Exotic Options

A Potpourri of options and equations !

- Compound, chooser, binary
- Path dependent: Barrier options, lookback, Asian
- Equations and numerics for Asians

Compound, chooser, binary

- Compound option: Call on a call: right to buy a 'call with maturity T and strike K' at time T_0 for the price K_0. Payoff:

$$CC(S, T_0, K_0, K, T) = \max [C(S, K, T) - K_0, 0]$$

- Chooser option: Gives the holder the right to choose whether the underlying option at time T_0 is a Call or a Put with the same strike K and maturity T. The payoff of a chooser option is

$$CH(S, K, T_0, T) = \max [C(S, K, T), P(S, K, T)]$$

- Binary: Cash or Nothing: Pays out Q at expiry T if option is in the money $S > K$, otherwise expires worthless. Payoff:

$$B(S, Q, K, T) = Q 1_{S \geq K}$$

variation: Asset or nothing, where Q is the asset itself.

Binaries or Digitals

- Binary options have a discontinuous payoff at expiry.
- An example for a binary call, is:
  The contract pays 1 at T, if the asset price is then greater than the exercise price E.
- The to the binary call belonging final condition is

$$V(S,T) = \mathcal{H}(S-E)$$

where $\mathcal{H}(\cdot)$ is the Heaviside function.

Path-dependency

- Options whose value depends on the asset history, but can still be written as $V(S, t)$ are said to be weakly path dependent.
- American options, with early exercise, are considered to be weakly path dependent. The next common reason for weak path dependence in a contract is a barrier.
- Strongly path-dependent contracts are of particular interest. These have payoff that depend on some property of the asset price path in addition to the value of the underlying at the present time. So, $V \neq V(S, t)$. The contract value is a function of at least one more independent variable, such as a 'running average' of asset prices.
- Weakly path-dependent contracts have the same number of dimensions as the non-path-dependent versions
- Strongly path dependent contracts are governed by an extra dimension. The new independent variable is a measure of the path-dependent quantity.

Path-dependent options

- Barrier options (US, 1967) are options that either come alive or die when predetermined trigger points (barriers) are reached.
  Down-and-out call: Option is knocked out if S hits a certain barrier H. Payoff:

$$C_{DO}(S, K, T) = \max(S - K, 0) \text{ if } S \geq H; \text{ else ceases to exist}$$

- Lookback options: Path dependent options whose payoffs depend on the max or min of the asset during a certain period (lookback period $[T_0, T]$).
  Payoff European fixed strike lookback call: $(\max_{[T_0,T]}(S_t) - K)^+$
  Payoff European fixed strike lookback put: $(K - \min_{[T_0,T]}(S_t))^+$

Barrier options

- Barrier options come in two main varieties, the ‘in’ barrier option (or knock-in) and the ‘out’ barrier option (knock-out). The former only have a payoff if the barrier level is reached before expiry and the latter only have a payoff if the barrier is not reached.
- Barrier options are popular for various reasons.
- Usually, a purchaser has very precise views about the direction of the market. If he wants the payoff from a call option but does not want to pay for all the upside potential, believing that the upward movement of the underlying will be limited prior to expiry, then he may choose to buy an up-and-out call. It will be cheaper than a similar vanilla call, since the upside is severely limited.
- Conversely, an ‘in’ option will be bought by someone who believes that the barrier level will be realized. Again the option is cheaper then the equivalent vanilla option.

Different types of barriers

- The out option only pays off if a level is not reached. If the barrier is reached the option is said to have knocked out.
- The in option pays off as long as a level is reached before expiry. If the barrier is reached then the option is said to have knocked in.
- If the barrier is above the initial asset value, we have an up option
- If the barrier is below the initial value, we have a down option
- The payoffs are the usual ones
- Barrier can be time dependent

