Multi-level Krylov: the Next Generation Helmholtz Solver

Fast Helmholtz Solvers Seminar, TU Delft, May 18th, 2015

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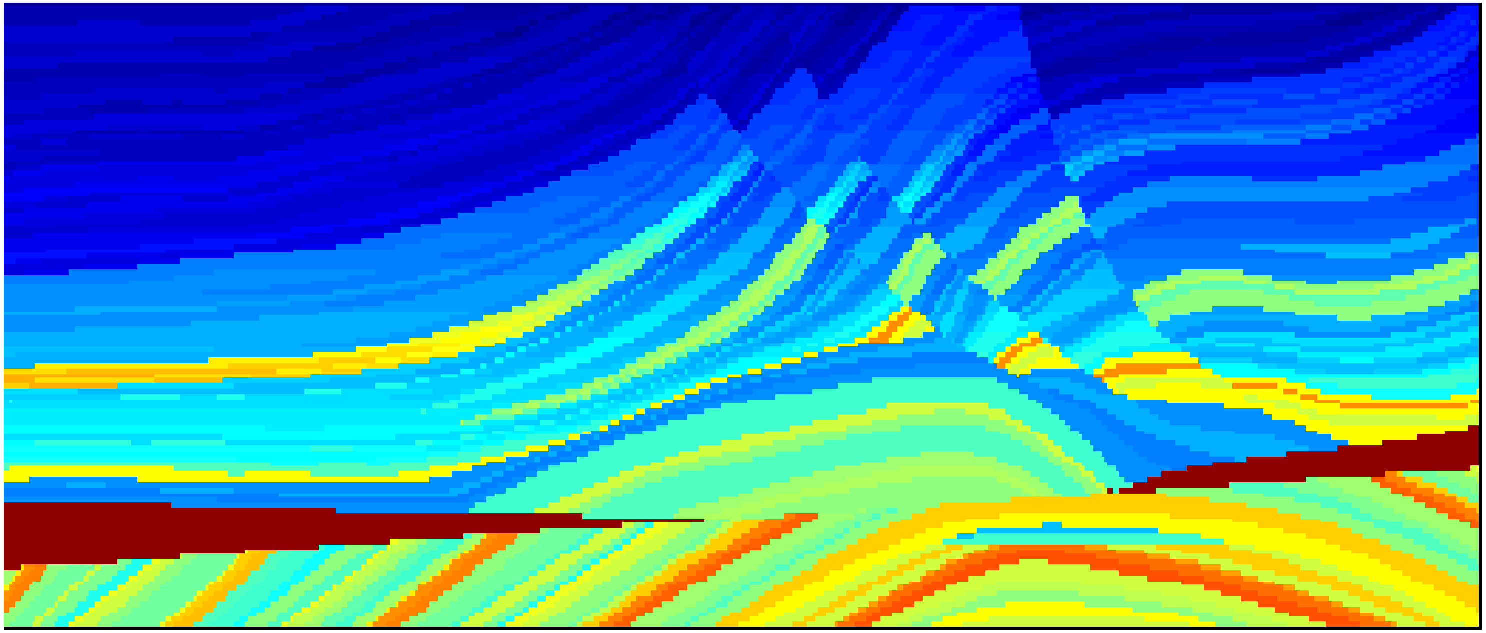
May 18, 2015

