Pricing Inflation Products with Stochastic Volatility and Stochastic Interest Rates

Stefan N. Singor[1,4,*,x], Lech A. Grzelak[2,4],
David D. B. van Bragt[3], and Cornelis W. Oosterlee[4,5]

May 2011

Abstract

We consider a Heston type inflation model in combination with a Hull-White model for nominal and real interest rates, in which all the correlations can be non-zero. Due to the presence of the Heston dynamics our derived inflation model is able to capture the implied volatility skew/smile, which is present in the inflation option market data. We derive an efficient approximate semi-closed pricing formula for two types of inflation dependent options: index and year-on-year inflation options. The derived pricing formulas allow for an efficient calibration of the inflation model. We also illustrate our approach using a real-life pension fund example, where the Heston Hull-White model is used to determine the value of conditional future indexations.

Keywords: Heston Hull-White model, Inflation, Affine diffusion processes, Monte Carlo simulation, Indexation provision, Pension fund.

JEL Classification: C02, C13, C58, C63, G12, G13, G22, G23.

---

[1]Corresponding author. Ortec Finance, Ortec Finance Research Center and of the Insurance Risk Management department, Boompjes 40, 3011 XB Rotterdam, The Netherlands.
[2]Rabobank, Derivatives Research and Validation Group. Jaarbeursplein 22, 3521 AP, Utrecht, The Netherlands.
[3]Aegon Asset Management, AEGONplein 50, 2591 TV, Den Haag, The Netherlands.
[4]Delft Institute of Applied Mathematics, Delft University of Technology.
[5]CWI - National Research Institute for Mathematics and Computer Science. Science Park 123, 1098 XG Amsterdam, The Netherlands.

[x]The authors would like to thank Mark Greenwood from the Royal Bank of Scotland for providing relevant inflation option market data. We appreciate the feedback of participants of the SIAM 2010 Conference on Financial Mathematics & Engineering in San Fransisco, California.

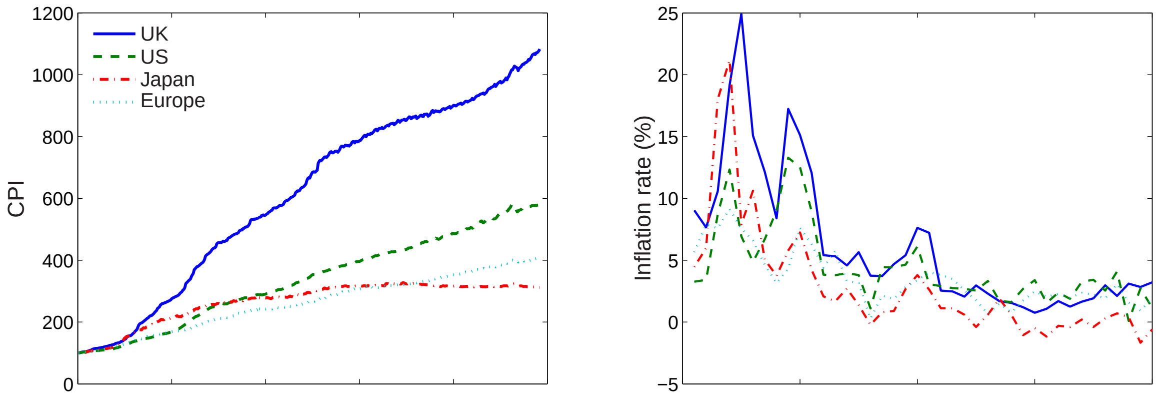
Contents

1 Introduction

Inflation-dependent derivatives are increasingly important in financial engineering. As a consequence, inflation[1] markets are becoming more active, liquid and transparent. Broker volumes increased substantially from late-2002, driven by a rise in the need to hedge, for example, retail products. Inflation derivatives have been traded for over a decade starting in the UK in the early 1990s. Since 2000, the market for inflation derivatives has seen a rapid growth in volumes and in types of products across various markets and linked to various domestic and regional inflation indices, such as, French CPI, Eurozone HICP, US CPI, etc (see Figure 1.1).

Figure 1.1: Historical overview of CPIs and inflation rates.

(a) CPIs (monthly data (31/01/1970-31/12/2010)) (b) Inflation rates (yearly data (31/12/1971-31/12/2010))



Many pension funds, (life) insurance companies[2] and banks trade these inflation-dependent derivatives. Pension funds are, for example, interested in the conditional future indexation of pension rights, which can be viewed as an exotic derivative depending on the CPI.

Modeling derivative products in finance often starts with the specification of a system of stochastic differential equations (SDEs). Such a SDE system consists of economic state variables like stock prices, inflation, nominal and real interest rates and volatility. By imposing a correlation structure (between the Brownian motions) on this system of SDEs one can define so-called hybrid models, and use them for pricing exotic derivatives, see, for example, Grzelak and Oosterlee (2009), Grzelak and Oosterlee (2010) or van Haastrecht and Pelsser (2009).

The well-known Fisher (1930) equation defines a relation between the nominal and real interest rates on the market and the break-even inflation rate[3]. Therefore, the use of stochastic nominal and real interest rates is crucial for an accurate inflation pricing model. Furthermore, as it turns out, according to Kruse (2007), there is a significant skew/smile present in the inflation option market data in the sense that the implied Black-Scholes (BS) volatilities are not constant for different strike levels and maturities (like in the stock or currency option markets). In Figure 1.2 the market implied volatility smile is clearly visible.

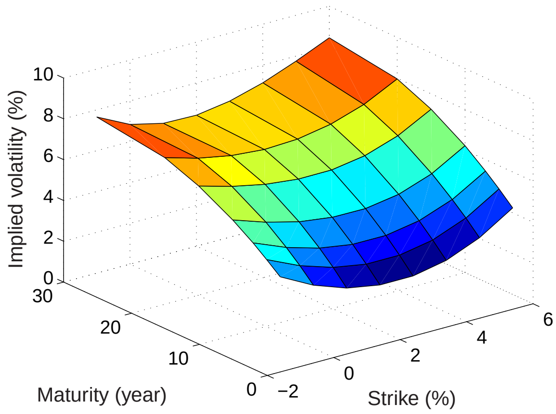
Because of this smile/skew effect in the inflation option market data, the Heston (1993) model is often used in

[1] Inflation is defined as a rise in the general level of prices of goods and services in an economy over a certain period of time (usually one year). The price level is usually measured by a so-called Consumer Price Index (CPI), which reflects the actual price level of a basket of typical consumer goods. The inflation rate is then defined as the percentage change of the CPI.

[2] For (life) insurance companies it is important, due to (among others) regulation and new accounting standards, to value their liabilities, which contain so-called (inflation dependent) 'embedded options', as market consistent as possible. Embedded options are rights in insurance policies or pension contracts that can provide a profit to policy holders but never a loss (see for more information, for example, van Bragt and Steehouwer (2007)).

[3] The break-even inflation rate is the yield spread between nominal and inflation-linked bonds and is a fundamental indicator of inflation expectations.

Figure 1.2: Market implied volatilities of (Euro) inflation indexed options as of September 30, 2010.



practice, as this model is capable of capturing this effect[4]. The variance process of the CPI is then modeled by a so-called Cox-Ingersoll-Ross (CIR) process (see Cox, Ingersoll, and Ross (1985)). Recently, much attention has also been devoted in the literature to stochastic volatility driven by a Schöbel-Zhu process (see for example van Haastrecht and Pelsser (2009)) in combination with stochastic interest rates to model the CPI. In van Haastrecht and Pelsser (2009) also a special case of the Heston model in combination with stochastic interest rates was investigated, where some correlations were assumed to be zero. However, the case of a full correlation structure is of particular interest in this article[5].

In this article we model the CPI by the Heston model, coupled with stochastic nominal and real interest rate processes that are driven by the one-factor Hull-White model[6]. Our focus is on the fast valuation of inflation index cap/floor options and year-on-year (YoY) inflation cap/floor options[7], because for these products the speed of valuation is crucial for calibration. We derive an efficient pricing engine for these options, so that calibration of our inflation model can be done relatively fast. The key to obtaining the pricing formulas is the derivation of the discounted log-CPI characteristic function (ChF) under the T-forward measure. Since the ChF to be derived contains expressions which have to be evaluated numerically, efficient numerical techniques are developed as well.

This paper is organized as follows. In Section 2 we discuss the coupled inflation-interest rate model and derive the model under the T-forward measure. In Section 3 we discuss the valuation of two inflation-dependent options: inflation index caps/floors and YoY inflation caps/floors. In Section 4 we present numerical results, which include calibration results. In Section 5 we illustrate our approach using a real-life pension fund example, where the Heston Hull-White model is used to determine the value of conditional future indexations. We conclude in Section 6.

[4]The Heston model is for example well established for pricing stock and currency derivatives, however, not yet for pricing inflation derivatives.

[5]It turns out that these correlation parameters can be influential when pricing exotic derivatives.

[6]A Hull-White model is a special case of a (multi-factor) Gaussian model (see Brigo and Mercurio (2006, Chap. 3 and 4)).

[7]YoY cap/floor options are defined as a series of forward starting call/put options.

2 Specification of the inflation model

We consider the Heston model in which interest rates are modeled by the one-factor Hull-White interest rate model (see Brigo and Mercurio (2006, p. 71-80)) to model the CPI. We call this inflation model the Heston Hull-White inflation (HHWi) model.

2.1 The Hull-White interest rate model

Term structure models, such as the Hull-White (HW) model, describe the evolution of the interest rate curve through time. Modeling the stochastic behavior of the interest rate term structure is particularly important when pricing interest rate-dependent derivatives. The HW model is an example of a 'no-arbitrage' model, because it is designed to exactly fit today's term structure by producing an interest rate behavior which is consistent with this term structure at all times.

Although the HW model allows for the occurrence of negative rates, it has many attractive features as well. For one, because of the underlying Gaussian distributions it is possible to derive explicit formulas for a number of financial instruments, like interest rate derivatives and bond prices. The different model parameters also provide flexibility and give insight into the dynamic behavior of the term structure.