PDE for Barrier options

- These options satisfy the Black-Scholes equation, on a special domain, with special boundary conditions
- The details of the barrier feature come in through the specification of the boundary conditions
- If the asset reaches the barrier S_u in an 'out' barrier option then the contract becomes worthless:

$$V(S_u, t) = 0 \text{ for } t < T$$

- If we have a down-and-out option with a barrier at S_d we solve for $S_d < S < \infty$ with

$$V(S_d, t) = 0$$

‘In’ Barriers

- An ‘in’ option only has a payoff if the barrier is triggered. If the barrier is not triggered we have $V(S,T) = 0$
- The value in the option is in the potential to hit the barrier. If the option is an up-and-in contract then on the upper barrier the contract must have the same value as a vanilla contract:

$$V(S_u, t) = \text{ value of vanilla contract, function of } t$$

So,

$$V(S_u, t) = V_{van}(S_u, t) \text{ for } t < T$$

Hedging barrier options

- Barrier options have discontinuous delta at the barrier

  For a knock-out, the option value is continuous, decreasing approximately continuously towards the barrier, then being zero beyond the barrier.

- A discontinuity in the delta means that the gamma is instantaneously infinite at the barrier. Delta hedging through the barrier is virtually impossible, and costly.
- There have been a number of suggestions made for ways to statically hedge barrier options. These methods try to mimic as closely as possible the value of a barrier option with vanilla calls and puts or with binary options.

Asian options

- Asian options: The payoff depends on the average of the underlying. Types of averages:
- Arithmetic average:

$$A = \frac{1}{n} \sum_{i=1}^{n} S_{t_i}$$

- Geometric average: $A = (\Pi_{i=1}^{n} S_{t_i})^{1/n}$
- Continuous average

$$A = \frac{1}{t} \int_0^t S_\tau d\tau$$

- All the above may be expressed as $A_t = \int_0^t f(S_\tau, \tau) d\tau$

Asian payoffs

- $(A-K)^+$: fixed strike call
- $(K-A)^+$: fixed strike put
- $(S_T-A)^+$: floating strike call
- $(A-S_T)^+$: floating strike put
- The PDE reads:

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}+f(S,t)\frac{\partial V}{\partial A}-rV=0$$

Discrete Averaging

- A discrete averaging is most used in practice. Let sampling times $t_1, \ldots, t_N$, and define the averages by:

$$A_n = \frac{1}{n} \sum_{i=1}^{n} S(t_i)$$

- Notice that $A_n = A_{n-1} + \frac{S(t_n) - A_{n-1}}{n}$
- Denoting t^+ and t^- the times before and after the date t_n:

$$A(S, t^+) = A(S, t^-) + \frac{S - A(S, t^-)}{n}$$

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Discrete Averaging (cont.)

- Simplifying notation: $A^+ = A^- + \frac{S-A^-}{n}$
- From no-arbitrage one has

$$V(S, A^+, t^+) = V(S, A^-, t^-)$$

- However, for fixed (S, A) this defines a jump across t_n
- Away from the observation dates one solves the plain BS equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

and apply at jump times the jump condition. Summarizing:

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Algorithm for fixed strike Asian put

Take a time slice $[t_N, T]$:

– Divide $[0, A_{max}]$ and $[0, S_{max}]$ with grid A_j and S_i

– Solve for each A_j the BS equation with final condition $(K - A)^+$ (say J equations simultaneously) and boundary conditions $(K - A_j)^+, S \to \infty$

– The surface obtained at time t_N (call it t_N^+) is then shifted by interpolation:

$$V(S, A, t_n^-) = V(S, A + \frac{S - A}{N}, t_N^+)$$

– The new surface $V(S, A, t_N^-)$ is used as new final condition for the new set of BS equations on time slice $[t_{N-1}, t_N]$ and so on...

Asian Options

Option depending on continuous average

- Previously details on Asian options with discrete averaging.
  Now, continuous averaging.
- The exercise price or the asset price is replaced by an average of the asset price:
- Final conditions for an arithmetic-average floating strike call:
  $u(S,T) = \max(S - \frac{1}{T}\int_0^T S(\tau)d\tau, 0)$
- With a new variable: $I(t) := \int_0^t S(\tau)d\tau$, one finds a similar Black-Scholes type equation for Asian options:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rs\frac{\partial V}{\partial S} + S\frac{\partial V}{\partial I} - rV = 0$$

- There is no diffusion term in the I-direction.

