan option III (Reference: $1.69516e+01$).

	Gaussian Quadrature	Extrapolation
Comp. Result	1.69511e+01	1.69525e+01
Absolute Error	5.0089e-04	9.2232e-04
Comp. Time	8810.7	2065.8

Table 4.13: American option (Reference: $7.9659e-01$).

	Extrapolation
Comp. Result	7.9616e-01
Absolute Error	1.9071e-04

The results presented in the extrapolation scheme under different cases of options show the applicability of the method.

4.5 Extrapolation in The Multi-Asset Option

The price of a 4-asset Bermudan put option is calculated with the multi-dimensional CONV method. The parameters used in the case is listed in Table (4.14).

Table 4.14: Parameters with Multi-Asset Option

$$S_0 = 40 \quad K = 40 \quad T = 1 \quad r = 0.06$$

$$\sigma = [0.2 \;\; 0.2 \;\; 0.2 \;\; 0.2]$$

$$D = [0.04 \;\; 0.04 \;\; 0.04 \;\; 0.04]$$

$$\rho = \begin{pmatrix} 1 & 0.25 & 0.25 & 0.25 \\ 0.25 & 1 & 0.25 & 0.25 \\ 0.25 & 0.25 & 1 & 0.25 \\ 0.25 & 0.25 & 0.25 & 1 \end{pmatrix}.$$

The extrapolation scheme takes the pre-extrapolation results calculated with 30, 38, 46, 54 and 60 points and gives the approximation in table (4.15).

Table 4.15: Extrapolation in the multi-asset case.

	Extrapolation	Reference
Value	1.653567	1.653787
Time (sec)	82.46	198.380
Error	$2.179e-04$	

The result shows the gain in the computational time with the multi-dimensional CONV method.

Chapter 5

Conclusions

In the thesis, we propose a extrapolation scheme to facilitate the Fourier transform method in option pricing. It integrates the interpolation method and the root-finding method together to estimate, adjust and obtain the extrapolated value. It is compared with the Gaussian quadrature method in performance when approximating an American option price with the Fourier transform method in Heston's model. The comparison shows that the extrapolation scheme results in a significant gain in the computational time and with enough accuracy as well. Some other options with various parameter and the multi-asset option are also evaluated, which leads to both accurate and fast result.

More research can be performed in the future. For example, error estimation of this scheme is worth to do. It may be also interesting to evaluate multi-asset options in Heston's model, where the characteristic function in the multi-asset case is not available yet. The Green function method, as an alternative way, may be used to evaluate Bermudan or American options.

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Appendix A

Appendix

A.1 Derivation of Itô's Lemma

Before we describe Itô's lemma, one result is needed. With probability 1, it says

$$dX^2 \to dt \qquad as \qquad dt \to 0. \tag{A.1.1}$$

It means if dt goes to 0, dX^2 tends to dt.

Suppose that $f(S)$ is a smooth function of S. If S varies by a small amount dS, it is clear that f will vary also by a small amount provided it is not close to singularities of f. According to Taylor's expansion, it gives

$$df = \frac{df}{dS}dS + \frac{1}{2}\frac{d^2 f}{dS^2}dS^2 + \cdots, \tag{A.1.2}$$

where the remainder terms are replaced by dots here.

With equation (2.1), it can be derived that

$$\begin{aligned} dS^2 &= (\sigma S dX + \mu S dt)^2 \\ &= \sigma^2 S^2 dX^2 + 2\sigma\mu S^2 dt dX + \mu^2 S^2 dt^2. \end{aligned} \tag{A.1.3}$$

In terms of (A.1.1), $dX = O(\sqrt{dt})$, which leads to

$$dS^2 = \sigma^2 S^2 dX^2 + \cdots,$$

and

$$dS^2 \to \sigma^2 S^2 dt.$$

After substitution into (A.1.2), the small variation in f gets to be

$$\begin{aligned} df &= \frac{df}{dS}(\sigma\mu S dX + \mu S dt) + \frac{1}{2}\sigma^2 S^2 \frac{d^2 f}{dS^2} dt \\ &= \sigma S \frac{df}{dS} dX + (\mu S \frac{df}{dS} + \frac{1}{2}\sigma^2 S^2 \frac{d^2 f}{dS^2}) dt. \end{aligned} \tag{A.1.4}$$

If f is considered to be the function of both the variables S and time, $f(S,t)$, the expansion of $f(S+dS, t+dt)$ will be

$$df = \sigma S \frac{\partial f}{\partial S} dX + (\mu S \frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t}) dt. \tag{A.1.5}$$

A.2 Derivation of Solution to Black-Scholes Equation

Here the derivation of the exact solution to the Black-Scholes equation is given.