The nominal and real interest rates, r_n and r_r, under the risk-neutral nominal and real economy measures $\mathbb{Q}_n$ and $\mathbb{Q}_r$, respectively, are modeled by one-factor HW models:

$$dr_l(t) = (\theta_l(t) - a_l r_l(t))dt + \eta_l dW^{r_l}(t), \quad r_l(0) \geq 0, \tag{2.1}$$

where a_l is a mean-reversion parameter and η_l a volatility parameter with $l \in \{n, r\}$. The time-dependent function $\theta_l(t)$ is determined by the nominal/real initial term structure as observed in the market via:

$$\theta_l(t) = \frac{\partial f_l(0,t)}{\partial t} + a_l f_l(0,t) + \frac{\eta_l^2}{2a_l}\left(1 - e^{-2a_l t}\right), \quad l \in \{n, r\}. \tag{2.2}$$

The time-dependent function $f_l(t,T)$ $(0 \leq t \leq T)$ denotes the instantaneous forward curve at time t for maturity T. See Brigo and Mercurio (2006, p. 73) for details.

Nowadays, the quadratic Gaussian and Libor Market Models (among others) are becoming increasingly important to model interest rates (see for example Bloch and Assefa (2009), Bloch (2009), Andersen and Andreasen (2002) and Grzelak and Oosterlee (2010)), because they can model an interest rate smile. However, the application of these models is left for our future work.

2.2 The Heston Hull-White inflation model

We model the evolution of the CPI, denoted by I, and the coupled stochastic variance factor ν by the Heston model under the nominal economy spot measure[8], $\mathbb{Q}_n$ (where the nominal and real interest rates follow a Hull-White model, see Eq. (2.1)). The dynamics are given by:

$$\begin{cases} dI(t) = (r_n(t) - r_r(t))I(t)dt + \sqrt{\nu(t)}I(t)dW^I(t), & I(0) \geq 0, \\ d\nu(t) = \kappa(\bar{\nu} - \nu(t))dt + \sigma_\nu\sqrt{\nu(t)}dW^\nu(t), & \nu(0) \geq 0, \end{cases} \tag{2.3}$$

where κ is a mean-reversion parameter, σ_ν a volatility parameter and $\bar{\nu}$ denotes the long-term variance level. The inflation rate is defined as the percentage change of the CPI, i.e. $\frac{I(t)}{I(\tilde{t})} - 1$ for $0 \leq \tilde{t} < t$.

Remark.

[8]In the nominal economy this measure is generated by the nominal money-savings account, $M_n(t)$, which evolves according to Eq. (2.4).

- *An analogy exists between our inflation model and the modeling of currencies, which is also remarked by Brigo and Mercurio (2006, p. 644-645). It turns out that the inflation model can be used to model currencies by replacing the real interest rate by the foreign interest rate. The CPI then denotes the exchange rate. See for example Grzelak and Oosterlee (2010) which employs a very similar model as our inflation model to model the exchange rate.*

- *We note that the instantaneous inflation, $(r_n(t) - r_r(t))\,dt$, in Eq. (2.3) is equal to the instantaneous break-even inflation, which is an important feature in our model.*

- *Seasonality in inflation rates can become important when modeling quarterly or monthly inflation rates. One way to model seasonality is to assume that we have already modeled the seasonally adjusted CPI, $I(t)$, using our inflation model. We can then add a seasonal component, say $\xi(t)$, to obtain the CPI value with seasonality, $\widetilde{I}(t)$. Different approaches can be used to estimate the $\xi(t)$ function, but this is outside the scope of the present article.*

We now need to determine the process for the real interest rate in the nominal economy. Therefore, we apply a change of measure (i.e. change of numéraire) from the risk-neutral real economy measure, $\mathbb{Q}_r$, to the nominal economy measure, $\mathbb{Q}_n$. Brigo and Mercurio (2006, p. 46) show that this change of measure is equivalent to a change of measure of the numéraire $M_r(t)$ to $M_n(t)/I(t)$, where $M_n(t)$ and $M_r(t)$ are money-savings accounts in the nominal and real economy, respectively, which evolve according to:

$$dM_l(t) = M_l(t)r_l(t)dt, \quad \text{with} \quad \mathrm{l} \in \{\mathrm{n}, \mathrm{r}\}. \tag{2.4}$$

By applying the two-dimensional version of Itô's lemma we derive the following SDE of the numéraire $M_n(t)/I(t)$ under $\mathbb{Q}_n$:

$$d\left(\frac{M_n(t)}{I(t)}\right) = \left(\frac{M_n(t}{I(t)}\right) r_r(t)dt - \left(\frac{M_n(t)}{I(t)}\right)\sqrt{\nu(t)}dW^I(t), \qquad I(0) \geq 0.$$

Using Brigo and Mercurio (2006, Prop. 2.3.1), we then obtain the following real interest rate dynamics under $\mathbb{Q}_n$:

$$dr_r(t) = (\theta_r(t) - \rho_{I,r}\eta_r\sqrt{\nu(t)} - a_r r_r(t))dt + \eta_r dW^{r_r}(t),\ r_r(0) \geq 0.$$

The correlation structure between the Brownian motions $d\mathbf{W}_t = \left(dW_t^I, dW_t^\nu, dW_t^{r_n}, dW_t^{r_r}\right)^{\mathrm{T}}$ is defined by the following symmetric instantaneous correlation matrix:

$$d\mathbf{W}_t\,(d\mathbf{W}_t)^{\mathrm{T}} = \begin{pmatrix} 1 & \rho_{I,\nu} & \rho_{I,n} & \rho_{I,r} \\ . & 1 & \rho_{\nu,n} & \rho_{\nu,r} \\ . & . & 1 & \rho_{n,r} \\ . & . & . & 1 \end{pmatrix} dt. \tag{2.5}$$

2.3 Inflation dynamics under the T-forward measure

To value inflation-dependent derivatives it is convenient to use the inflation model under the T-forward nominal economy measure (instead of the spot measure), which we denote by $\mathbb{Q}_n^T$ (see for example van Haastrecht et al. (2009)). This measure is generated by the nominal zero-coupon bond, $P_n(t,T)$. In other words, under the T-forward measure the forward CPI, I_T, is a martingale, i.e.

$$P_n(t,T)\mathbb{E}^T\left[I_T(T) \mid \mathcal{F}_t\right] = P_n(t,T)I_T(t) = P_r(t,T)I(t), \tag{2.6}$$

where $P_n(t,T)$ and $P_r(t,T)$ are nominal and real zero-coupon bonds, respectively. The inflation model under this T-forward measure is given in Proposition 2.1.

Proposition 2.1. *The inflation model under the T-forward nominal economy measure $\left(\mathbb{Q}_n^T\right)$, with a full matrix of correlations, is given by:*

$$\begin{cases} dI_T(t) = I_T(t)\left(\sqrt{\nu(t)}dW_T^I(t) + \eta_n B_n(t,T)dW_T^n(t) - \eta_r B_r(t,T)dW_T^r(t)\right), \\ d\nu(t) = \left(\kappa(\bar{\nu} - \nu(t)) - \sigma_\nu\eta_n\rho_{\nu,n}B_n(t,T)\sqrt{\nu(t)}\right)dt + \sigma_\nu\sqrt{\nu(t)}dW_T^\nu(t), \end{cases}$$

where I_T denotes the forward CPI under the T-forward measure. The interest rate processes are given by:

$$\begin{cases} dr_n(t) = \left(\theta_n(t) - \eta_n^2 B_n(t,T) - a_n r_n(t)\right) dt + \eta_n dW_T^{r_n}(t), \\ dr_r(t) = \left(\theta_r(t) - \rho_{I,r}\eta_r\sqrt{\nu(t)} - \eta_n\eta_r\rho_{n,r}B_n(t,T) - a_r r_r(t)\right) dt + \eta_r dW_T^{r_r}(t), \end{cases}$$

where the time-dependent function $\theta_l(t)$ is given by Eq. (2.2) and $B_l(t,T) = \frac{1}{a_l}\left(1 - e^{-a_l(T-t)}\right)$, for $l \in \{n, r\}$.

The dynamics of the forward CPI are simplified by changing to logarithmic transformed coordinates, where we define $x_T(t) := \log I_T(t)$[9]:

$$\begin{aligned} dx_T(t) &= -\frac{1}{2}\Big(\nu(t) + \eta_n^2 B_n^2(t,T) + \eta_r^2 B_r^2(t,T) + 2\rho_{\nu,n}\eta_n B_n(t,T)\sqrt{\nu(t)} \\ &\quad - 2\rho_{\nu,r}\eta_r B_r(t,T)\sqrt{\nu(t)} - 2\rho_{n,r}\eta_n\eta_r B_n(t,T)B_r(t,T)\Big) dt \\ &\quad + \sqrt{\nu(t)}dW_T^I(t) + \eta_n B_n(t,T)dW_T^n(t) - \eta_r B_r(t,T)dW_T^r(t). \end{aligned}$$

Proof. The general outline of the proof is as follows. From Eq. (2.6) it follows that

$$I_T(t) = I(t)\frac{P_r(t,T)}{P_n(t,T)}, \tag{2.7}$$

where the dynamics of $I(t)$ are given in Section 2.1. The dynamics of $I_T(t)$ are obtained by applying Itô's lemma to Eq. (2.7) in combination with the dynamics of $I(t)$ and the dynamics of the real and nominal zero-coupon bonds, $P_r(t,T)$ and $P_n(t,T)$, under the nominal economy measure ($\mathbb{Q}_n$). Expressing the full model in terms of independent Brownian motions simplifies the derivation of the Radon-Nikodým derivative (see Brigo and Mercurio (2006, p. 45 and 911)). By computing the Itô derivative of this Radon-Nikodým derivative the Girsanov kernel for the transition from $\mathbb{Q}_n$ to $\mathbb{Q}_n^T$ is derived and finishes the proof. For the full proof we refer to Grzelak and Oosterlee (2010, Appendix A). □

Remark. *From Proposition 2.1 we note that under the T-forward nominal economy measure $\left(\mathbb{Q}_n^T\right)$ the forward CPI does not depend[10] directly on the real and nominal interest rate processes, $r_r(t)$ and $r_n(t)$, but only depends on the Brownian motions $dW^{r_n}(t)$ and $dW^{r_r}(t)$.*

[9] Note that this transformation is well defined since $I(0) > 0$ and, thus, $I_T(0) > 0$.

[10] Note that actually the forward CPI depends on all the Brownian motions since the correlations can be non-zero. The key is the independence of the state variables.

