Physics-Informed Machine Learning Embedded Into Isogeometric Analysis

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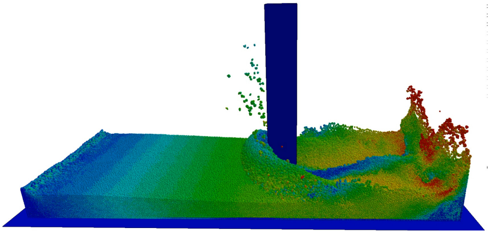
Lorentz Center Workshop
"Computational Mathematics and Machine Learning"

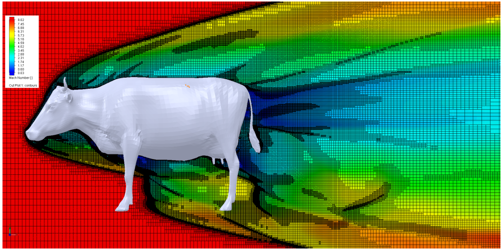
1st November 2021

Life before PINNs

Simulation-based analysis of PDEs with numerical methods has a long tradition

- particle methods: PIC (1955), SPH (1977), DPD (1992), RKPM (1995), ...
- hybrid particle-mesh methods: MPM (1990s), ...
- mesh-based methods: FEM (1940s), FDM (1950s), FVM (1971), IGA (2005), ...





Left: wave-structure interaction, LS-DYNA; right: supersonic flow around a cow, Siemens FloEFD

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  - a priori/ a posteriori error estimates, practical error indicators
  - strategies for adaptive hp-mesh refinement

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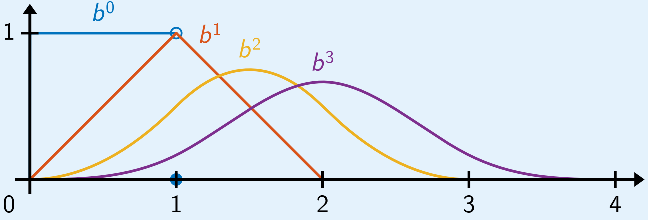
    - strategies for adaptive hp-mesh refinement
    - unified framework for computer-aided design and finite element analysis

B-spline basis functions



knot vector $\Xi = [0, 1, 2, 3, 4]$

$$b_\ell^0(\xi) = \begin{cases} 1 & \text{if } \xi_\ell \leq \xi < \xi_{\ell+1} \\ 0 & \text{otherwise} \end{cases}$$

$$b_\ell^p(\xi) = \frac{\xi - \xi_\ell}{\xi_{\ell+p} - \xi_\ell} b_\ell^{p-1}(\xi) + \frac{\xi_{\ell+p+1} - \xi}{\xi_{\ell+p+1} - \xi_{\ell+1}} b_{\ell+1}^{p-1}(\xi)$$

T.J.R. Hughes, J.A.Cottrell, Y.Bazilevs: Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement. CMAME 194(39–41), 2005.

Isogeometric Analysis

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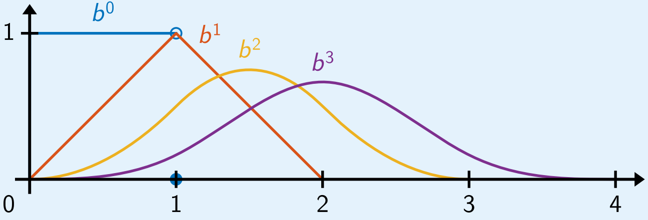

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Many good properties: compact support $[\xi_\ell, \xi_{\ell+p+1})$, positive function values over support interval, derivatives of B-splines are combinations of lower-order B-splines, ...