Path dependency and the integral

- We start by assuming that the underlying asset follows the lognormal random walk:

$$dS = \mu S dt + \sigma S dW$$

Imagine a contract that pays off at maturity T an amount that is a function of the path taken by the asset between zero and maturity

- Suppose that this path-dependent quantity can be represented by an integral of some function of the asset over the period zero to T:

$$I(T) = \int_0^T f(S, \tau) d\tau$$

- Most path-dependent quantities in exotic derivative contracts can be written in this form with a suitable choice of $f(S, t)$.

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- Prior to maturity we have information about the possible final value of S (at time T) in the present value of S at time t. For example, the higher S is today, the higher it will probably end up at maturity.

  Similarly we have information about the possible final value of I in the value of the integral to date:

$$I(t) = \int_0^t f(S, \tau) d\tau$$

  As we get closer to maturity, we become more confident about the final value of I.

- The value of the option is therefore not only a function of S and t, but also a function of I; I will be our new independent variable, called a state variable

- As we will use Itô's lemma, we need to know the stochastic differential equation satisfied by I. This is simply (incrementing t by dt we find that):

$$dI = f(S,t)dt$$

- I is thus a smooth function, and the equation for dI does not contain stochastic terms

Examples

- An Asian option has a payoff that depends on the average of asset price over some period. If that period is from time zero to maturity and the average is arithmetic then:

$$I = \int_0^t S d\tau$$

- The payoff may then be, (a floating strike put), for example,

$$\max\left(\frac{I}{T} - S, 0\right)$$

- Another example: Imagine a contract that pays off a function of the square of the underlying asset, but only counts those times for which the asset is below S_u. Then

$$I = \int_0^t S^2 \mathcal{H}(S_u - S) d\tau$$

where $\mathcal{H}$ is the Heaviside function.

Continuous sampling

The pricing equation

- We derive the pricing PDE for a contract that pays some function of new variable I
- The value of the contract is now a function of the three variables: $V(S, I, t)$.
- Set up a portfolio containing one of the path-dependent options and short a number Δ of the underlying asset:

$$\Pi = V(S, I, t) - \Delta S$$

- The change in the value of this portfolio is given by

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \frac{\partial V}{\partial I} dI + \left(\frac{\partial V}{\partial S} - \Delta \right) dS$$

- Choosing $\Delta = \partial V / \partial S$ to hedge the risk, we find:

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + f(S, t)\frac{\partial V}{\partial I} \right) dt$$

- This change is risk free and thus earns the risk-free rate of interest r, leading to the PDE:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + f(S,t)\frac{\partial V}{\partial I} + rS\frac{\partial V}{\partial S} - rV = 0$$

- This is to be solved subject to

$$V(S, I, t) = \text{ payoff}$$

- This completes the formulation of the valuation problem.

Higher dimensions

- The methods outlined are not restricted to a single path-dependent quantity. Any finite number of path-dependent variables can be accomodated, theoretically.
- Imagine that a contract pays off the difference between a continuous geometric and a continuous arithmetic average. To price this one would need to introduce I_g and I_a, defined by

$$I_g = \int_0^t \log(S) d\tau, \quad I_a = \int_0^t S d\tau$$

- The solution would be a function of four variables $V(S, I_g, I_a, t)$.
- This growth in dimensionality may be difficult for solving a PDE with numerical techniques !

Similarity reductions

- Some options have a particular structure that permits a reduction in the dimensionality of the problem by use of a similarity variable.
- The dimensionality of the continuously-sampled artihmetic floating strike option can be reduced from three to two.
- The payoff for the call option is

$$\max\left(S - \frac{1}{T}\int_0^T S(\tau)d\tau, 0\right)$$

- We can write the running payoff for the call option as

$$I\max\left(R - \frac{1}{t}, 0\right), \text{ where } I = \int_0^t S(\tau)d\tau, \text{ and } R = \frac{S}{\int_0^t S(\tau)d\tau}$$

- The payoff at maturity may then be written as

$$I \max\left(R - \frac{1}{T}, 0\right)$$

- In view of the form of the payoff, it is plausible that the option value takes the form:

$$V(S,R,t) = IW(R,t). \text{ with } R = \frac{S}{I}$$

- We find that W satisfies:

$$\frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2 R^2 \frac{\partial^2 W}{\partial R^2} + R(r-R)\frac{\partial W}{\partial R} - (r-R)W = 0$$