3 Pricing formulas

In this section we discuss the pricing of two inflation dependent options. The pricing of inflation index options is discussed in Section 3.1 and the pricing of YoY inflation options is discussed in Section 3.2. In Section 3.3 we show numerical results of the derived pricing formulas of forward starting options.

3.1 Inflation indexed options

In this section we briefly discuss the pricing of inflation indexed cap and floor options. The inflation model, which we use for option pricing, is given in Section 2.1 under the measure $\mathbb{Q}_n$ and by Proposition 2.1 under the measure $\mathbb{Q}_n^T$.

The price of an inflation indexed cap/floor option maturing at time T with strike level[11] $K := (1+\tilde{k})^{\mathrm{T}}$ (the expression $(1+\tilde{k})^{\mathrm{T}}$ means $1+\tilde{k}$ to the power T) written on the inflation index (the CPI) (with $\omega = 1$ for a cap option and $\omega = -1$ for a floor option) is given by

$$\overline{\Pi}(t, T, \tilde{k}, \omega) := M_n(t)\mathbb{E}^{\mathbb{Q}_n}\left[\frac{\max\left(\omega\left(I(T) - K\right), 0\right)}{M_n(T)} \mid \mathcal{F}_t\right], \tag{3.1}$$

where $M_n(t)$ indicates the nominal money-savings account, which evolves according to Eq. (2.4). Since the stochastic expressions $M_n(T)$ and $\max\left(\omega\left(I(T) - K\right), 0\right)$ are not independent, the computation of the expectation under the $\mathbb{Q}_n$ measure is rather involved.

It turns out that the complexity of the problem is greatly reduced under the T-forward measure. We then get the following pay-off structure:

$$\overline{\Pi}(t, T, \tilde{k}, \omega) = P_n(t, T)\,\mathbb{E}^{\mathbb{Q}_n^T}\left[\max\left(\omega\left(I_T(T) - K\right), 0\right) \mid \mathcal{F}_t\right]. \tag{3.2}$$

From the two pay-off structures in Eqs. (3.1) and (3.2) we note that the pay-off structure under the T-forward measure has a simpler form since the price of the pure discount bond at time $t = 0$ is directly observable in the market.

$\overline{\Pi}(t, T, \tilde{k}, \omega)$ in Eq. (3.2) can also be formulated in integral form:

$$\overline{\Pi}(t, T, \tilde{k}, \omega) = P_n(t, T)\int_{\mathbb{R}} \max\left(\omega K\left(e^y - 1\right), 0\right)\widetilde{f}(y|x)dy, \tag{3.3}$$

where $\widetilde{f}(y|x)$ denotes the probability density function of $y := \log\left(\frac{I_T(T)}{K}\right)$ given
$x := \log\left(\frac{I_T(t)}{K}\right)$.

Fourier-based methods[12] can be used to compute these integrals in the case that the density function is not known in advance. These methods rely on the existence of the ChF. The derivation of the ChF for this particular option is discussed in Grzelak and Oosterlee (2010). We denote the corresponding approximation of the full-scale HHWi model by HHWi-i. For this model we can employ Fourier-based methods for efficient pricing of inflation index options.

3.2 Year-on-year inflation options

In this section we discuss the pricing of YoY inflation cap/floor options by describing the general pricing methodology[13]. In general, a cap/floor option, $\widetilde{\Pi}$, is defined by a series of so-called caplet/floorlet options, $\widehat{\Pi}$, i.e:

$$\widetilde{\Pi}(\omega, t, \tau, T, \overline{k}) = \sum_{k=1}^{n} \widehat{\Pi}(\omega, t, T_{k-1}, T_k, \overline{k}),$$

[11] The strike level $\tilde{k}$ is (market data) input. More information can be found in Oman (2005, p. 5).
[12] See for example Carr and Madan (1999) and Fang and Oosterlee (2008).
[13] More information can be found in Oman (2005, p. 5).

where $\omega = 1$ for a cap/caplet option and $\omega = -1$ for a floor/floorlet option. Furthermore, $\tau := T_k - T_{k-1}$ defines the tenor parameter with $T_0 = 0$ and $T_n = T$. The integer n denotes the number of caplets/floorlets in the cap/floor option. This integer is dependent on the tenor parameter, which is in practice often a fixed interval. The strike level is given by $\bar{k}$. So, the pricing of a YoY inflation cap/floor option reduces to the pricing of a series of YoY inflation caplet/floorlet options.

The price of a YoY inflation caplet/floorlet option starting at time T_{k-1} ($0 \leq t \leq T_{k-1}$) and maturing at time T_k ($T_{k-1} \leq T_k$), written on the inflation index, is given by

$$\widehat{\Pi}(\omega, t, T_{k-1}, T_k, \bar{k}) = M_n(t)\mathbb{E}^{\mathbb{Q}_n}\left[\frac{\max\left(\omega\left(\frac{I(T_k)}{I(T_{k-1})} - (\bar{k}+1)\right), 0\right)}{M_n(T_k)} \mid \mathcal{F}_t\right],$$

where $M_n(t)$ indicates the nominal money-savings account, which evolves according to Eq. (2.4).

By changing the measure from $\mathbb{Q}_n$ to the T_k-forward measure, $\mathbb{Q}_n^{T_k}$, with $k = 1, \ldots, n$, and by using $K^* := 1 + \bar{k}$, we arrive at the following pricing problem:

$$\widehat{\Pi}(\omega, t, T_{k-1}, T_k, \bar{k}) = P_n(t, T_k)\mathbb{E}^{T_k}\left[\max\left(\omega\left(\frac{I(T_k)}{I(T_{k-1})} - K^*\right), 0\right)\middle|\mathcal{F}_t\right].$$

Since the T_k-forward CPI, $I_{T_k}(t) = \frac{P_r(t,T_k)}{P_n(t,T_k)}I(t)$, under the T_k-forward measure is a martingale with numéraire $P_n(t, T_k)$ and $I_{T_k}(T_k) = I(T_k)$, we can simply write:

$$\widehat{\Pi}(\omega, t, T_{k-1}, T_k, \bar{k}) = P_n(t, T_k)\mathbb{E}^{T_k}\left[\max\left(\omega\left(\frac{P_r(T_{k-1}, T_k)}{P_n(T_{k-1}, T_k)}\frac{I_{T_k}(T_k)}{I_{T_k}(T_{k-1})} - K^*\right), 0\right)\middle|\mathcal{F}_t\right].$$

The dynamics for $I_{T_k}(t)$ under the T_k-forward measure are given by Proposition 2.1.

Remark. *For numerical experiments we make use of the put-call parity to price options of call type, so in this case caplet options. In other words, when for example a floorlet option, $\widehat{\Pi}(-1, t, T_1, T_2, \bar{k})$, with strike $\bar{k}$ and times $0 \leq t \leq T_1 < T_2$, is computed, the price of the corresponding caplet option $\widehat{\Pi}(1, t, T_1, T_2, \bar{k})$ is computed by:*

$$\widehat{\Pi}(1, t, T_1, T_2, \bar{k}) = \widehat{\Pi}(-1, t, T_1, T_2, \bar{k}) + P_n(t, T_1)P_r(T_1, T_2) - P_n(t, T_2)\left(1 + \bar{k}\right),$$

where P_n and P_r are nominal and real zero-coupon bonds, respectively.

As already mentioned, to apply Fourier-based pricing methods we have to derive the (forward) ChF belonging to this option, which is the topic of the next subsection.

Derivation of the (forward) characteristic function

By setting

$$X(T_{k-1}, T_k) = \frac{P_r(T_{k-1}, T_k)}{P_n(T_{k-1}, T_k)}\frac{I_T(T_k)}{I_T(T_{k-1})}, \quad \text{for} \quad \text{k} = 1, \ldots, \text{n},$$

we perform the log-transformation:

$$\begin{aligned} x(T_{k-1}, T_k) &:= \log X(T_{k-1}, T_k) = \log\left(\frac{P_r(T_{k-1}, T_k)}{P_n(T_{k-1}, T_k)}\frac{I_{T_k}(T_k)}{I_{T_k}(T_{k-1})}\right), \\ &= \log I_{T_k}(T_k) - \log I_{T_k}(T_{k-1}) + \log P_r(T_{k-1}, T_k) - \log P_n(T_{k-1}, T_k). \end{aligned}$$

We derive the forward ChF for the process $x(T_{k-1}, T_k)$:

$$\phi_{YoY}(u, t, x(T_{k-1,}, T_k)) := \mathbb{E}^{T_k}\left[\mathrm{e}^{iux(T_{k-1}, T_k)}\middle|\mathcal{F}_t\right]. \tag{3.4}$$

By substitution we have:

$$\phi_{YoY}(u, t, x(T_{k-1}, T_k)) = \mathbb{E}^{T_k}\left[\mathrm{e}^{iu\left(\log I_{T_k}(T_k) - \log I_{T_k}(T_{k-1}) + \log P_r(T_{k-1}, T_k) - \log P_n(T_{k-1}, T_k)\right)}\middle|\mathcal{F}_t\right].$$

Now, by iterated expectations we find:

$$\phi_{YoY}(u,t,x(T_{k-1},T_k)) = \\ \mathbb{E}^{T_k}\left[\mathbb{E}^{T_k}\left[e^{iu\left(\log I_{T_k}(T_k)-\log I_{T_k}(T_{k-1})+\log P_r(T_{k-1},T_k)-\log P_n(T_{k-1},T_k)\right)}\Big|\mathcal{F}_{k-1}\right]\Big|\mathcal{F}_t\right].$$

Since $I_T(T_{k-1})$, $P_n(T_{k-1},T_k)$ and $P_r(T_{k-1},T_k)$ are $I_{T_{k-1}}$ measurable[14], we can write:

$$\begin{aligned}\phi_{YoY}(u,t,x(T_{k-1},T_k)) &= \mathbb{E}^{T_k}\left[e^{-iu\left(\log I_{T_k}(T_{k-1})-\log P_r(T_{k-1},T_k)+\log P_n(T_{k-1},T_k)\right)}\times\right.\\ &\quad \left.\mathbb{E}^{T_k}\left[e^{iu\log I_{T_k}(T_k)}\Big|\mathcal{F}_{k-1}\right]\Big|\mathcal{F}_t\right].\end{aligned}$$

The last expectation equals the characteristic function for $\log I_{T_k}(T_k)$, i.e.

$$\phi_i(u,\log I_T(T_k),T_{k-1},T_k) := \mathbb{E}^{T_k}\left[e^{iu\log I_{T_k}(T_k)}\Big|\mathcal{F}_{k-1}\right].$$

In Grzelak and Oosterlee (2010) an affine approximation is found for this ChF, i.e.:

$$\phi_{i,1} := e^{A(u,T_k-T_{k-1})+iu\log I_{T_k}(T_{k-1})+C(u,T_k-T_{k-1})v(T_{k-1})}, \tag{3.5}$$

with functions $A(u,\tau)$ and $C(u,\tau)$ given by Eqs. (3.6) and (3.7). By subscripts (like the ,1 in Eq. (3.5)) we indicate subsequent approximations.