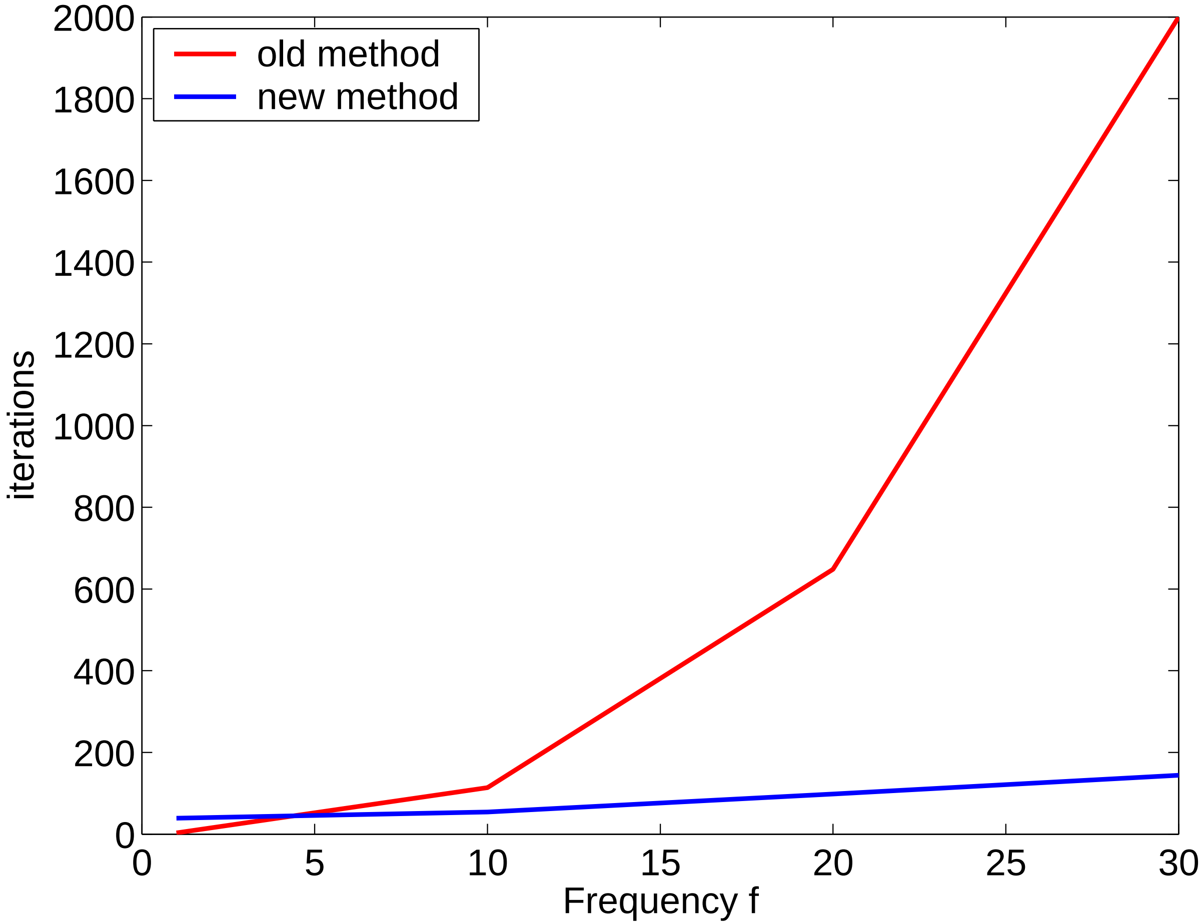
Application: geophysical survey

hard Marmousi Model



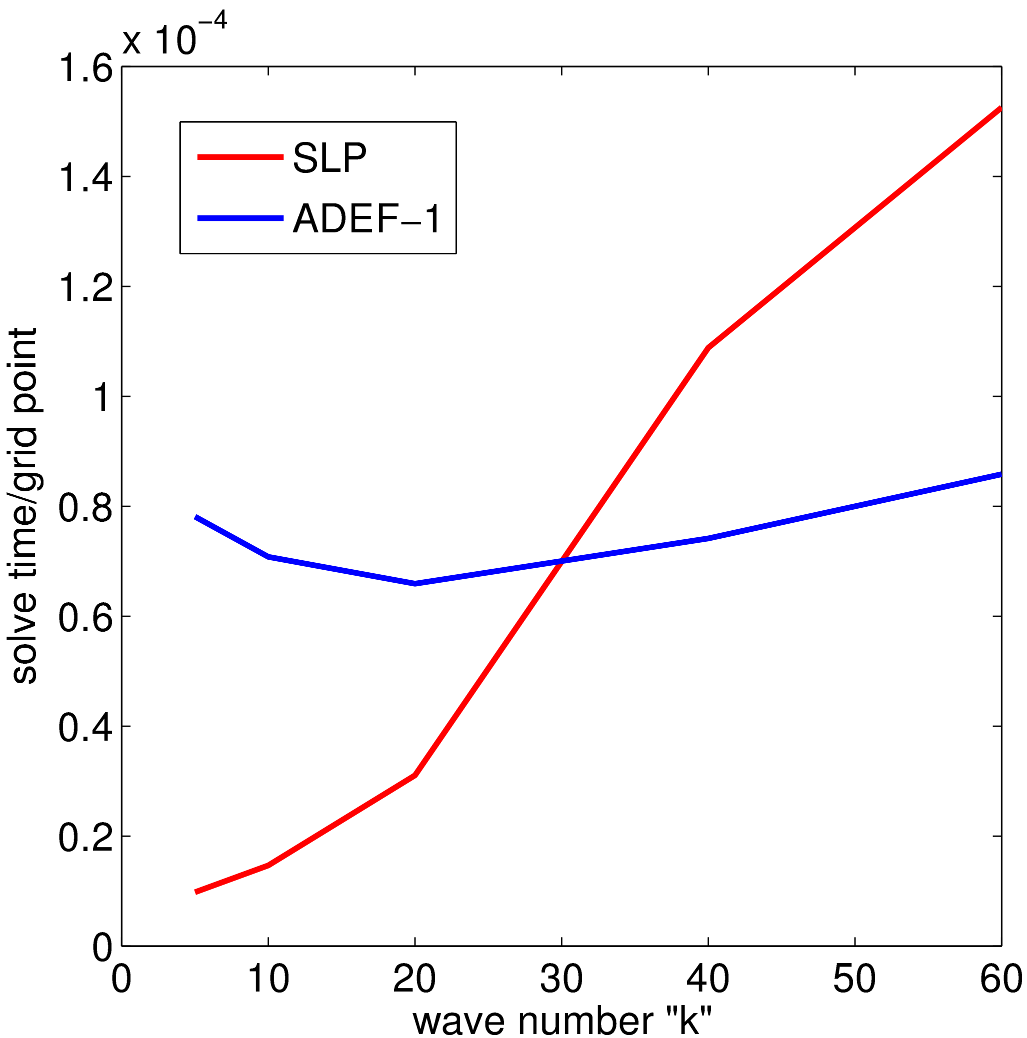
Application: geophysical survey

hard Marmousi Model (2005)



Application: geophysical survey

Cube with constant k (2015)



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1. Introduction

The Helmholtz equation without damping

$$-\Delta \mathbf{u}(x,y) - k^2(x,y)\mathbf{u}(x,y) = \mathbf{g}(x,y) \ \text{ in } \Omega$$

$\mathbf{u}(x,y)$ is the pressure field,

$\mathbf{k}(x,y)$ is the wave number,

$\mathbf{g}(x,y)$ is the point source function and

Ω is the domain. Absorbing boundary conditions are used on Γ.

$$\frac{\partial \mathbf{u}}{\partial n} - \iota \mathbf{u} = 0$$

n is the unit normal vector pointing outwards on the boundary.