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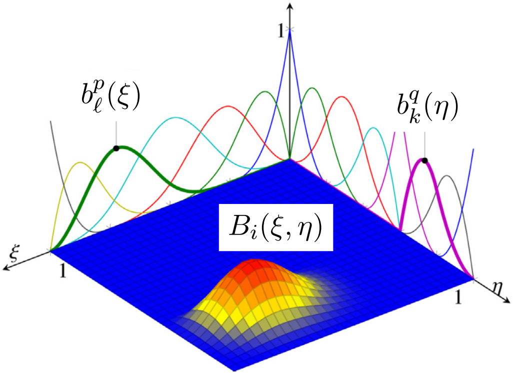
Isogeometric Analysis

Paradigm: represent 'everything' in terms of tensor products of B-spline basis functions

$$B_i(\xi,\eta) := b_\ell^p(\xi)\cdot b_k^q(\eta), \qquad i := (k-1)\cdot n_\ell + \ell, \quad 1\le \ell \le n_\ell, \quad 1 \le k \le n_k,$$



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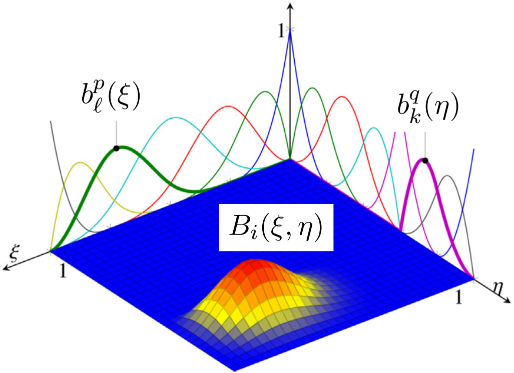
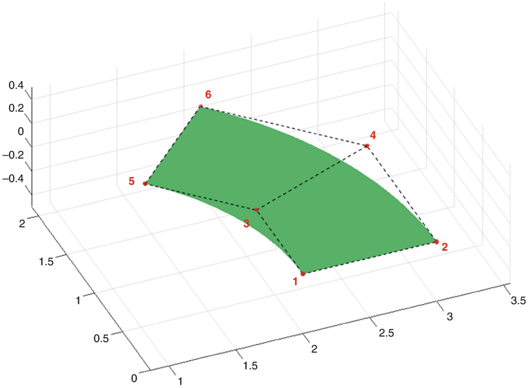


Many more good properties: partition of unity $\sum_{i=1}^{n} B_i(\xi,\eta) \equiv 1$, C^{p-1} continuity, ...

Isogeometric Analysis

Geometry: bijective mapping from the unit square to the physical domain $\Omega_h \subset \mathbb{R}^d$

$$\mathbf{x}_h(\xi,\eta) = \sum_{i=1}^{n} B_i(\xi,\eta) \cdot \mathbf{x}_i \qquad \forall(\xi,\eta) \in [0,1]^2 =: \hat{\Omega}$$



- the shape of Ω_h is fully specified by the set of **control points** $\mathbf{x}_i \in \mathbb{R}^d$

Isogeometric Analysis

Geometry: bijective mapping from the unit square to the physical domain $\Omega_h \subset \mathbb{R}^d$

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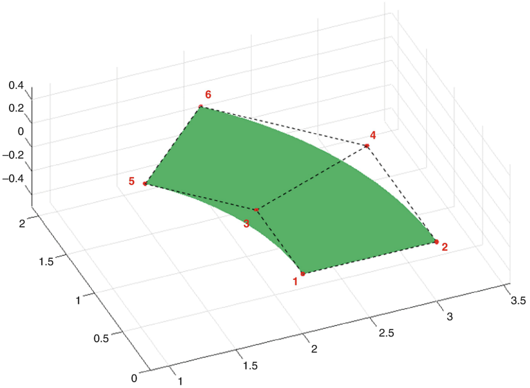

- the shape of Ω_h is fully specified by the set of **control points** $\mathbf{x}_i \in \mathbb{R}^d$
- interior control points must be chosen such that 'grid lines' do not fold as this violates the bijectivity of $\mathbf{x}_h : \hat{\Omega} \to \Omega_h$