Remark. *In Grzelak and Oosterlee (2010, Chap. 2.3) it is noted that the Kolmogorov backward partial differential equation, for which ϕ in Eq. (3.4) is the solution, contains non-affine $\sqrt{v}$-terms, so that finding the solution is nontrivial. Approximation of these $\sqrt{v}$-terms by a linearization technique leads to an approximating closed-form solution of the ChF.*

The functions $A(u,\tau)$ and $C(u,\tau)$ in Eq. (3.5) are given by:

$$\begin{aligned}A(u,\tau) &:= \int_0^\tau \left(\kappa\bar{v}-\rho_{v,n}\sigma_v\eta_n\varphi(s)B_n(s)(1-iu)-\rho_{v,r}\sigma_v\eta_r\varphi(s)B_r(s)\right)C(s)ds\\ &\quad +(u^2+iu)\int_0^\tau \Psi(s,\varphi(s))ds, \end{aligned}\tag{3.6}$$

$$C(u,\tau) := \frac{1-e^{-d\tau}}{\sigma_v^2(1-ge^{-d\tau})}\left(\kappa-\rho_{I,v}\sigma_v iu-d\right), \tag{3.7}$$

where $\varphi(t) := \mathbb{E}\left[\sqrt{v(t)}\right]$[15], $d := \sqrt{(\kappa-\rho_{I,v}\sigma_v iu)^2-\sigma_v^2 iu(iu-1)}$ and $g := \frac{\kappa-\rho_{I,v}\sigma_v iu-d}{\kappa-\rho_{I,v}\sigma_v iu+d}$. Furthermore,

$$\begin{aligned}\Psi(t,\varphi(t)) &:= (\rho_{I,r}\eta_r B_r(t,T)-\rho_{I,n}\eta_n B_n(t,T))\varphi(t)+\rho_{n,r}\eta_n\eta_r B_n(t,T)B_r(t,T)\\ &\quad -\frac{1}{2}\left(\eta_n^2B_n^2(t,T)+\eta_r^2B_r^2(t,T)\right).\end{aligned}$$

The ChF, ϕ, is then approximated by:

$$\phi_{YoY,1} = \mathbb{E}^{T_k}\left[e^{-iu\log I_{T_k}(T_{k-1})+iu\log P_r(T_{k-1},T_k)-iu\log P_n(T_{k-1},T_k)}\phi_{i,1}\Big|\mathcal{F}_t\right].$$

Due to Eq. (3.5) we have:

$$\begin{aligned}\phi_{YoY,1} = &\quad \mathbb{E}^{T_k}\left[e^{A(u,T_k-T_{k-1})+C(u,T_k-T_{k-1})v(T_{k-1})}e^{-iu\log P_n(T_{k-1},T_k)}\times\right.\\ &\quad \left.e^{iu\log P_r(T_{k-1},T_k)}\Big|\mathcal{F}_t\right].\end{aligned}\tag{3.8}$$

As the underlying nominal interest-rate model is the Hull-White model, the zero-coupon bond (ZCB) $P_n(T_{k-1},T_k)$ is given by, see (Brigo and Mercurio 2006, p. 75-78):

$$P_n(T_{k-1},T_k) = e^{A_n(T_{k-1},T_k)-B_n(T_{k-1},T_k)r_n(T_{k-1})}, \tag{3.9}$$

[14] See Brigo and Mercurio (2006, Appendix C).

[15] In Grzelak and Oosterlee (2009) approximations are proposed for $\mathbb{E}\left[\sqrt{v(t)}\right]$, which are also used in this article for the numerical experiments.

with analytically known functions $A_n(T_{k-1},T_k)$ and $B_n(T_{k-1},T_k)$. However, since we work under the nominal economy measure $\mathbb{Q}_n$, the dynamics of the real interest rate are not affine and, as a consequence, the dynamics of P_r are not affine. Hence, the derivation of the dynamics of P_r is nontrivial.

By approximating the variance process under $\mathbb{Q}_n$ (see Section 2.1) by its expectation, the process of the real interest rate, conditional on $\mathcal{F}_s$, is affine and normally distributed. Following the approach as outlined in (Brigo and Mercurio 2006, Chap. 3.3) we derive:

$$\begin{aligned} A_r(T_{k-1},T_k) = \quad & \log\frac{P_r(0,T_k)}{P_r(0,T_{k-1})}(B_r(T_{k-1},T_k)f_r(0,T_{k-1}) + \Lambda(T_{k-1},T_k) \\ & -\frac{\eta_r^2}{4a_r}(1-e^{-2a_rT_{k-1}})B_r(T_{k-1},T_k)^2), \\ B_r(T_{k-1},T_k) = \quad & \frac{1}{a_r}\left(1-e^{-a_r(T_k-T_{k-1})}\right), \end{aligned}$$

where

$$\begin{aligned} \Lambda(T_{k-1},T_k) = \quad & \mathbb{E}\left[\sqrt{v(T_k)}\right]\frac{\rho_{I,r}\eta_r}{a_r}(T_k-T_{k-1}-B_r(T_{k-1},T_k)-B_n(T_{k-1},T_k) \\ & +\frac{1}{a_n+a_r}(1-e^{-(a_n+a_r)(T_k-T_{k-1})})). \end{aligned}$$

By substituting the nominal and real ZCB expressions into the expression in Eq. (3.8) the approximating ChF in Eq. (3.8) is now given by:

$$\begin{aligned} \phi_{YoY,1} = \mathrm{e}^{iu(A_r(T_{k-1},T_k)-A_n(T_{k-1},T_k))}\mathrm{e}^{A(u,T_k-T_{k-1})}\times \\ \mathbb{E}^{T_k}\left[\mathrm{e}^{C(u,T_k-T_{k-1})v(T_{k-1})}\mathrm{e}^{iu(B_n(T_{k-1},T_k)r_n(T_{k-1})-B_r(T_{k-1},T_k)r_r(T_{k-1}))}\middle|\mathcal{F}_t\right]. \end{aligned} \tag{3.10}$$

The Laplace transform in Eq. (3.10) is of a very complicated form. In order to find a closed-form solution for Eq. (3.10), additional assumptions of independence between processes are required.

A basic approximation to Eq. (3.10) is given by:

$$\begin{aligned} \phi_{YoY,2} = \quad & \mathrm{e}^{iu(A_r(T_{k-1},T_k)-A_n(T_{k-1},T_k))+A(u,T_k-T_{k-1})}\cdot\mathbb{E}^{T_k}\left[\mathrm{e}^{C(u,T_k-T_{k-1})v(T_{k-1})}\middle|\mathcal{F}_t\right]\times \\ & \mathbb{E}^{T_k}\left[\mathrm{e}^{iu(B_n(T_{k-1},T_k)r_n(T_{k-1})-B_r(T_{k-1},T_k)r_r(T_{k-1}))}\middle|\mathcal{F}_t\right]. \end{aligned} \tag{3.11}$$

The approximation above consists of two expectations under the T_k-forward measure. Since the nominal and real interest rates, $r_n(T_{k-1})$ and $r_r(T_{k-1})$, are normally distributed, the sum of these two normally distributed random variables is also normally distributed and the ChF of this sum can be found analytically. Furthermore, since $v(T_{k-1})$ is noncentral chi-square distributed the corresponding ChF can also be found analytically. Result 3.1 and Lemma 3.1 provide these solutions.

Result 3.1. *For given times $0\le s\le t\le T$, nominal and real interest rate processes r_n and r_r, as defined in Proposition 2.1, and $Y(t,T):=B_n(t,T)r_n(t)-B_r(t,T)r_r(t)$, the following holds:*

$$\mathbb{E}^T\left[\mathrm{e}^{iuY(t,T)}|\mathcal{F}_s\right]\approx\exp\left(iu\mathbb{E}^T\left[Y(t,T)|\mathcal{F}_s\right]-\frac{1}{2}u^2\mathbb{V}\mathrm{ar}^T\left(Y(t,T)|\mathcal{F}_s\right)\right),$$

where r_n evolves under $\mathbb{Q}_n^T$ according to Proposition 2.1. To ensure that the real interest rate process is normally distributed under $\mathbb{Q}_n^T$, we assume that it evolves according to

$$\mathrm{d}r_r(t)=\left(\theta_r(t)-\rho_{I,r}\eta_r\mathbb{E}\left[\sqrt{v(t)}\right]-\eta_n\eta_r\rho_{n,r}B_n(t,T)-a_rr_r(t)\right)\mathrm{d}t+\eta_r\mathrm{d}W_T^{r_r}(t).$$

The random variable $Y(t,T)$ is then normally distributed with expectation and variance given by:

$$\begin{aligned} \mathbb{E}^T\left[Y(t,T)|\mathcal{F}_s\right] &= B_n(t,T)\mathbb{E}^T\left[r_n(t)|\mathcal{F}_s\right]-B_r(t,T)\mathbb{E}^T\left[r_r(t)|\mathcal{F}_s\right], \\ \mathbb{V}\mathrm{ar}^T\left[Y(t,T)|\mathcal{F}_s\right] &= B_n^2(t,T)\mathbb{V}\mathrm{ar}^T\left[r_n(t)|\mathcal{F}_s\right]+B_r^2(t,T)\mathbb{V}\mathrm{ar}^T\left[r_r(t)|\mathcal{F}_s\right] \\ & \quad -2B_n(t,T)B_r(t,T)Cov^T\left[r_n(t),r_r(t)|\mathcal{F}_s\right], \end{aligned}$$

with

$$Cov^T\left[r_n(t),r_r(t)|\mathcal{F}_s\right]=\rho_{n,r}\sqrt{\mathbb{V}\mathrm{ar}^T\left[r_n(t)|\mathcal{F}_t\right]\mathbb{V}\mathrm{ar}^T\left[r_r(t)|\mathcal{F}_s\right]}.$$

Proof. By approximating the variance process under $\mathbb{Q}_n$ (see Section 2.1) by its expectation the process of the real interest rate, conditional on $\mathcal{F}_s$, is normally distributed.