Perfectly Matched Layer (PML) and Absorbing Boundary Layer (ABL)

Problem description

- Second order Finite Difference stencil:

$$\begin{bmatrix} & -1 & \\ -1 & 4-k^2h^2 & -1 \\ & -1 & \end{bmatrix}$$

- Linear system $Au = g$: properties
  Sparse & complex valued
  Symmetric & Indefinite for large k
- For high resolution a very fine grid is required: $10 - 20$ gridpoints per wavelength $\rightarrow A$ is extremely large!

2. Shifted Laplace Preconditioning

Equivalent linear system $M_1^{-1} A M_2^{-1} \tilde{x} = \tilde{b}$, where $M = M_1 \cdot M_2$ is the preconditioning matrix and

$$\tilde{x} = M_2 x, \quad \tilde{b} = M_1 b.$$

Requirements for a preconditioner

- better spectral properties of $M^{-1}A$
- cheap to perform $M^{-1}r$.

Spectrum of A is $\{\mu_i - k^2\}$, with k a given constant and μ_i are the eigenvalues of the Laplace operator. Note that $\mu_1 - k^2$ may be negative.

Preconditioning (Laplace type)

Laplace operator	Bayliss and Turkel, 1983
Definite Helmholtz	Laird, 2000
Shifted Laplace	Y.A. Erlangga, C. Vuik and C.W.Oosterlee, 2003

Shifted Laplace preconditioner (SLP)

$$M \equiv -\Delta - (\beta_1 - \mathrm{i}\beta_2)k^2, \quad \beta_1, \beta_2 \in \mathbb{R}.$$

$\rightarrow \beta_1, \beta_2 = 0$	:	Bayliss and Turkel
$\rightarrow \beta_1 = -1, \beta_2 = 0$	:	Laird
$\rightarrow \beta_1 = 1, \beta_2 = 0.5$	:	Y.A. Erlangga, C. Vuik and C.W.Oosterlee

Numerical experiments

Example with constant k in Ω

Iterative solver: Bi-CGSTAB

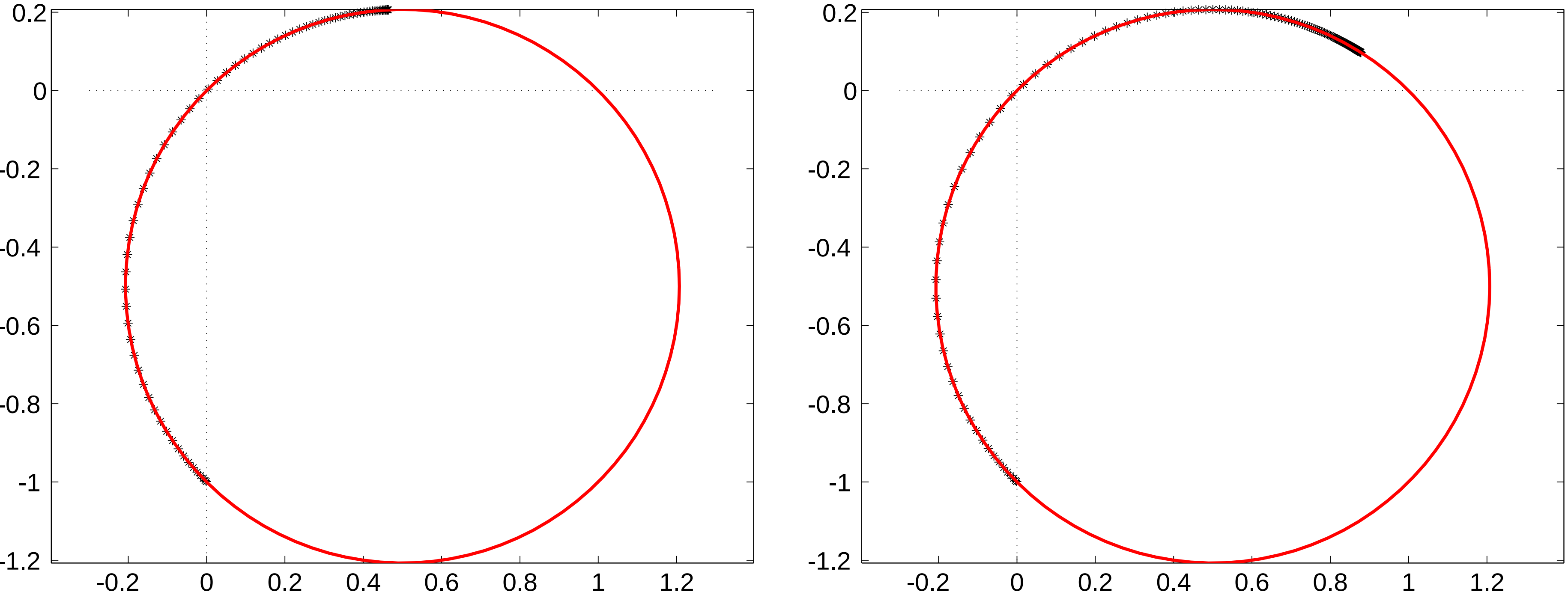
Preconditioner: Shifted-Laplace operator, discretized using the same method as the Helmholtz operator.