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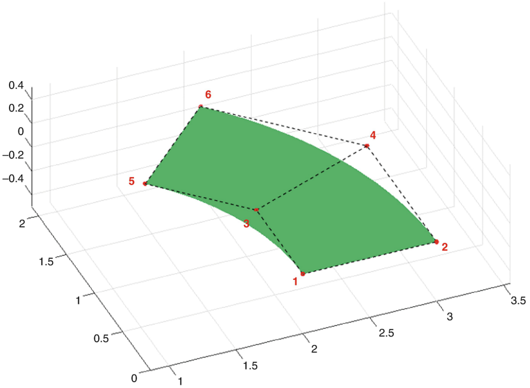

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- interior control points must be chosen such that 'grid lines' do not fold as this violates the bijectivity of $\mathbf{x}_h : \hat{\Omega} \to \Omega_h$
- refinement in h (knot insertion) and p (order elevation) preserves the shape of Ω_h and can be used to generate finer computational 'grids' for the analysis

Isogeometric Analysis

Data, boundary conditions, and solution: forward mappings from the unit square

(r.h.s vector) $$f_h \circ \mathbf{x}_h(\xi,\eta) = \sum_{i=1}^{n} B_i(\xi,\eta)\cdot f_i \qquad \forall(\xi,\eta)\in[0,1]^2$$

(boundary conditions) $$g_h \circ \mathbf{x}_h(\xi,\eta) = \sum_{i=1}^{n} B_i(\xi,\eta)\cdot g_i \qquad \forall(\xi,\eta)\in\partial[0,1]^2$$

(solution) $$u_h \circ \mathbf{x}_h(\xi,\eta) = \sum_{i=1}^{n} B_i(\xi,\eta)\cdot u_i \qquad \forall(\xi,\eta)\in[0,1]^2$$

Isogeometric Analysis

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$$\text{(r.h.s vector)} \qquad f_h \circ \mathbf{x}_h(\xi, \eta) = \sum_{i=1}^{n} B_i(\xi, \eta) \cdot f_i \qquad \forall (\xi, \eta) \in [0, 1]^2$$

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$$\text{(solution)} \qquad u_h \circ \mathbf{x}_h(\xi, \eta) = \sum_{i=1}^{n} B_i(\xi, \eta) \cdot u_i \qquad \forall (\xi, \eta) \in [0, 1]^2$$

Model problem: Poisson's equation

$$-\Delta u_h = f_h \quad \text{in} \quad \Omega_h, \qquad u_h = g_h \quad \text{on} \quad \partial\Omega_h$$

Isogeometric Analysis

Different solution approaches

- Galerkin-type IGA (Hughes *et al.* 2005 and many more)
- Isogeometric collocation methods (Reali, Hughes, 2015)
- Variational collocation method (Gomez, De Lorenzis, 2016)

Isogeometric Analysis

Different solution approaches

- Galerkin-type IGA (Hughes *et al.* 2005 and many more)
  - matrix assembly is computational expensive
    $\rightarrow$ weighted quadrature (Calabrò *et al.* 2016),
    $\rightarrow$ low-rank approximation (Mantzaflaris *et al.* 2019), ...
  - condition number of system matrix is exponential in p
    $\rightarrow$ h-multigrid solvers (Hofreither 2016, Takacs *et al.* 2017, de la Riva 2018),
    $\rightarrow$ p-multigrid solver (Tielen *et al.* 2018/20),
    $\rightarrow$ preconditioners (da Veiga *et al.* 2012/13, Sangalli 2016, Tani 2017, Cho 2018/19), ...
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Different solution approaches

- Galerkin-type IGA (Hughes *et al.* 2005 and many more)
- Isogeometric collocation methods (Reali, Hughes, 2015)
- Variational collocation method (Gomez, De Lorenzis, 2016)
    - choice of optimal collocation points is not trivial
    - convergence theory not fully developed (Schillinger *et al.* 2013)

