DELFT UNIVERSITY OF TECHNOLOGY
Faculty of Electrical Engineering, Mathematics and Computer Science

Numerical Analysis CII (wi4014)
Numerical Methods PDE (wi3001)
2003/2004 Take Home Exams, Second Series

1. On the bounded domain $\Omega \subset I\!R^2$ we consider the minimization problem (the so called minimal surface problem)

$$\min_u J[u] = \int_\Omega \sqrt{1 + u_x^2 + u_y^2}\, d\Omega, \tag{1}$$

On a part of the boundary, Γ_1, u is prescribed: $u = g$.

Show by the same method as in Theorem 4.3, that the Euler-Lagrange equation for this problem is

$$\text{div}\, \left([1 + u_x^2 + u_y^2]^{-\frac{1}{2}} \text{grad}\, u \right) = 0 \tag{2}$$

Derive also from the minimization problem a boundary condition for Γ_2, the part of the boundary where u has not been prescribed.

2. Consider the minimization problem:

$$\min_{u \in \Sigma} \int_\Omega \frac{1}{2} \left(a_{11} u_x^2 + 2a_{12} u_x u_y + a_{22} u_y^2 \right) - uf\, d\Omega \tag{3}$$

in which $\Omega \subset I\!R^2$ is a bounded domain with boundary $\Gamma = \Gamma_1 \cup \Gamma_2$ and

$$\Sigma : \{ u \in C^1(\Omega) | u = g, \quad \forall (x, y) \in \Gamma_1 . \} \tag{4}$$

The coefficients a_{ij} depend on (x,y) and the symmetric matrix A with coefficients a_{ij} is positive definite for each $(x, y) \in \Omega$.
We solve this problem by using the Finite Element Method and we use piecewise quadratic elements.

(a) Derive the Euler-Lagrange equations with corresponding boundary conditions.

(b) Show that the Newton-Côtes formula for the quadratic element is given by

$$\int_e Int(\boldsymbol{x})\, de = \frac{|\Delta|}{6} \left[Int(\boldsymbol{x}_{23}) + Int(\boldsymbol{x}_{13}) + Int(\boldsymbol{x}_{12}) \right] \tag{5}$$

(c) Construct a table of the values of $\phi_{i_x}, \phi_{i_y}, \phi_{ij_x}, \phi_{ij_y}$ in the integration points of an element. Show how to calculate the integrals $\int_e a_{11} \phi_{i_x} \phi_{j_x}\, de$ and $\int_e a_{11} \phi_{i_x} \phi_{ij_x}\, de$ using this table. (Use Newton-Côtes formulae.)

(d) Compute the element vector.

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