k	ILU(0.01)	M_0	M_1	M_i
5	9	13	13	13
10	25	29	28	22
15	47	114	45	26
20	82	354	85	34
30	139	> 1000	150	52

Eigenvalues for Complex preco $k = 100$

spectrum is independent of the grid size

75 grid points | 150 grid points



Inner iteration

Possible solvers for solution of $Mz = r$:

- ILU approximation of M
- inner iteration with ILU as preconditioner
- Multigrid

Multigrid components

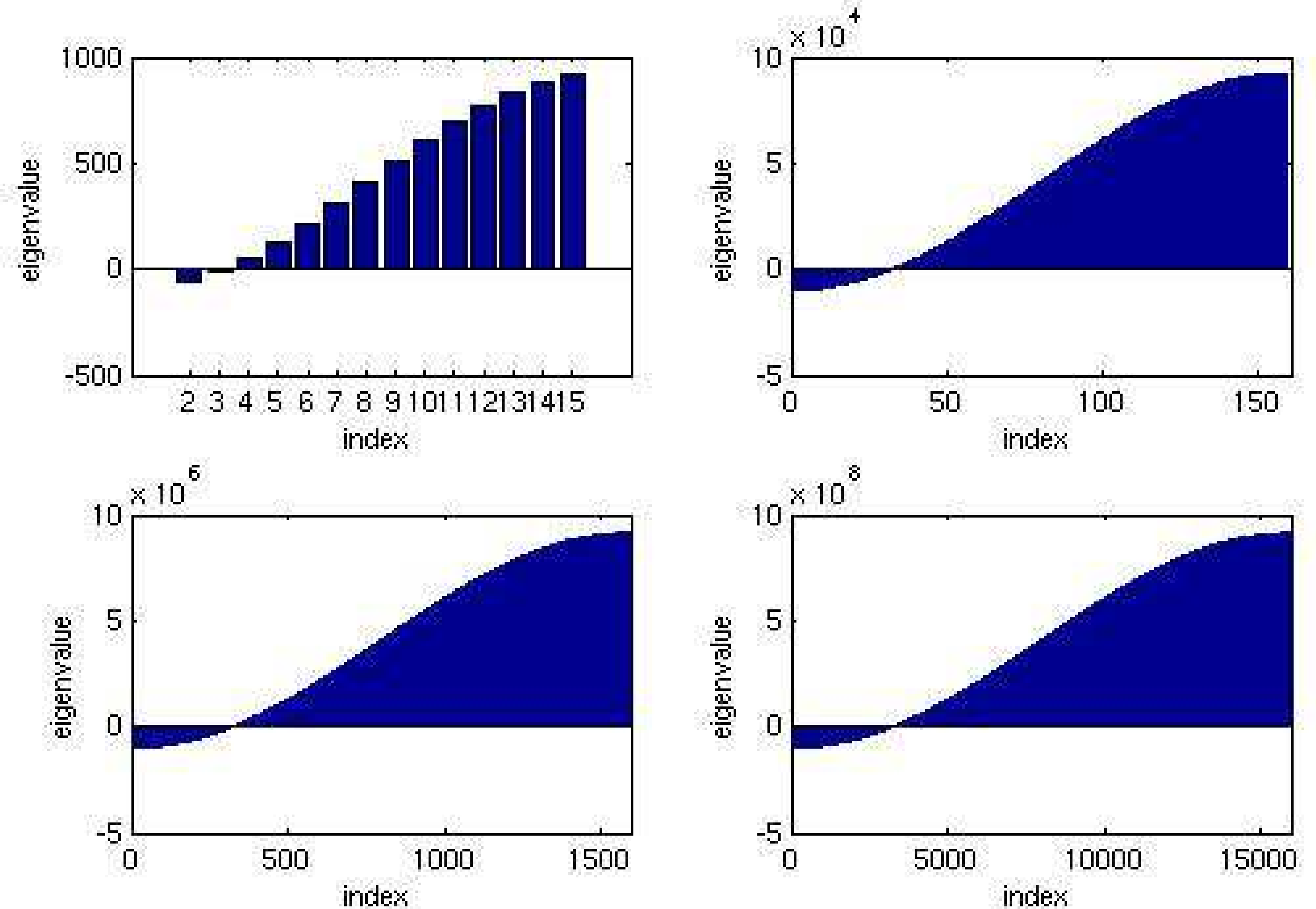
- geometric multigrid
- Gauss-Seidel with red-black ordering
- matrix dependent interpolation, full weighting restriction