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Abstract representation

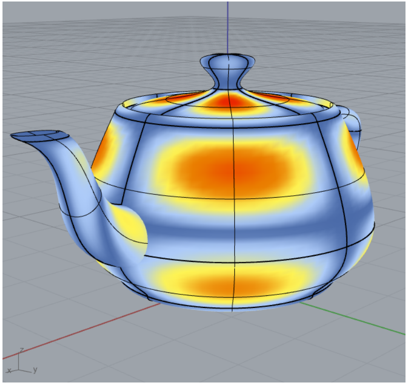
Given $\mathbf{x}_i$ (geometry), f_i (r.h.s. vector), and g_i (boundary conditions), **compute**

$$\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = A^{-1}\left(\begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_n \end{bmatrix}\right) \cdot b\left(\begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_n \end{bmatrix}, \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}, \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}\right)$$

Any point of the solution can afterwards be obtained by a simple **function evaluation**

$$(\xi, \eta) \in [0,1]^2 \quad \mapsto \quad u_h \circ \mathbf{x}_h(\xi, \eta) = [B_1(\xi, \eta), \dots, B_n(\xi, \eta)] \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

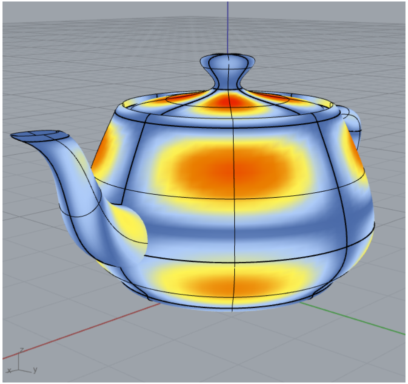
Isogeometric Analysis



- multi-patch IGA for complex geometries
- paradigm can be extended to volumetric splines
- local adaptation using THB- (Gianelli *et al.* 2012), LR- (Dokken *et al.* 2014), T- (Sederberg *et al.* 2004), or U-splines (Scott 2018)
- IGA enables ***integrated*** but ***not (yet) interactive*** **computer-aided design-and-analysis workflows**

CSE-minor BSc project, TU Delft

Isogeometric Analysis



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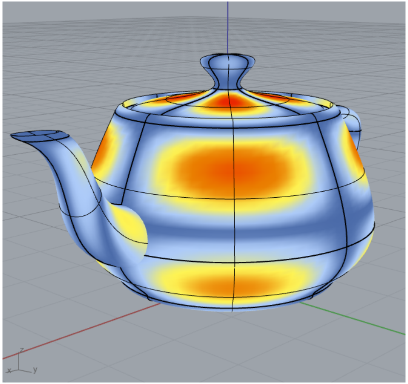
IGA-PINN paradigm: replace **computation** by **physics-informed machine learning**

$$\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = A^{-1}\left(\begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_n \end{bmatrix}\right) \cdot b\left(\begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_n \end{bmatrix}, \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}, \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}\right)$$

Isogeometric Analysis



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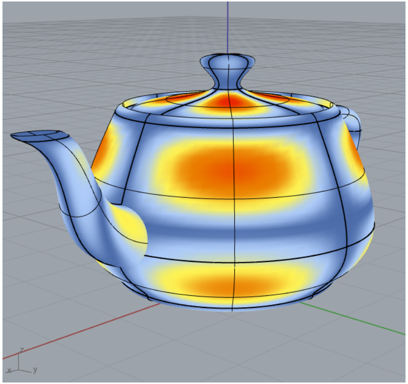
IGA-PINN paradigm: replace **computation** by **physics-informed machine learning**

$$\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \text{PINN}\left(\begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_n \end{bmatrix}, \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}, \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}; (\xi_k, \eta_k)_{k=1}^{N_{\text{samples}}} \right)$$