First of all, it is easy to show that one of the solutions to the heat equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

in

$$u(x, \tau) = \frac{1}{2\sqrt{\pi\tau}} e^{-\frac{x^2}{4\tau}}.$$

When $\tau \sim 0$, this function goes to its limit: the delta function $\delta(\tau)$. Therefore the function

$$u_\delta(x, \tau) = \frac{1}{2\sqrt{\pi\tau}} e^{-\frac{(x-s)^2}{4\tau}}.$$

is also a solution if we use either x or s as the spatial independent variable. Meanwhile, it is easy to see

$$u_\delta(x - s, \tau) = u_\delta(s - x, \tau).$$

If x is regarded as the spatial variable, then

$$u_0(s) u_\delta(s - x, \tau)$$

satisfies the heat equation. Based on the linearity of the heat equation and the law of superposition, the function

$$u(x, \tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} u_0(s) e^{-\frac{(x-s)^2}{4\tau}} ds$$

is also a solution to the heat equation, and in particular $u(x, 0) = \int_{-\infty}^{\infty} u_0(s)\delta(s - x)ds = u_0(x)$. This results in a general solution to a parabolic equation with the initial condition $u_0(x)$.

Secondly, the Black-Scholes equation with respect to a call option

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rS\frac{\partial V}{\partial S} - rV = 0,$$

with

$$V(0,t) = 0, \qquad V(S,t) \sim S \quad as \quad S \to \infty,$$

and

$$V(S,T) = (S-K,0)^+$$

can be transformed to a standard parabolic partial differential equation. To do this, it is first made dimensionless by

$$S = Ke^x, \qquad t = T - \frac{\tau}{\frac{1}{2}\sigma^2}, \qquad V = Kz(x,\tau).$$

This leads to the equation

$$\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2} - (k-1)\frac{\partial z}{\partial x} - kz = 0,$$

where $k = \frac{r}{\frac{1}{2}\sigma^2}$. Now the initial condition becomes

$$z(x,0) = (e^x - 1, 0)^+.$$

By introducing

$$z = e^{\alpha x + \beta\tau} u(x,\tau),$$

where

$$\alpha = -\frac{1}{2}(k-1), \qquad \beta = -\frac{1}{4}(k+1)^2,$$

the transformation gives the standardized parabolic equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}.$$

The initial condition is

$$U_0(x) = u(x,0) = (e^{\frac{1}{2}(k+1)} - e^{\frac{1}{2}(k-1)}, 0)^+.$$

Then the solution to the Black-Scholes equation can be formed as

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} u_0(s) e^{-\frac{(x-s)^2}{4\tau}} ds.$$

If the variable is changed by using $y = (s - x)/\sqrt{2\tau}$, the formula above will give

$$\begin{aligned} u(x,\tau) &= \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} u_0(y\sqrt{2\tau}+x) e^{-\frac{1}{2}y^2} dy \\ &= \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} e^{\frac{1}{2}(k+1)(x+y\sqrt{2\tau})} e^{\frac{1}{2}y^2} dy \\ &\quad - \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} e^{\frac{1}{2}(k-1)(x+y\sqrt{2\tau})} e^{\frac{1}{2}y^2} dy \\ &= I_1 - I_2. \end{aligned} \tag{A.2.6}$$

I_1 can be evaluated by

$$\begin{aligned} I_1 &= \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} u_0(y\sqrt{2\tau}+x) e^{-\frac{1}{2}y^2} dy \\ &= \frac{e^{\frac{1}{2}(k+1)x}}{2\sqrt{\pi\tau}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{1}{2}(k+1)^2\tau} e^{-\frac{1}{2}(y-\frac{1}{2}(k+1)\sqrt{2\tau})^2} dy \\ &= \frac{e^{\frac{1}{2}(k+1)x+\frac{1}{4}(k+1)^2\tau}}{2\sqrt{\pi\tau}} \int_{-x/\sqrt{2\tau}-\frac{1}{2}(k+1)\sqrt{2\tau}}^{\infty} e^{-\frac{1}{2}\rho^2} d\rho \\ &= \frac{e^{\frac{1}{2}(k+1)x+\frac{1}{4}(k+1)^2\tau}}{2\sqrt{\pi\tau}} N(d_1), \end{aligned} \tag{A.2.7}$$

where

$$d_1 = x/\sqrt{2\tau} + \frac{1}{2}(k+1)\sqrt{2\tau}$$

and

$$N(d_1) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{d_1} e^{-\frac{1}{2}s^2} ds.$$

Finally, with back transformations, we find

$$z(x,\tau) = e^{-\frac{1}{2}(k+1)x-\frac{1}{4}(k+1)^2\tau} u(x,\tau)$$

and $x = \log(S/K)$, $\tau = \frac{1}{2}\sigma^2(T-t)$ and $V = Kz(x,\tau)$, which results in

$$V(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2), \tag{A.2.8}$$

where

$$\begin{aligned} d_1 &= \frac{\log(\frac{S}{K}) + (r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= \frac{\log(\frac{S}{K}) + (r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}. \end{aligned}$$

This is the explicit expression for the value of an European call option.