- Galerkin coarse grid approximation

Numerical results for a wedge problem

k_2	10	20	40	50	100
grid	32^2	64^2	128^2	192^2	384^2
No-Prec	201(0.56)	1028(12)	5170(316)	–	–
ILU(A,0)	55(0.36)	348(9)	1484(131)	2344(498)	–
ILU(A,1)	26(0.14)	126(4)	577(62)	894(207)	–
ILU(M,0)	57(0.29)	213(8)	1289(122)	2072(451)	–
ILU(M,1)	28(0.28)	116(4)	443(48)	763(191)	2021(1875)
MG(V(1,1))	13(0.21)	38(3)	94(28)	115(82)	252(850)

Spectrum as function of k



3. Second Level Preconditioning

Deflation (or two-grid method), a projection preconditioner

$$P = I - AQ, \quad \text{with} \quad Q = ZE^{-1}Z^T \text{ and } E = Z^TAZ$$

where,

$$Z \in R^{n\times r}, \quad \text{with deflation vectors} \quad Z = [z_1, ..., z_r], \quad rank(Z) = r \leq n$$

Along with a traditional preconditioner M, deflated preconditioned system reads

$$PM^{-1}Au = PM^{-1}g.$$

Deflation vectors shifted the eigenvalues to zero.

Deflation for Helmholtz

With choice of multigrid inter-grid transfer operator (Prolongation) as deflation matrix, i.e. $Z = I_h^{2h}$ and $Z^T = I_{2h}^h$ then

$$P_h = I_h - A_h Q_h, \quad \text{with} \quad Q_h = I_h^{2h} A_{2h}^{-1} I_{2h}^h \text{ and } A_{2h} = I_{2h}^h A_h I_h^{2h}$$

where

P_h can be interpreted as a coarse grid correction and

Q_h as the coarse grid operator

Deflation: ADEF1

Deflation can be implemented combined with SLP M_h,

$$M_h^{-1} P_h A_h u_h = M_h^{-1} P_h g_h$$

$A_h u_h = g_h$ is preconditioned by the two-level preconditioner $M_h^{-1} P_h$.

For large problems, A_{2h} is too large to invert exactly.
Inversion of A_{2h} is sensitive, since P_h deflates the spectrum to zero.

To do: Solve A_{2h} iteratively to a required accuracy on certain levels, and shift the deflated spectrum to λ_h^{max} by adding a shift in the two level preconditioner. This leads to the **ADEF1** preconditioner

$$P_{(h,ADEF1)} = M_h^{-1} P_h + \lambda_h^{max} Q_h$$

Deflation: MLKM

Multi Level Krylov Method [a], take $\hat{A}_h = M_h^{-1} A_h$, and define $\hat{P}_h$ by using $\hat{A}_h$ (instead of A_h) will be

$$\hat{P}_h = I_h - \hat{A}_h \hat{Q}_h,$$

where

$$\hat{Q}_h = I_h^{2h} \hat{A}_{2h}^{-1} I_{2h}^h \text{ and } \hat{A}_{2h} = I_{2h}^h \hat{A}_h I_h^{2h} = I_{2h}^h (M_h^{-1} A_h) I_h^{2h}$$

Construction of coarse matrix A_{2h} at level $2h$ costs inversion of preconditioner at level h.