Isogeometric Analysis



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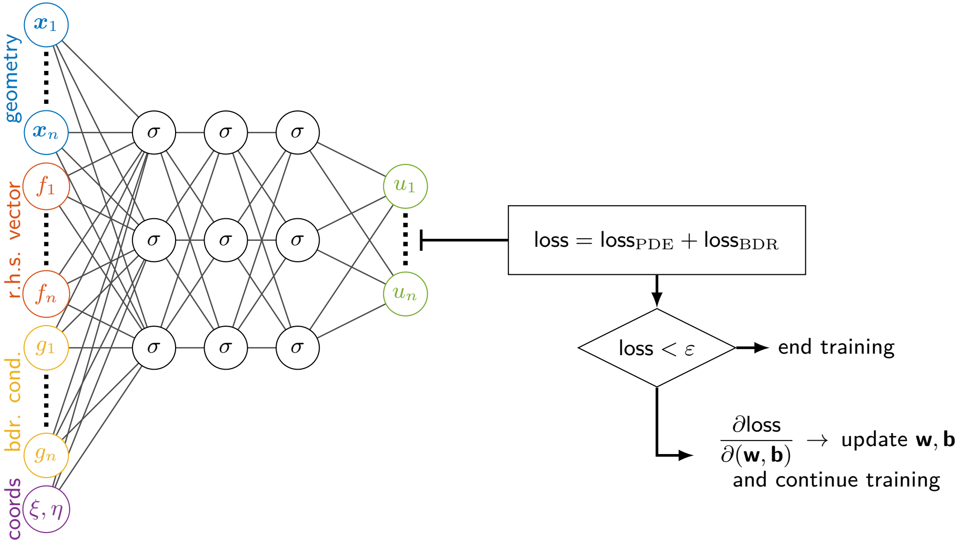
$$u_h(\xi,\eta) \approx [B_1(\xi,\eta),\dots,B_n(\xi,\eta)] \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \text{PINN}\left(\begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_n \end{bmatrix}, \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}, \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}; (\xi,\eta) \right)$$

IGA-PINN


geometry
$\boldsymbol{x}_1$
$\boldsymbol{x}_n$
r.h.s. vector
f_1
f_n
bdr. cond.
g_1
g_n
coords
ξ, η
σ
u_1
u_n
loss = loss$_{\mathrm{PDE}}$ + loss$_{\mathrm{BDR}}$
loss $< \varepsilon$
end training
$\frac{\partial \text{loss}}{\partial(\mathbf{w}, \mathbf{b})} \rightarrow$ update $\mathbf{w}, \mathbf{b}$
and continue training

Loss function

$$\text{loss}_{\mathrm{PDE}} = \frac{\alpha}{N_\Omega} \sum_{k=1}^{N_\Omega} \left| \Delta[u_h \circ \mathbf{x}_h(\xi_k, \eta_k)] - f_h \circ \mathbf{x}_h(\xi_k, \eta_k) \right|^2$$

$$\text{loss}_{\mathrm{BDR}} = \frac{\beta}{N_\Gamma} \sum_{k=1}^{N_\Gamma} \left| u_h \circ \mathbf{x}_h(\xi_k, \eta_k) - g_h \circ \mathbf{x}_h(\xi_k, \eta_k) \right|^2$$

Express derivatives with respect to physical space variables using the Jacobian J, the Hessian H and the matrix of squared first derivatives Q (Schillinger *et al.* 2013):

$$\begin{bmatrix} \frac{\partial^2 B}{\partial x^2} \\ \frac{\partial^2 B}{\partial x \partial y} \\ \frac{\partial^2 B}{\partial y^2} \end{bmatrix} = Q^{-\top} \left(\begin{bmatrix} \frac{\partial^2 B}{\partial \xi^2} \\ \frac{\partial^2 B}{\partial \xi \partial \eta} \\ \frac{\partial^2 B}{\partial \eta^2} \end{bmatrix} - H^\top J^{-\top} \begin{bmatrix} \frac{\partial B}{\partial \xi} \\ \frac{\partial B}{\partial \eta} \end{bmatrix} \right)$$

Two-level training strategy

For $[\mathbf{x}_1, \dots, \mathbf{x}_n] \in \mathcal{S}_{\text{geo}}$, $[f_1, \dots, f_n] \in \mathcal{S}_{\text{rhs}}$, $[g_1, \dots, g_n] \in \mathcal{S}_{\text{bcond}}$ **do**