Next, since the random variable $Y(t,T)$ is defined as a (weighted) sum of normally distributed random variables[16], $Y(t,T)$ is also normally distributed. The characteristic function for any normally distributed random variable X, $X \sim N(\mu,\sigma)$ with expectation μ and variance σ^2 is given by

$$\phi_X(u) = \mathbb{E}[\exp(iuX] = \exp\left(iu\mu - \frac{1}{2}\sigma^2 u^2\right).$$

The proof is finished by the appropriate substitutions. □

Lemma 3.1. *For $0 \le s \le t \le T$ the Laplace transform of $\mathbb{E}^T\left[e^{C(u,T-t)v(t)}\middle|\mathcal{F}_s\right]$ is given by:*

$$\begin{aligned}\mathbb{E}^T\left[e^{C(u,T-t)v(t)}|\mathcal{F}_s\right] &= \psi(u,s,t,T)^{\frac{2\kappa\bar{v}}{\gamma^2}}\cdot \\ &\quad \exp\left(\psi(u,s,t,T)e^{-\kappa(t-s)}C(u,T-t)v(t)\right),\end{aligned} \tag{3.12}$$

provided that

$$\psi(u,s,t,T) := \frac{1}{1-\frac{2\gamma^2}{4\kappa}\left(1-e^{-\kappa(t-s)}\right)C(u,T-t)} \ge 0.$$

The function $C(u,T-t)$ is given in Eq. (3.7).

Proof. Since the variance process $v(t)$, conditional on $\mathcal{F}_s$, is distributed as a constant $c := \frac{\sigma_v^2\left(1-e^{-\kappa(t-s)}\right)}{4\kappa}$ times a noncentral chi-square distribution with $d := \frac{4\kappa\bar{v}}{\sigma_v^2}$ degrees of freedom and non-centrality parameter $\lambda := \frac{4\kappa e^{-\kappa(t-s)}}{\sigma_v^2\left(1-e^{-\kappa(t-s)}\right)}$, the proof is straightforward, see Cox, Ingersoll, and Ross (1985). □

We denote the approximation in Eq. (3.11) of the full-scale HHWi model by HHWi-YoY. For this model we can employ Fourier-based methods for efficient pricing of YoY inflation options.

3.3 Numerical experiment

To analyze the performance of the approximations introduced for the YoY inflation options we compute the initial ($t = 0$) implied Black-Scholes volatilities for different strike levels using the full-scale HHWi model and the HHWi-YoY model. This is done by inverting the characteristic function using Fourier-based methods. We consider two test cases:

- Case I: the forward starting option starts at $T_1 = 4$ and matures at $T_2 = 5$.
- Case II: the forward starting option starts at $T_1 = 29$ and matures at $T_2 = 30$.

For the generation of risk-neutral (RN) scenarios we make use of an advanced simulation scheme including exact simulation (also called unbiased simulation) for the interest rate and variance processes (see, for example, Andersen (2007) and Broadie and Kaya (2006)). To reduce the variance of the MC estimator we use 100.000 scenarios in combination with two variance reduction techniques (i) antithetic sampling and (ii) Empirical Martingale Simulation (EMS) (see respectively Glasserman (2004) and Duan and Simonato (1995)).

As the base parameter setting we use the parameters as specified in Eq. (3.13):

$$\kappa = 0.3, \quad v(0) = 0.04, \quad \bar{v} = 0.04, \quad \sigma_v = 0.6, \quad \rho_{I,v} = -0.7, \tag{3.13}$$

with interest rate volatilities $\eta_n = 0.0089$, $\eta_r = 0.0084$ and correlations $\rho_{I,n} = \rho_{I,r} = \rho_{v,n} = \rho_{v,r} = 0$ and $\rho_{n,r} = 0$ (unless stated otherwise). To test the pricing accuracy we use an extreme test case, i.e. the Feller condition, $2\kappa\bar{v} > \sigma_v^2$, is not satisfied, so that inflation volatilities can attain zero. These parameters are not calibrated to

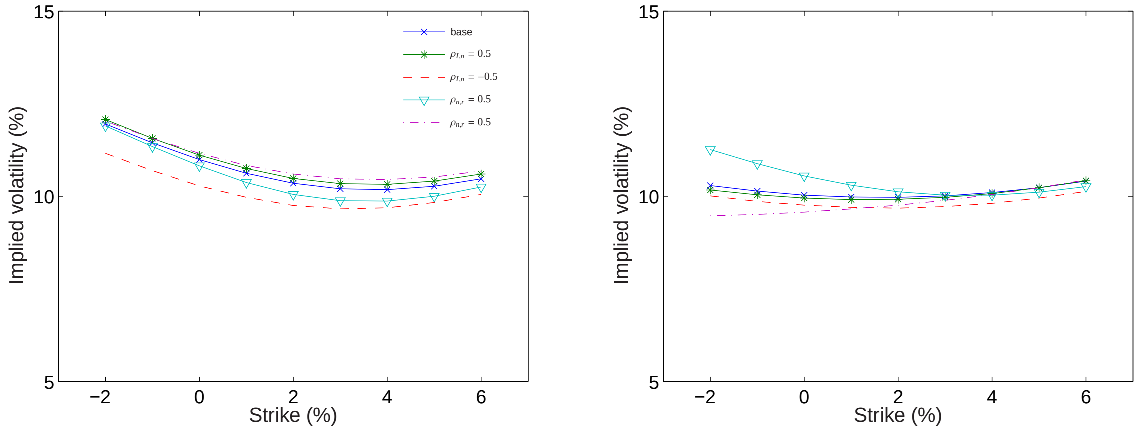
[16] $r_n(t)$ and $r_r(t)$, conditional on $\mathcal{F}_t$, are normally distributed Brigo and Mercurio (2006, Chap. 3.3.1).

market data; this topic will be discussed in Section 4.

First we investigate the sensitivity of the pricing of YoY inflation options to the correlation parameters by performing a MC simulation. We therefore vary the correlations $\rho_{I,n}$ and $\rho_{n,r}$. The results for cases I and II are presented in Figures 3.1a and 3.1b, respectively.

Figure 3.1: Sensitivity to correlations using a Monte Carlo simulation to the full-scale HHWi model.

(a) Case I (b) Case II

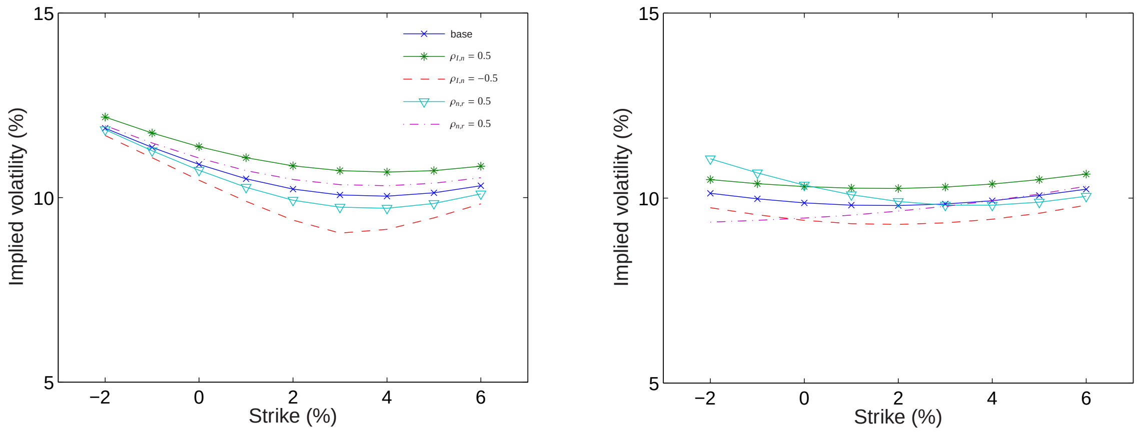


Observing Figures 3.1a and 3.1b we conclude that for this parameter setting the correlation parameters $\rho_{I,n}$ and $\rho_{n,r}$ are influential regarding the change in implied volatility.

Next, we perform the same experiment using the HHW-YoY model. The results for cases I and II are presented in Figures 3.2a and 3.2b, respectively.

Figure 3.2: Sensitivity to correlations using the HHWi-YoY model.

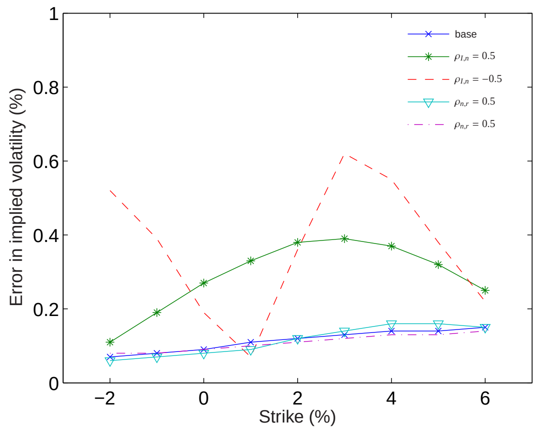
(a) Case I (b) Case II



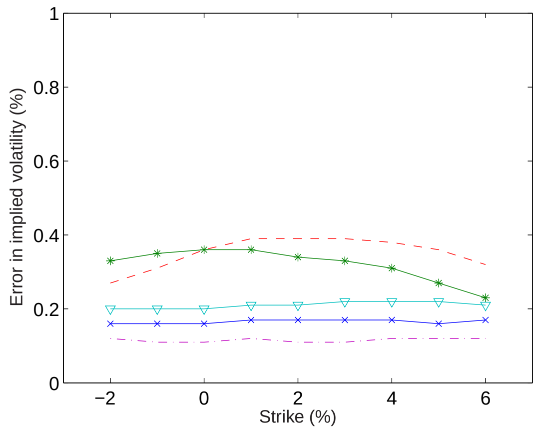
Figures 3.3a and 3.3b show the difference in implied volatility between the full-scale HHWi and HHWi-YoY model.