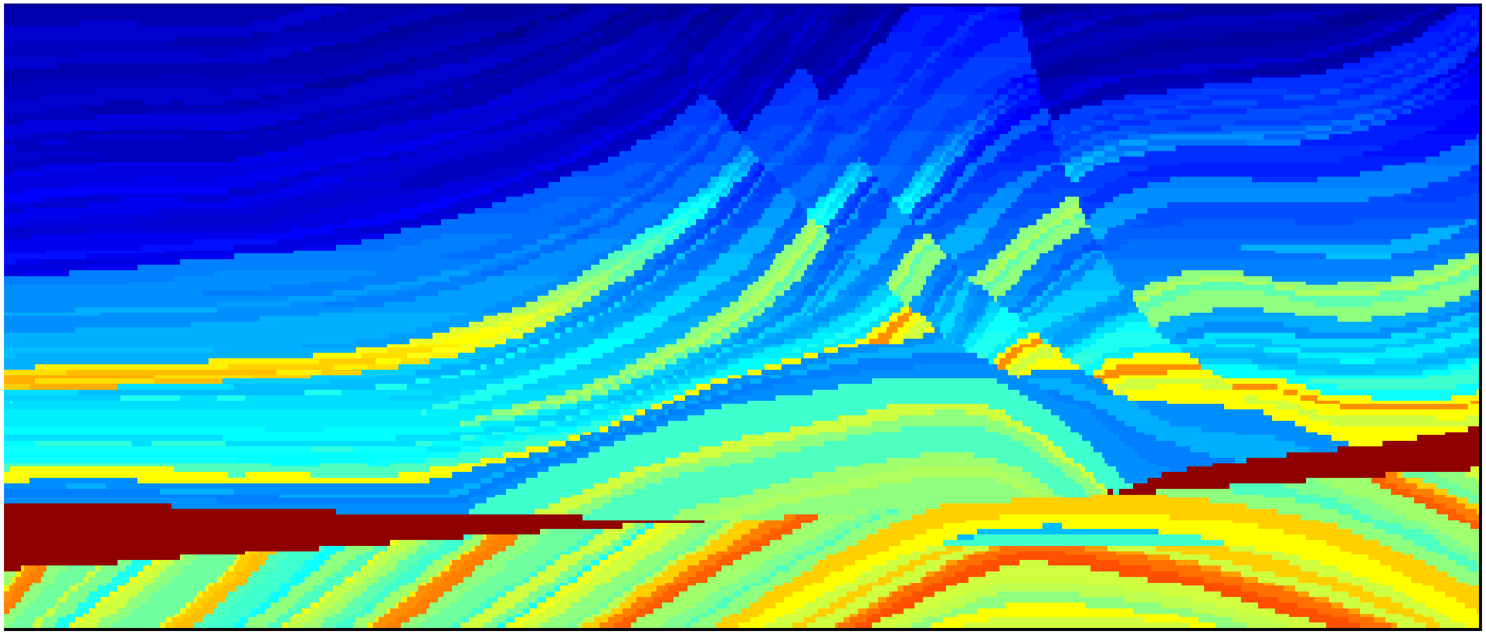
Approximate A_{2h}

Ideal	**Practical**
$\hat{A}_{2h} = I_{2h}^h (M_h^{-1} A_h) I_h^{2h}$	$\hat{A}_{2h} = I_{2h}^h (M_h^{-1} A_h) I_h^{2h}$
	$\hat{A}_{2h} \approx I_{2h}^h I_h^{2h} M_{2h}^{-1} A_{2h}$

[a] Erlangga, Y.A and Nabben R., ETNA 2008

4. Numerical results

hard Marmousi Model



Application: geophysical survey

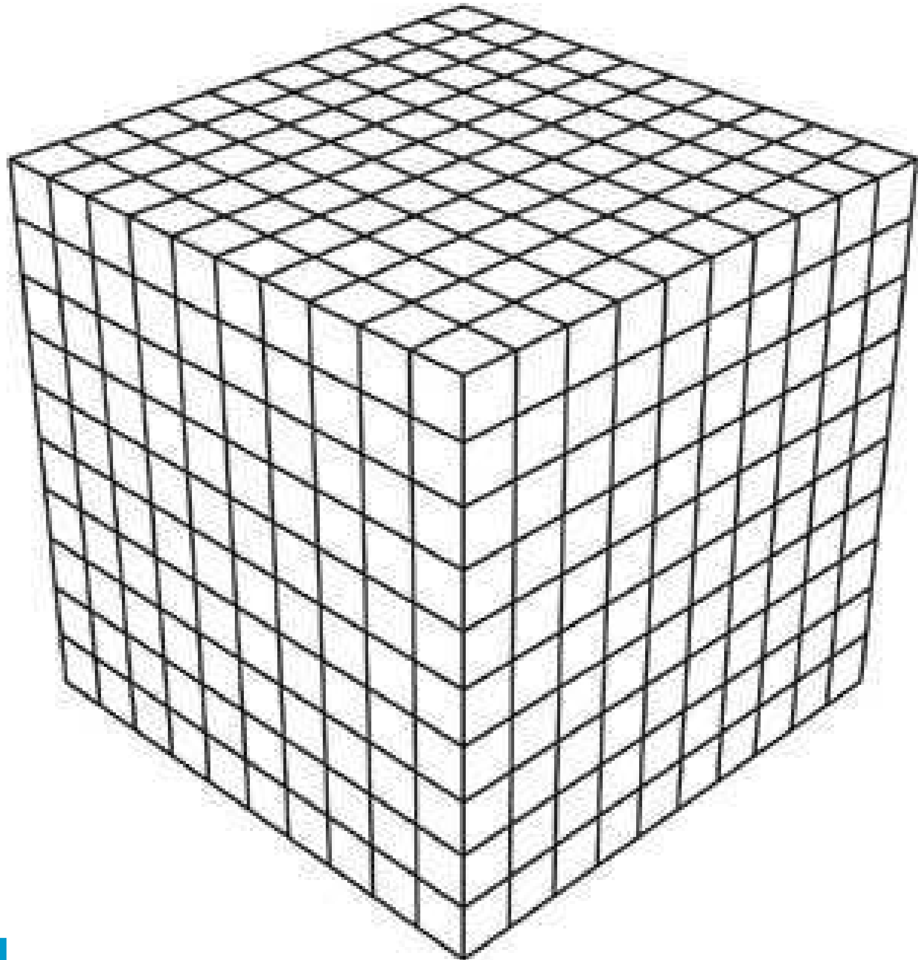
hard Marmousi Model, PETSc solver

$kh = 0.39$, Bi-CGSTAB for SLP, FGMRES(20) for ADEF1(8,2,1)

Frequency f	Solve Time		Iterations	
	SLP-F	ADEF1-F	SLP-F	ADEF1-F
1	1.22	5.07	13	7
10	10.18	9.43	112	13
20	72.16	60.32	189	22
40	550.20	426.79	354	39