For a batch of randomly sampled $(\xi_k, \eta_k) \in [0,1]^2$ **do**

$$\text{Train PINN} \left(\begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_n \end{bmatrix}, \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}, \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}; (\xi_k, \eta_k)_{k=1}^{N_{\text{samples}}} \right) \mapsto \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

EndFor

EndFor

IGA details: 7×7 bi-cubic tensor-product B-splines for $\mathbf{x}_h$ and u_h, C^2-continuous

PINN details: TensorFlow 2.6, 7-layer neural network with 50 neurons per layer and ReLU activation function (except for output layer), Adam optimizer, 30.000 epochs, training is stopped after 3.000 epochs w/o improvement of the loss value

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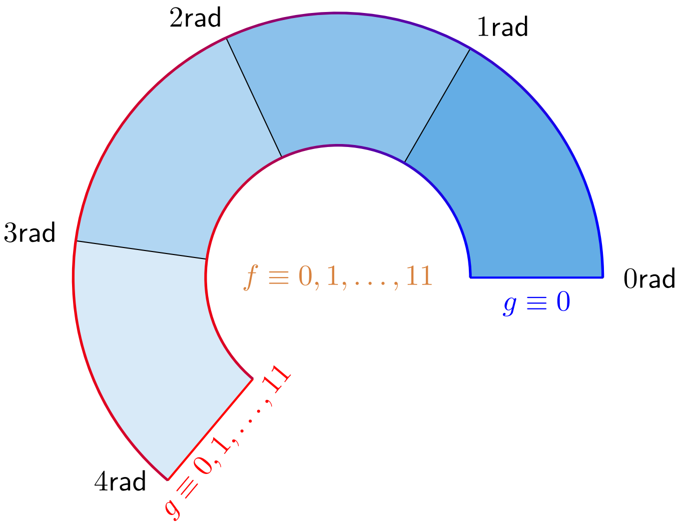
Test case: Poisson's equation on a variable annulus



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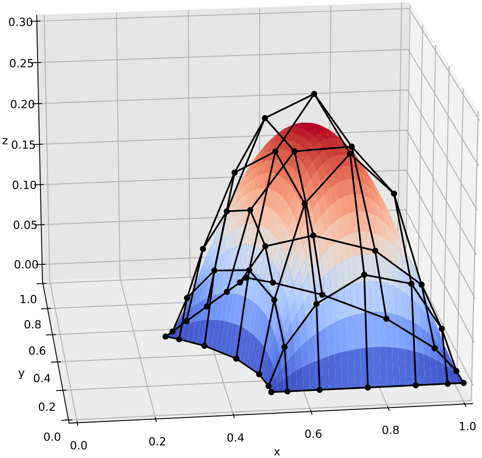
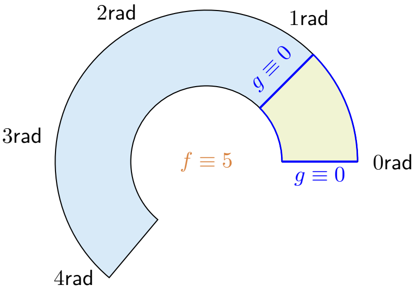
Preliminary results





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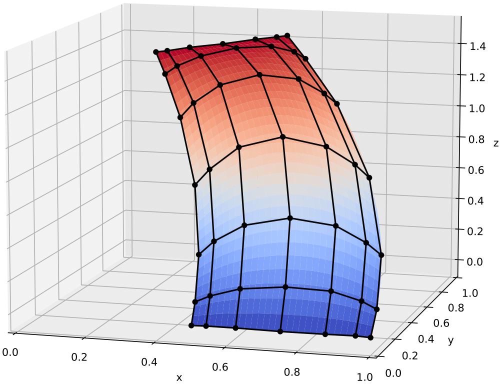