Figure 3.3: Difference between the HHWi and HHWi-YoY model.

(a) Case I



(b) Case II



From Figures 3.3a and 3.3b we can conclude that the maximum error for cases I and II is equal to 0.6% point and 0.4% point in terms of implied volatilities, respectively. In both cases we considered $\tau = 1$, which is common when YoY forward starting options are considered.

4 Calibration results

Calibration is an optimization procedure to estimate the model parameters in such a way that (relevant) market prices are replicated by the model as good as possible. Hence, a calibration procedure consists of the computation of $\min_{\Omega}\left\{||C-\widehat{C}||\right\}$, where C denotes the market price, $\widehat{C}$ the model price, Ω the set of parameters (including constraints) and $||\cdot||$ some norm. In our case market data are available for pairs $\left(\overline{T},\overline{K}\right)$, with $\overline{T}$ denoting the option maturity and $\overline{K}$ the strike level. For the norm we take the Euclidean one, so that calibration in our case consists of computing:

$$\min_{\Omega}\left\{||C-\widehat{C}||_p\right\}=\min_{\Omega}\left\{\left(\sum_{j=1}^{m}\sum_{k=1}^{n}\left|C(\overline{T}_j,\overline{K}_k)-\widehat{C}(\overline{T}_j,\overline{K}_k)\right|^p\right)^{\frac{1}{p}}\right\}, \tag{4.1}$$

where we use $p=2$. We note that also the p-norm of the difference of market and model implied volatilities could be minimized. However, since then in every iteration step of the optimization procedure an extra numerical inversion has to be performed, which may lead to numerical difficulties, this is not the method of choice. Market prices of plain vanilla options are often used for calibration, because this data is available.

Remark. *In the calibration procedure it is possible to incorporate both types of inflation options in the calibration procedure. This is easily done when we specify the market option price C (and, thus, also the corresponding model value $\widehat{C}$) as an inflation indexed cap/floor or a YoY inflation caplet/floorlet with corresponding strike level and maturity. It is also possible to assign different weights to different calibration points.*

The minimization problem in Eq. (4.1) is solved iteratively using a numerical minimization algorithm. We first sample random starting points and then we refine this solution using the well-known Levenberg-Marquardt least-squares algorithm, which is a local minimization method. This procedure is repeated and the best solution is kept.

In this section we show calibration results for the full-scale HHWi model (see Section (2.1)). The calibration procedure used is the following:

1. We calibrate the one-factor Hull-White interest rate model to interest rate options, like swaptions and/or interest rate cap/floor options, see Brigo and Mercurio (2006, Chap. 2 and 3), to determine the interest rate model parameters a_n, a_r, η_n and η_r (see Section 2.1).
2. Conditional on the parameters of the interest rate model, we calibrate the inflation model to inflation indexed cap/floor options and/or YoY inflation caplet/floorlet options with Fourier-based methods.

Remark. *For the correlation parameters we perform the following calibration:*

1. *The correlation parameters between 'observable' variables, i.e. $\rho_{I,n}=0.36$, $\rho_{I,r}=-0.29$ and $\rho_{n,r}=0.78$ are determined using historical information*[17] *in the sample period $1985-2009$.*
2. *The correlation parameter $\rho_{I,v}$ is determined in the calibration process. Appropriate bounds for this parameter are used in the calibration process so that the correlation matrix remains positive definite.*
3. *The correlation parameters, $\rho_{r,v}$ and $\rho_{n,v}$ are derived from a conditional sampling method.*

Because of the procedure mentioned above, we start the calibration with the following correlation matrix, which is defined in Eq. (2.5):

$$\begin{pmatrix} 1 & \rho_{I,v} & \rho_{I,n} & \rho_{I,r} \\ . & 1 & \rho_{v,n} & \rho_{v,r} \\ . & . & 1 & \rho_{n,r} \\ . & . & . & 1 \end{pmatrix} = \begin{pmatrix} 1 & \rho_{I,v} & 0.36 & -0.29 \\ . & 1 & \rho_{v,n} & \rho_{v,r} \\ . & . & 1 & 0.78 \\ . & . & . & 1 \end{pmatrix}, \tag{4.2}$$

where the correlation parameters $\rho_{I,v}$, $\rho_{v,n}$ and $\rho_{v,r}$ are to be determined.

[17] This is industrial practice.

The inflation option market data, as of September 30, 2010, which are used in this section for calibration consists of two inflation option products, namely inflation index caps/floors and YoY inflation caps/floors. For both options market data is available for a whole range of strikes and maturities and prices are quoted in terms of base points (bp.). To compare calibration results option prices are expressed here in terms of implied Black-Scholes volatilities.

Since YoY inflation caps/floors are essentially a series of YoY caplets/floorlets we perform a so-called stripping method, which is explained in Brigo and Mercurio (2006, p. 682), to obtain the market data for YoY inflation caplets/floorlets. Obviously, performing a calibration to YoY caplets/floorlets instead of to YoY caps/floors reduces the computation time significantly.

4.1 Calibrating the interest rate model

For the calibration of the Euro nominal interest rate model we use the zero-coupon interest rate curve of September 30, 2010. The zero-coupon real interest rate curve as of September 30, 2010 is constructed using available information about zero-coupon break-even inflation as derived from index-linked swaps[18] (as of September 30, 2010).

We then obtain an estimate of the initial real zero-coupon curve by applying the Fisher equation

$$r_r(t) = \frac{1 + r_n(t)}{1 + bei(t)} - 1, \tag{4.3}$$

where bei denotes the break-even inflation, r_r the real interest rate and r_n the nominal interest rate. The resulting interest rate curves are shown in Figure 4.1a.

We calibrate the one-factor Hull-White model using market prices as of September 30, 2010 of forward-at-the-money options on Euro swap contracts (Euro swaptions). We calibrate the two parameters of the model, the mean-reversion and the volatility parameter, using a large set of swaptions, with option maturities ranging from 1 to 15 years and swap maturities ranging from 1 to 10 years. Swaptions with long maturities, > 15 years, and swap lengths, > 10 years, have deliberately been omitted from the calibration set. Liquidity for such contracts is often limited, which may result in not very representative market quotes. The optimal mean-reversion parameter is 0.0300; the optimal volatility parameter is 0.0089. A comparison between the model and market prices is shown in Figure 4.1b, where prices are expressed in terms of implied Black volatilities.

[18] The maturities of these swaps range from 1 to 50 years. We set the short break-even inflation equal to the 1-year break-even inflation. Missing maturities are approximated by linear interpolation.

Figure 4.1: Calibration results of interest rates.

(a) Overview of interest rate curves (b) Quality of fit of the calibrated Hull-White model

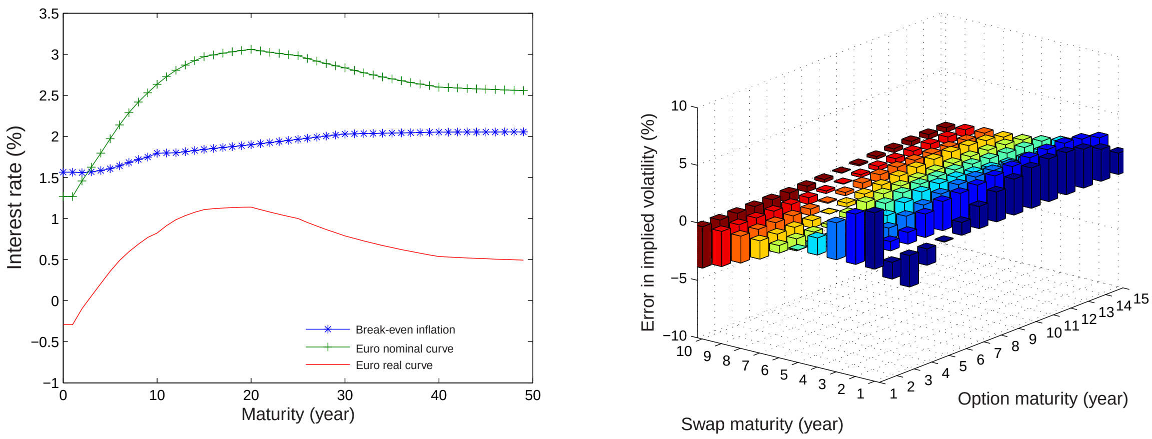


Figure 4.1b shows that the difference between model and market prices is limited. The average absolute error is 1.5% point. The fit is less good for short maturing options. This is due to the used objective function $\|C-\widehat{C}\|_2$ in our optimization procedure. Since the values of long maturing options are higher than the values of short maturing options the long maturing options automatically have a 'higher weight' in the optimization procedure. This can be overcome by introducing weights in the calibration procedure, however this refinement is outside the scope of this article.

Option markets for real interest rates are still very limited. Therefore we set the mean-reversion parameter of the real interest rate model equal to the mean-reversion parameter of the nominal interest rate model.

Remark. *The choice of equal mean reversion parameters is justified when we estimate a Vasicek model (see Brigo and Mercurio (2006, Chap. 3.2.1)) using a maximum likelihood estimation to historical nominal and real interest rates in the sample period* $1985-2009$. *It turns out that the resulting mean reversion parameters are of the same order. The results are, however, left outside the article.*

The volatility parameter of the real interest rate model parameter is determined by a scaling factor based on the volatility of historical nominal and real interest rates. The correlation parameter $\rho_{n,r}$ is also based on historical data (see Eq. (4.2)). The resulting parameters of the interest rate model are found to be:

$$a_n = 0.0300,\ a_r = 0.0300,\ \eta_n = 0.0089,\ \eta_r = 0.0084 \quad \text{and} \quad \rho_{\text{n,r}} = 0.78.$$

4.2 Calibration to inflation market data

The calibration of the inflation model can be performed using inflation market data. The specific inflation options were already explained in Section 3. To derive a reliable set of parameters, we use relevant liquid market data so that market conditions are captured well. We perform a calibration to YoY inflation caplets/floorlets. In the calibration routine the approximate model HHWi-YoY is applied.