- This reduction is not possible for American variants

Put-Call Parity for European Floating Strike

- The payoff at maturity for a portfolio of one European floating strike call held long and one put held short is

$$I \max{(R - \frac{1}{T}, 0)} - I \max{(\frac{1}{T} - R, 0)}$$

- Whether R is greater or less than T at maturity, this payoff is simply:

$$S - \frac{I}{T}$$

- The value of this portfolio is identical to one consisting of one asset and a financial product whose payoff is $-I/T$

- In order to value this product find a solution of the floating strike equation of the form

$$W(R,t) = b(t) + a(t)R$$

and with $a(T) = 0$ and $b(T) = -1/T$; such a solution would have the required payoff of $-I/T$.

- Substitution and satisfying the boundary conditions, we find that

$$a(t) = \frac{-1}{rT}\left(1 - e^{-r(T-t)}\right), \quad b(t) = -\frac{1}{T}e^{-r(T-t)}$$

- We conclude that

$$V_c - V_p = S - \frac{S}{rT}(1 - e^{-r(T-t)}) - \frac{1}{T}e^{-r(T-t)}\int_0^t S(\tau)d\tau$$

where V_c and V_p are the values of the European arithmetic floating strike call and put.

- With a new variable: $A(t) := (\int_0^t S(\tau)d\tau)/t$, one obtains the following Black-Scholes type equation for Asian options:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rs\frac{\partial V}{\partial S} + \left(\frac{S - A}{t}\right)\frac{\partial V}{\partial A} - rV = 0$$

- There is no diffusion term in the A-direction.

$\Rightarrow$ In CFD, this situation often occurs and appropriate discretizations for such terms have been developed there.

$\Rightarrow$ Ultraparabolic equation.

- Moreover, American-style Asian options exist (combination of convection dominance and free boundary aspects)
- We cannot transform the equation to a diffusion equation anymore. We need to discretization and solve the original equation.

Definition of the problem

Examples of multi-asset options

- A basket option is an option whose payoff depends on the value of a portfolio (or basket) of assets. Basket options are growing in popularity as a means of hedging the risk of a portfolio and are highly interesting for banks nowadays.
- They are attractive because an option on a basket is cheaper than buying options on the individual assets. Furthermore, their payoff profile replicates the changes in a portfolio's value more closely than any combination of options on the underlying assets.
- Basket options: $u(\mathbf{S}(T), T) = \left(\sum_{i=1}^{d} w_i S_i - K\right)^+$
- Call option on the minimum of the underlying assets $u(\mathbf{S}(T), T) = (\min_i S_i - K)^+$
- Put option on the maximum of the underlying assets $u(\mathbf{S}(T), T) = (K - \max_i S_i)^+$
- Exchange option (two-asset): $u(\mathbf{S}(T), T) = (S_1 - S_2)^+$

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Multi-asset options

Problem definition

- Multi-asset options are multi-dimensional. Using numerical techniques, the number of grid points grows exponentially $\Rightarrow$ Curse of dimensionality. Problems are not solvable on nowadays machines unless advanced techniques are used.
- Sparse grid methods reduces the number of grid points per dimension, so larger problems can be computed. In finance, the sparse grid method for solving PDE's is used for the first time by C. Reisinger [4].
- Partitioning/Splitting and parallelization of the method reduces memory usage.

PDE methods

Multi-d Black-Scholes equation

The PDE-method is based on the solution of the multi-dimensional Black-Scholes equation:

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sum_{i=1}^{d}\sum_{i=1}^{d}\rho_{ij}\sigma_i\sigma_j S_i S_j \frac{\partial^2 u}{\partial S_i \partial S_j} + \sum_{i=1}^{d}(r-\delta_i) S_i \frac{\partial u}{\partial S_i} - ru = 0 \qquad (1)$$

With

- S_i, the value of underlying asset i
- σ_i volatility of asset i
- ρ_{ij} correlation between asset i and j
- r risk-free interest rate
- δ_i continuous dividend yield

Dividends

- The owner of a stock theoretically owns a piece of the company. This ownership can only be turned into cash, if he owns so many of the stocks that he can take over the company and keep all profits for himself, which is unrealistically (for most of us).
- To the average investor the value in holding the stock comes from the dividends and any growth in the stock's value. Dividends are the lump payments, paid out every quarter or every six months, to the holder of the stock.
- The amount of dividend varies from year to year depending on the profitability of the company. Companies like to try to keep the level of dividends about the same.
- The amount of dividend is decided by the board of directors of the company and is usually set a month or so before the dividend is actually paid.
- When the stock is bought it either comes with its entitlement to the next dividend (cum) or not (ex). There is a date at around the time of dividend payment when the stock goes from cum to ex. The original holder of a stock gets the dividend but the person who buys it obviously does not.