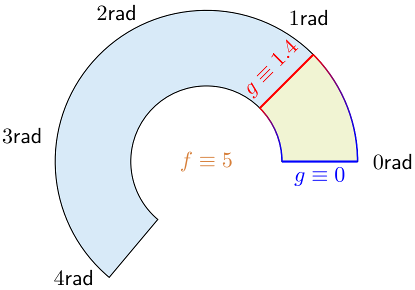

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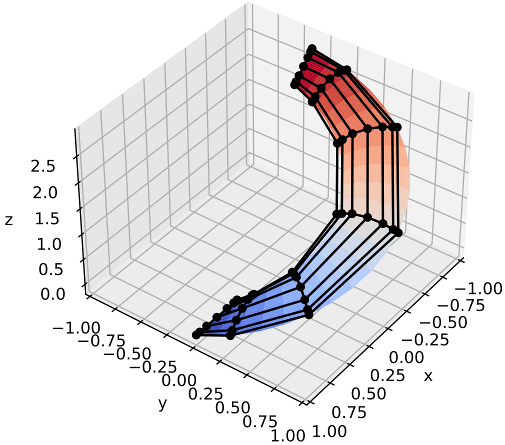

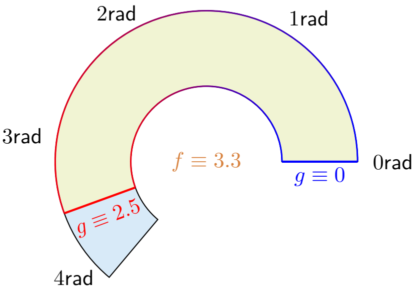

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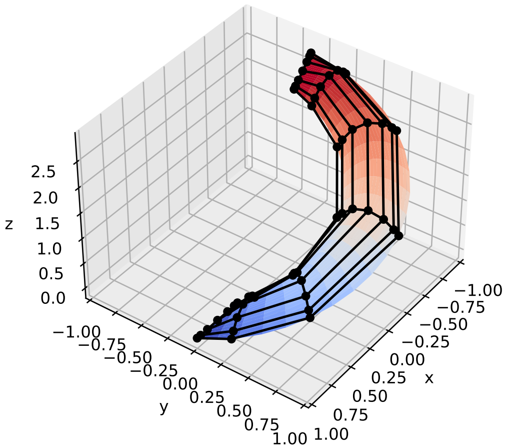

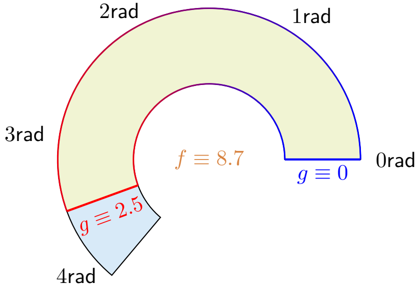

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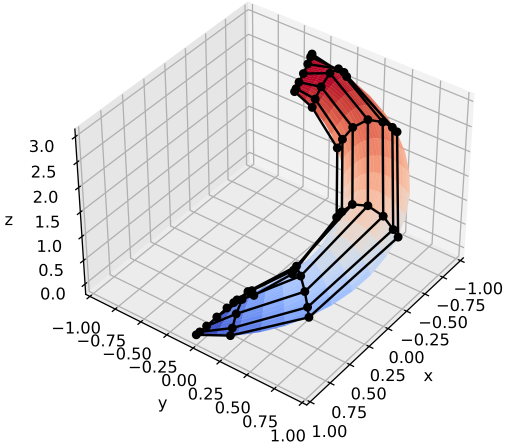

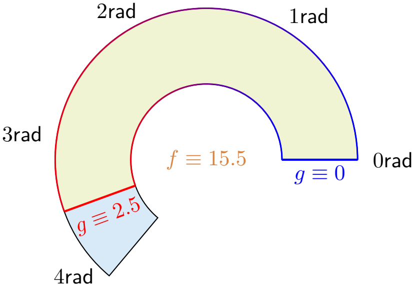

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Conclusion and outlook

IGA-PINNs combine the best of both worlds and may finally enable
integrated and interactive computer-aided design-and-analysis workflows

Todo

- performance and hyper-parameter tuning
- extension to multi-patch topologies
- use of IGA and IGA-PINNs in concert
- transfer learning upon refinement of basis functions

Thank you for your attention!