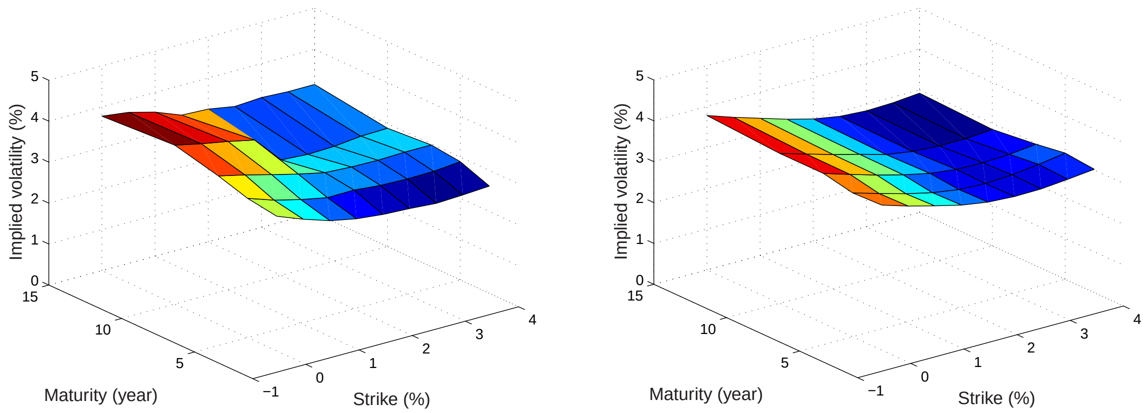
Remark. *Note that a combined calibration to two different sets of inflation market data, namely to inflation index caps/floors and YoY inflation caplets/floorlets, can also be performed. In such a calibration routine the approximate models HHWi-i and HHWi-YoY would be applied. It depends, however, on the 'problem at hand', which calibration is preferable*[19].

In Figures 4.2a and 4.2b the calibration results of the calibration to YoY inflation options are presented.

[19] For example, when one is interested in pricing an out-of-the-money (inflation dependent) option, one should calibrate the inflation model to out-of-the-money options.

Figure 4.2: Quality of fit of the calibrated inflation model to YoY inflation options.

(a) Market implied volatilities (%) (b) HHWi implied volatilities (%)



The calibration errors are very low; the average absolute error is 0.16% point and the maximum absolute error is 0.4% point, which indicates that the inflation model can be well calibrated to YoY inflation options.

The calibration results in the following model parameters:

$$\kappa = 0.03271, \quad \nu(0) = 3.9665 \cdot 10^{-4}, \quad \overline{\nu} = 7.5306 \cdot 10^{-4}, \quad \sigma_\nu = 0.0100,$$
$$\rho_{I,\nu} = -0.1900, \quad \rho_{\nu,r} = 0.0551 \text{ and } \rho_{\nu,\mathrm{n}} = -0.0684.$$

Observing this parameter setting, we note that the Feller condition, $2\kappa\overline{\nu} > \sigma_\nu^2$, is not satisfied, hence,

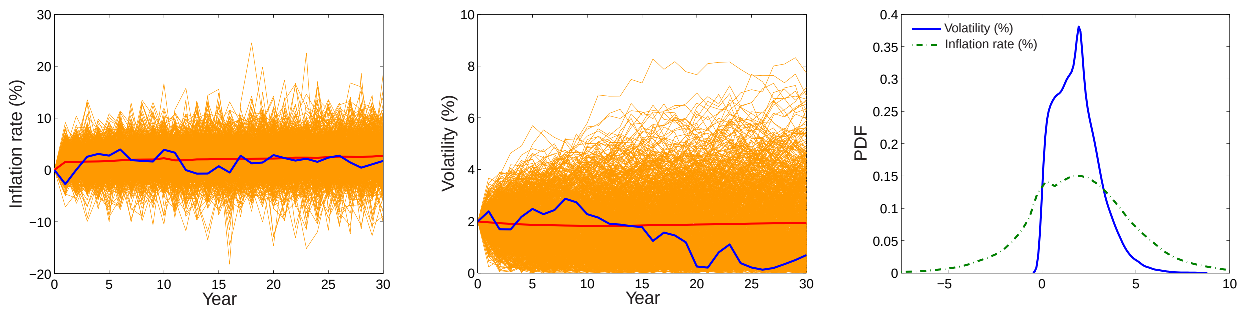
$$\mathbb{P}\left(\nu(t) = 0 \,|\, t > 0\right) > 0.$$

This implies that the variance process has a fat tailed distribution.

To get an impression of the results, 1.000 scenarios for the inflation rate[20] and the volatility process are visualized in Figures 4.3a and 4.3b; the red line represents the average value over all scenarios and the blue line represents a scenario. In Figure 4.3c the probability density function (PDF) is plotted of the inflation rate and volatility scenarios to gain more insight in the results.

Figure 4.3: Graphical impression of the generated risk-neutral scenarios of the inflation rate and the volatility process for a horizon of 30 years.

(a) Inflation rate process (b) Volatility process (c) Probability density function



The average inflation rate is approximately equal to the difference between the (average) nominal and real rates. The volatility of changes in the price inflation is high (approximately 2.8% per year). As a result, the

[20] As already mentioned, the inflation rate is defined as the percentage change of the CPI.

probability of negative inflation (deflation) is high (up to 20%). The fat tailed distribution of the volatility process is clearly visible in Figure 4.3c.

5 Valuation of the indexation provision of a pension fund

Risk-neutral (RN) scenarios are mainly used for valuation purposes. Such special purpose scenarios can, for example, be used for a market-consistent valuation of premiums, benefits, and indexations of a pension fund (PF), to support strategic decision-making and provisioning. This valuation of premiums, benefits, and indexations is becoming increasingly important for risk management to assess the consequences of policy changes to the different stakeholders of a PF[21]. Furthermore, the valuation of indexations is important for hedging strategies. See for similar experiments, for example, van Bragt and Steehouwer (2007), Possen and van Bragt (2009) and van Bragt, Waalwijk, and Steehouwer (2010).

In this section we perform several MC simulations to obtain a value for the conditional indexations provision[22] of a PF. For this numerical experiment we use a stylized PF. The liabilities of this PF can be viewed as a general liability setting in the Netherlands. The initial funded ratio (FR) is equal to 110%. The PF makes use of a conditional indexation policy. Indexation is linear when the FR is between 105% and 115%; when the FR is below 105% pension rights are not indexed. We assume that the PF invests in three main investment categories, 20% MSCI Europe stocks, 10% Euro direct real estate (RE) and 70% Euro government bonds.

Note that the inflation rate is the main driver of the initial indexation provision of a PF. We assume that indexation follows the price inflation for the inactive members of the PF and the wage inflation for the active members. In order to obtain the initial indexation provision we generate a consistent set of RN scenarios, so that all future indexation cash flows can be discounted with the nominal risk-free interest rate. The option price is then computed by:

$$\frac{1}{N}\sum_{k=1}^{N}\left(\sum_{t^*=t}^{T}\frac{M_{k,n}(t)}{M_{k,n}(t^*)}\widetilde{C}_k(t^*)\right),$$

where $t \leq t^* \leq T$, N denotes the number of scenarios, $\widetilde{C}_k(t^*)$ denotes the indexation cash flow in year t^* and scenario k, and $M_{k,n}$ denotes the nominal money-savings account (see Eq. (2.4)) in scenario k. We note that for this experiment we assume yearly time steps, i.e. $t^*, t, T \in \mathbb{N}$. In order to obtain an accurate option value, the number of scenarios N should be chosen as high as possible.

Since liquid inflation option market data only recently became available we take as the benchmark the fact that the price inflation model is calibrated to historical data. The historical volatility of the inflation rate is equal to 0.81%, which results in the following Heston parameters: $\kappa = 1$, $\nu(0) = \bar{\nu} = 0.46$ and $\sigma_\nu = 0$, as benchmark parameter setting. The full matrix of correlations is then also calibrated to historical data so that numerical inconsistencies are avoided.

Wage inflation, which is used for the (conditional) indexation of pension rights of active members, is modeled as price inflation plus 1% point. Furthermore, direct RE is modeled using a special purpose model, which is based on the Heston Hull-White model, where we explicitly model auto-correlation in the returns (see for more information van Bragt et al. (2009)). The investment category MSCI Europe stocks is also modeled by a Heston Hull-White model and is, for simplicity, calibrated to the historical volatility. Furthermore, an appropriate underlying bond portfolio is used for the investment category government bonds.

We perform the following two numerical experiments:

Experiment I Since the indexation provision of the PF is based on the price and wage inflation we apply several calibrations of our price inflation model[23]. Besides the inflation market data as of Q3 2010 we use a shifted set of market inflation option prices w.r.t. the market data of Q3 2010, using factors $\pm 10\%$ and 30%. The calibrated models are then used for a market consistent valuation of the indexation provision. The results are shown in Figure 5.1a.

[21] Stakeholders of a PF are for example: pensioners, the sponsor and employees.

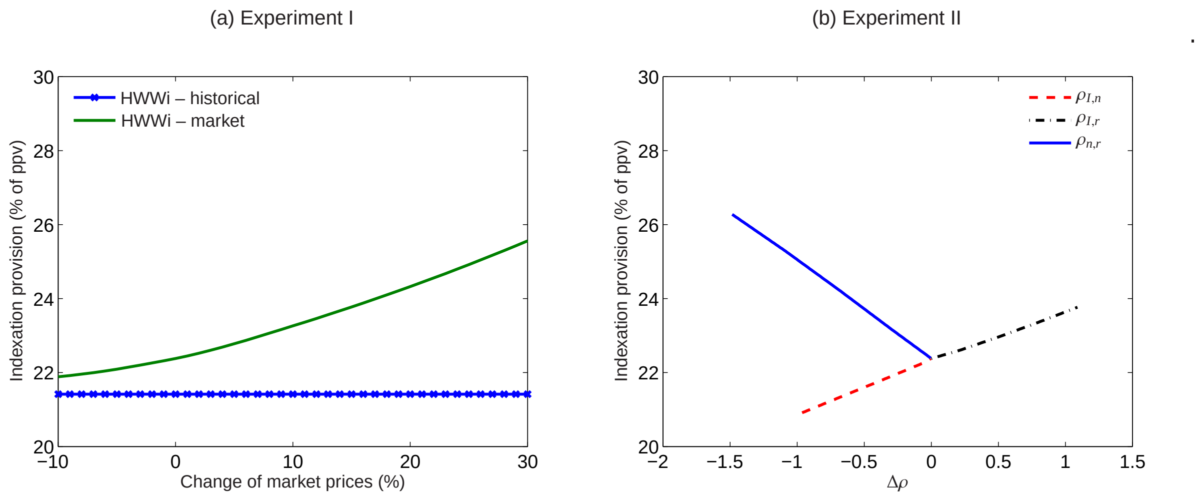
[22] The indexation provision of a PF can be viewed as an 'embedded option' on the balance sheet of the PF.