Options on dividend-paying equities

- A stock that is cum dividend is better than one that is ex dividend. Thus at the time that the dividend is paid there will be a drop in the value of the stock. (The jump in asset price is more complex in practice.)
- The price of an option on an dividend-paying asset is affected by these payments. Therefore we must modify the Black-Scholes analysis.
- Different structures are possible for the dividend payment
  - payments may be deterministic or stochastic
  - payments may be made continuously or at discrete times
- Here, we only consider deterministic dividends, whose amount and timing are known.
- Let's consider that in a time dt the underlying asset pays out a dividend $DSdt$ with D constant. The payment is independent of time except through the S-dependence. It represents a continuous and constant dividend yield.
- This structure is a good model for index options.

- Arbitrage considerations show that the asset price must fall by the amount of dividend payment. This is modeled as: $dS = \sigma S dw + (\mu - D)S dt$.
- The dividend payment also has its effect on the hedged portfolio: Since we receive $DSdt$ for every asset held and we hold $-\Delta$ of the underlying, the portfolio changes by an amount $-DS\Delta dt$. Therefore, we add to our $d\Pi$ from before this amount:

$$d\Pi = dV - \Delta dS - DS\Delta dt.$$

  We find after similar reasoning as for European options that dividend is included in the following formulation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D)S\frac{\partial V}{\partial S} - rV = 0$$

- This model is also applicable to options on foreign currencies, though only for short dated options. Since holding an amount of foreign currency yields interest at the foreign rate r_f, in this case $D = r_f$
- A nonzero dividend yield also has an effect on the boundary and final conditions.

Options on dividend-paying equities

- At the time that a dividend is paid there will be a drop in the value of the stock.
- The price of an option on an dividend-paying asset is affected by these payments.
- Different structures are possible for the dividend payment (deterministic or stochastic with payments continuously or at discrete times)
- We consider discrete deterministic dividends, whose amount and timing are known.
- Arbitrage arguments require:

$$V(S, t_d^-) = V(S - D, t_d^+)$$

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Final/Boundary conditions

- European Call option: Right to buy assets at maturity $t = T$ for exercise price K.
- Final condition: $V(S,T) = \max(S - K, 0)$
- Boundary conditions $S = 0$: $V(0,t) = 0$,

  for $S \to \infty$: $V(S_{max}, t) = S_{max} - Ke^{-r(T-t)} - De^{-r(t_d - t)}$ or $V_{ss} = 0$.
- The strategy to solve the Black-Scholes equation numerically is as follows
  - Start solving from $t = T$ to $t = t_d$ with the usual pay-off.
  - Apply an interpolation to calculate the new asset and option price on the grid discounted with D.
  - Restart the numerical process with the PDE from the interpolated price as final condition from t_d to $t = 0$.

Example

- Multiple discrete dividends: analytic solution not available
- Parameters: $s_0 = K = 100,\ r = 0.06, \sigma_c = 0.25$, multiple dividends of 4 (ex-dividend date is each half year), $T = 1, 2, 3, 4, 5, 6$. Grid: $s_{max} = RK(3 \leq R \leq 7), \mu = 0.15$

Grid	$T = 1$	Grid	$T = 2$	Grid	$T = 3$
20×20	10.660	20×40	15.202	20×80	18.607
40×40	10.661	40×80	15.201	40×160	18.600
Lewis (Wilmott Mag. 2003)	10.661		15.199		18.598
Grid	$T = 4$	Grid	$T = 5$	Grid	$T = 6$
20×80	21.370	20×100	23.697	20×120	25.710
40×160	21.362	40×200	23.691	40×240	25.698
Lewis	21.364		23.697		25.710