Application: geophysical survey

Cube with constant k

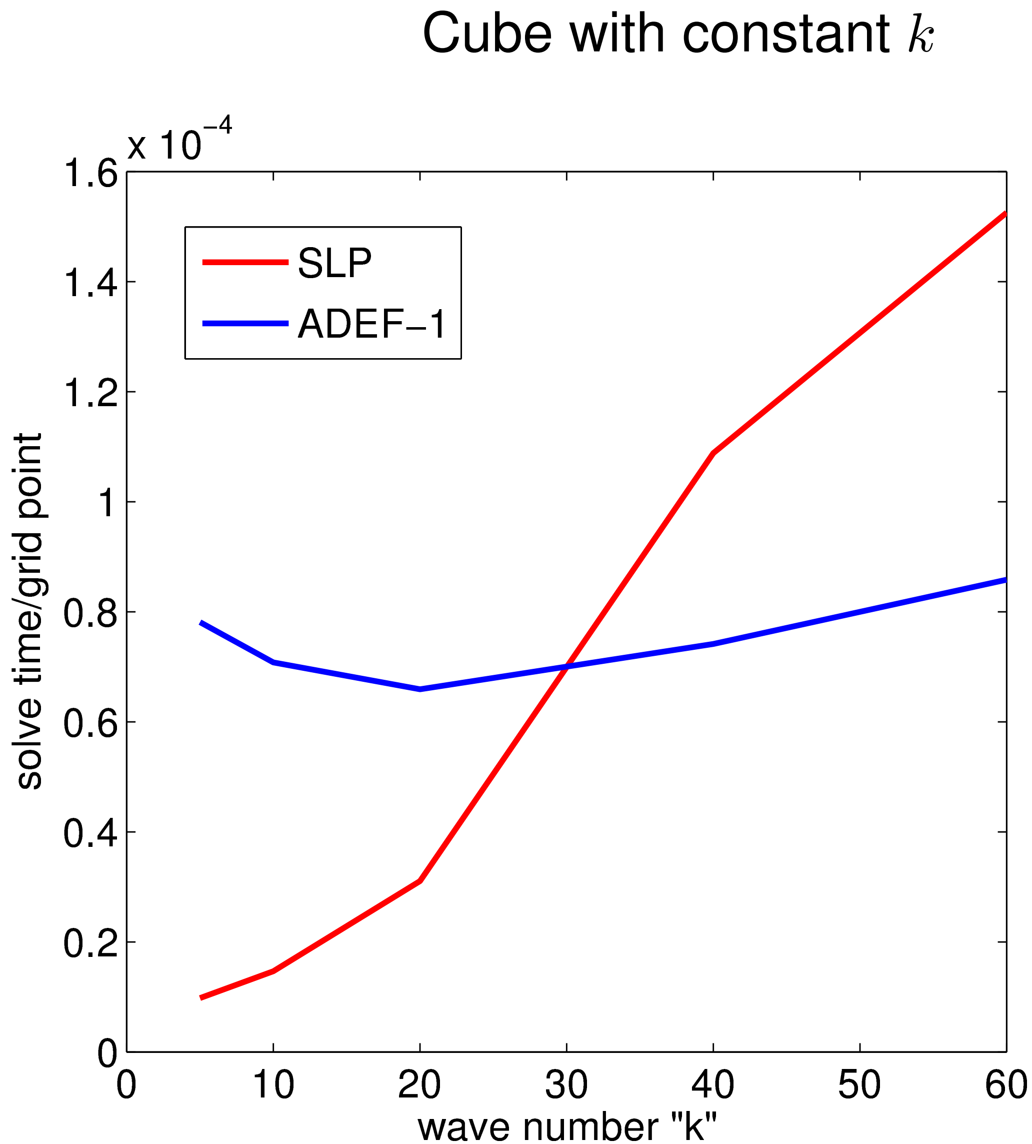


Application: geophysical survey

Cube with constant k

Wave number	Solve Time		Iterations	
k	SLP-F	ADEF1-F	SLP-F	ADEF1-F
5	0.04	0.32	7	8
10	0.48	2.32	9	9
20	8.14	17.28	20	9
40	228.29	155.52	70	10
60	1079.99	607.45	97	11

Application: geophysical survey



Application: geophysical survey

Cube with variable k

k	CLSP(time)	ADEF1(time)	CLSP	ADEF1
5	0.09	0.24	9	11
10	1.07	1.94	15	12
20	16.7	18.9	32	16
40	1304	214	331	24

5. Conclusions

- Without deflation, when imaginary shift is increased in SLP, spectrum remains bounded above 1, but lower part moves to zero.
- With deflation the convergence is nearly independent of the imaginary shift.
- With deflation the convergence is initially weakly depending on k. For very large k it scales again linearly.
- With deflation the CPU time is less than without deflation.
- The convergence of ADEF1 and the practical variant of MLKM are similar.

References

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- http://ta.twi.tudelft.nl/nw/users/vuik/numanal/sheikh.pdf Thesis