[23] In Section 4.2 we have shown that our inflation model can be well calibrated to inflation option market data, so that market conditions are replicated well and, therefore, a realistic (market consistent) value of the indexation provision can be obtained.

Experiment II To show the effect of different correlation parameters on the indexation provision, we perform a valuation of the indexation provision using different correlation values for $\rho_{I,n}$ $\rho_{I,r}$ and $\rho_{n,r}$. As a starting point we use the calibrated inflation model (see Section 4.2). The results are shown in Figure 5.1b.

For our numerical experiment we use $N = 10.000$ scenarios[24] so that sufficiently accurate results are obtained (other specifics of the MC simulation can be found in Section 3.3). Since the horizon of the liabilities is long we use as a simulation horizon $T = 80$ years so that all indexation cash flows are included in the MC simulation.

Figure 5.1: Overview of numerical results. Option values are expressed in terms of the pension fund provision (ppv).



Observing Figure 5.1a we can conclude that calibrating the inflation model to inflation option market data results in different indexation provisions compared to the benchmark inflation models. The benchmark inflation models are insensitive to a change of the inflation option market prices, which justifies the usefulness of calibrating the inflation model to inflation option market data.

Observing Figure 5.1b we can conclude that changing the correlation parameters can have a significant effect on the indexation provision; especially when $\rho_{n,r}$ changes. When for example the correlation, $\rho_{n,r}$, changes from 0.78 to -0.7 then the indexation provision changes from 22.4% to 26.3%. Therefore, we can conclude that the indexation provision is influenced by the correlations, which confirms that all correlations should indeed be incorporated in a valuation model.

[24] See Appendix A for validation experiments.

6 Conclusion

We derived an approximate closed-form solution of inflation indexed cap/ floor options and year on year inflation caplet/floorlet options, where the CPI follows a Heston model in which the nominal and real interest rates are modeled by one-factor Hull-White models. Using Fourier-based methods calibration can be done highly efficiently.

Using the developed models we have performed a calibration of the inflation model to year-on-year inflation options. Our inflation model is able to model the market implied volatility skew accurately, so that market conditions are replicated well.

Furthermore, using the calibrated inflation model we performed a market consistent valuation of the conditional indexation provision of a stylized pension fund. It turns out that the results change significantly when performing a calibration to market inflation option data instead to historical data, so it is recommendable to use market data instead of historical data for valuation purposes. By changing the correlation parameters, indexation provisions change significantly, which justifies the use of a full correlation matrix.

References

Andersen, L. 2007. "Simple and Efficient Simulation of the Heston Stochastic Volatility Model." *Working Paper, Bank of America Securities, New York*.

Andersen, L. B. G., and J. Andreasen. 2002. "Volatile Volatilities." *Risk* 15:163–168.

Bloch, D. A. 2009. "Fast Calibration of Interest Rate Claims in the Quadratic Gaussian Model: 2 the Swaptions." *Available at SSRN: http://ssrn.com/abstract=1441187*.

Bloch, D. A., and S. Assefa. 2009. "Fast Calibration of Interest Rate Claims in the Quadratic Gaussian Model: 1 The Caplets." *Available at SSRN: http://ssrn.com/abstract=1361609*.

Brigo, D., and F. Mercurio. 2006. *Interest Rate Models - Theory and Practice: With Smile, Inflation and Credit*. Springer Finance (Second edition).

Broadie, M., and O. Kaya. 2006. "Exact Simulation of Stochastic Volatility and other Affine Jump Processes." *Operations Research*, vol. 54.

Carr, P., and D. B. Madan. 1999. "Option Valuation Using the Fast Fourier Transform." *Journal of Computational Finance* 2:61–73.

Cox, J. C., J. E. Ingersoll, and S. A. Ross. 1985. "A Theory of the Term Structure of Interest Rates." *Econometrica* 53:385–407.

Duan, J. C., and J. G. Simonato. 1995. "Empirical Martingale Simulation for Asset Prices." *Management Science* 44:1218–1233.

Fang, F., and C. W. Oosterlee. 2008. "A Novel Pricing Method for European Options Based on Fourier-Cosine Series Expansions." *SIAM J. Sci. Comput.* 31:826–848.

Fisher, I. 1930. "The Theory of interest." *Basingstoke: MacMillan Press Ltd.*

Glasserman, P. 2004. *Monte Carlo Methods in Financial Engineering*. Springer Verslag.

Grzelak, L. A., and C. W. Oosterlee. 2009. "On the Heston Model with Stochastic Interest Rates." *Techn. Report 09-05, Delft Univ. Techn., The Netherlands, Forthcoming in SIAM J. Finan. Math*.

———. 2010. "On Cross-Currency Models with Stochastic Volatility and Correlated Interest Ratess." *Techn. Report 10-13, Delft Univ. Techn., The Netherlands, SSRN working paper*.

Heston, S. L. 1993. "A Closed-form Solution for Options with Stochastic volatility with Applications to Bond and Currency Options." *Review of Financial Studies* 6 (2): 327–343.

Kruse, S. 2007. "Pricing of Inflation-Indexed Options under the Assumption of a Lognormal Inflation Index as well as under Stochastic Volatility." *Available at http://ssrn.com/abstract=948399*.

Oman, M. 2005. "Inflation derivatives: A Users Guide." *Barclays Capital Research, London*.

Possen, T., and D. D. B. van Bragt. 2009. "Market-Consistent Valuation of Life Cycle Unit-Linked Contracts." *OFRC Applied Working Paper No. 2009-03*.

van Bragt, D. D. B., M. K. Francke, B. Kramer, and A. Pelsser. 2009. "Risk-Neutral Valuation of Real Estate Derivatives." *OFCR Technical Working Paper No. 2009-02*.

van Bragt, D. D. B., and H. Steehouwer. 2007. "Recent Trends in Asset and Liability Modelling for Life Insurers." *OFRC Methodological Working Paper No. 2007-01*.

van Bragt, D. D. B., B. Waalwijk, and H. Steehouwer. 2010. "Market-Consistent ALM for Life Insurers - Steps Toward Solvency II." *The Geneva Papers on Risk and Insurance - Issues and Practice*, vol. 35.

van Haastrecht, A., and A. Pelsser. 2009. "Generic pricing of FX, inflation and stock options under stochastic interest rates and stochastic volatility." *Journal of Property Research* 17:93–108.

van Haastrecht, A., A. Pelsser, R. Lord, and D. Schrager. 2009. "Pricing Long-Maturity Equity and FX Derivatives with Stochastic Interest Rates and Stochastic Volatility." *Ins.: Mathematics Econ.* 45:436–448.

Appendix A

Validation of the Monte Carlo simulation

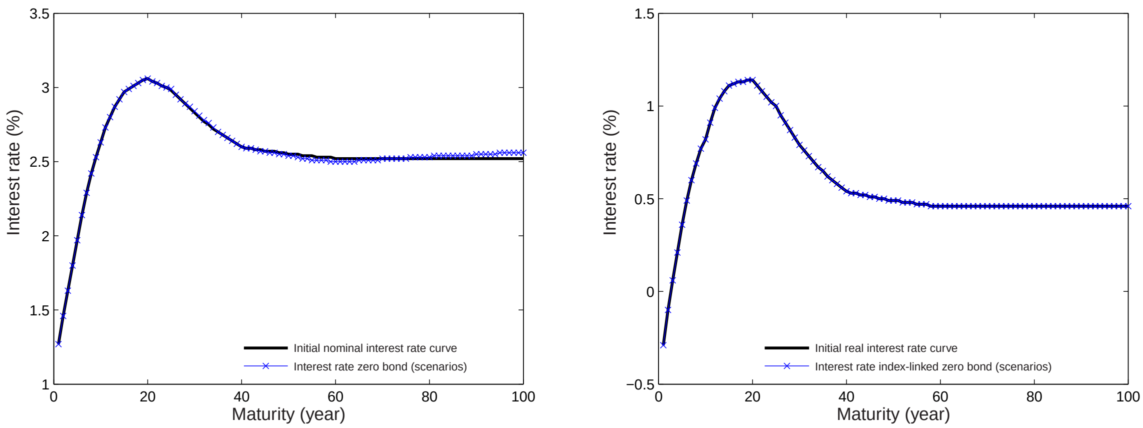
To illustrate the fact that the MC simulation (using 10.000 scenarios and a horizon of 100 years) of the HHWi model performed in Section 5 fulfils the martingale condition we perform two martingale tests. We first perform a (simple) MC experiment in which we price a series of zero-coupon bonds with different maturities. The payoff of this experiment is obviously equal to the principal of each bond for all scenarios. This payoff is then discounted back along the path of the short nominal interest rate for each scenario. The average discounted value (over all scenarios) then yields the MC price of each bond. This price can be converted into an equivalent interest rate for each maturity. If the generated scenario set is indeed arbitrage free, these interest rates should coincide with the initial nominal interest rate curve. The results are shown in Figure 6.1a.

As a second test, we price a series of index-linked zero-coupon bonds with different maturities. The principal of each bond is now indexed at the end of each year with the price inflation. The final payoff is then again discounted back along the path of the short nominal interest rate for each scenario. The average discounted value (over all scenarios) then yields the MC price of each index-linked bond. This price can subsequently be converted into an real interest rate for each maturity. If the generated scenario set is indeed arbitrage free, these interest rates should coincide with the initial real interest rate curve.

This comparison is made in Figure 6.1b. Note the perfect agreement between the real interest rates as implied by the scenario set and the initial real interest rate curve. This is due to the application of the empirical martingale simulation technique, which detects and corrects deviations from the desired martingale property.

Figure 6.1: Comparison between the nominal/real interest rates as determined by the scenario set and the initial nominal/real interest rate curve.

(a) Nominal interest rate (b) Real interest rate



Observing Figure 6.1a a good agreement between the nominal interest rates as implied by the scenario set and the initial nominal curve is visible. This indicates that the interest rate scenarios are arbitrage free with respect to the initial nominal interest rate curve. The remaining differences will further diminish when a larger scenario set is used.