  http://ahsheikh.github.io/adef1.html Software

  http://ta.twi.tudelft.nl/nw/users/vuik/pub_it_helmholtz.html Publications

Fourier Analysis of two-level methods

Dirichlet boundary conditions for analysis.
With above deflation,

$$\textbf{spec}(PM^{-1}A) = f(\beta_1, \beta_2, k, h)$$

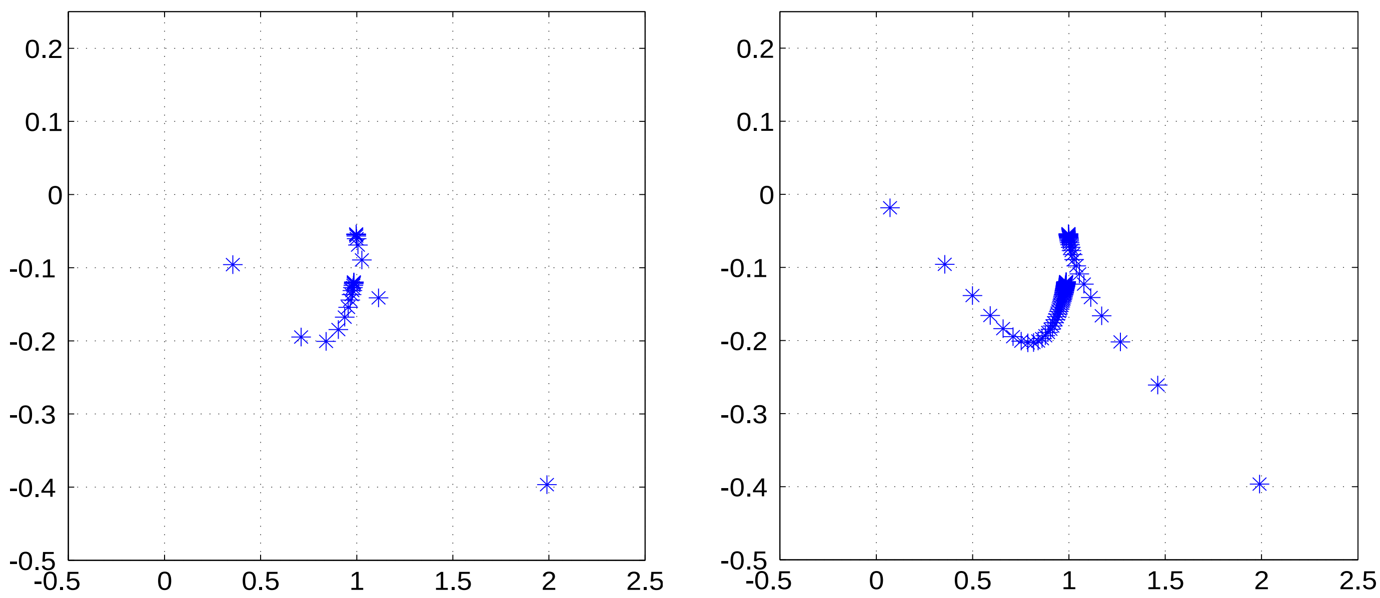
is a complex valued function.
Setting $kh = 0.625$,

- Spectrum of $PM^{-1}A$ with shifts $(1, 0.5)$ is clustered around 1 with a few outliers.
- Spectrum remains almost the same, when the imaginary shift for the preconditioner is varied from 0.5 to 1.

Fourier Analysis

ADEF1: Analysis shows spectrum clustered around 1 with few outliers.

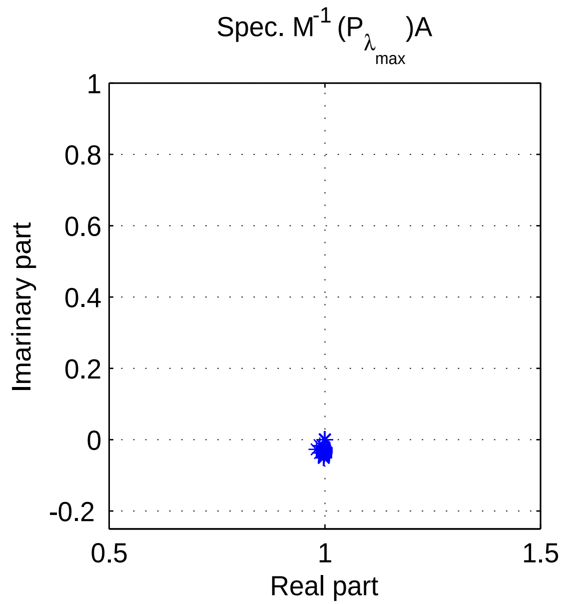
$k = 30$ $\quad$ $k = 120$



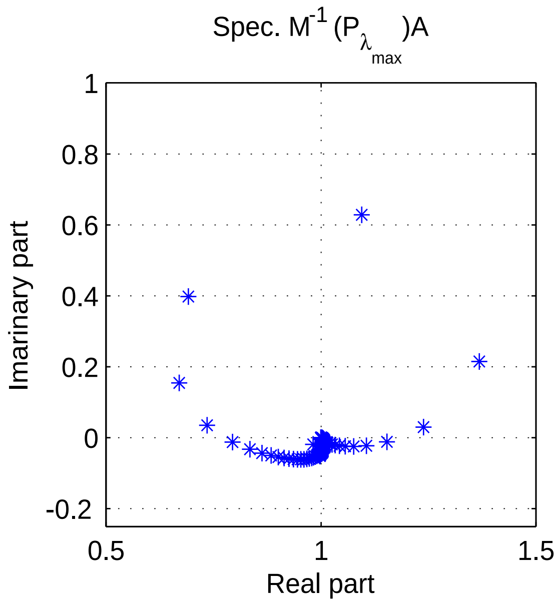
Fourier Analysis

Spectrum of Helmholtz preconditioned by MLKM [b],
$k = 160$ and 20 gp/wl

Ideal



Practical



[b] Two-level