On the design of high-resolution finite element schemes for coupled problems with application to an idealized Z-pincho implosion model

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Motivation: *Z-pincho implosion*

- Phenomenological model by Banks and Shadid
  
  *compressible Euler equations + source term*

  *coupled with tracer equation for Lorentz force*

- Mathematical challenges
  
  *high-resolution scheme for time-dependent conservation laws;*
  
  *positivity-preservation of density and pressure; failsafe strategy*
High-resolution schemes for $\partial_t U + \nabla \cdot F = 0$

- **High-order scheme**
  \[ MC \frac{dU^H}{dt} = KU^H \]

- **Low-order scheme**
  \[ ML \frac{dU^L}{dt} = LU^L, \quad L = K + D \]

- **Predictor:** Compute low-order solution
  \[ ML \frac{dU^L}{dt} = LU^L \Rightarrow \dot{U}^L \approx ML^{-1} LU^L \]

- **Corrector:** Apply limited antidiffusion
  \[ MLU = MLU^L + \overline{F}, \quad F = [ML - MC] \dot{U}^L - DU^L \]
High-resolution schemes for \( \partial_t U + \nabla \cdot F = 0 \)

- **High-order scheme**
  \[ M_C \frac{dU^H}{dt} = KU^H \]

- **Low-order scheme**
  \[ M_L \frac{dU^L}{dt} = LU^L, \quad L = K + D \]

- **Predictor:** Compute low-order solution
  \[ M_L \frac{dU^L}{dt} = LU^L \quad \Rightarrow \quad \dot{U}^L \approx M_L^{-1} LU^L \]

- **Corrector:** Apply limited antidiffusion
  \[ M_L U = M_L U^L + \overline{F}, \quad F = [M_L - M_C]\dot{U}^L - DU^L \]

⚠️ Low-order scheme must satisfy physical constraints
Perform flux correction such that mass is conserved.
Design principles of FCT schemes

Perform flux correction such that \textit{mass is conserved}.

- Conservative flux decomposition

\[ m_i U_i = m_i U_i^L + \sum_{j \neq i} F_{ij}, \quad F_{ji} = -F_{ij} \]
Design principles of FCT schemes

Perform flux correction such that mass is conserved.

- Conservative flux decomposition and limiting

\[ m_i U_i = m_i U_i^L + \sum_{j \neq i} \alpha_{ij} F_{ij}, \quad F_{ji} = -F_{ij}, \quad \alpha_{ji} = \alpha_{ij} \]

- high-order approximation \((\alpha_{ij} = 1)\) to be used in smooth regions
- low-order approximation \((\alpha_{ij} = 0)\) to be used near steep fronts
Design principles of FCT schemes, cont’d

Perform flux correction such that certain physical quantities are bounded by the local extrema of the low-order solution.
Perform flux correction such that certain **physical quantities**

*are bounded* by the local extrema of the low-order solution.

\[
    u_i^{\min} \leq u_i^L + \frac{1}{m_i} \sum_{j \neq i} \alpha_{ij}^u f_{ij}^u \leq u_i^{\max}
\]
Fixed fraction flux limiter

- Set \( \alpha_{ij}^{(0)} := 1 \) and repeat \( r = 1, \ldots, R \)

- Mark all nodes \( i \) that violate the local FCT constraint

\[
 u_{i\min} \leq u_i^L + \frac{1}{m_i} \sum_{j \neq i} \alpha_{ij}^{(r-1)} f_{ij} u_{ij} \leq u_{i\max}
\]

- Eliminate fixed fraction of unacceptable antidiffusion

\[
 \alpha_{ij}^{(r)} := \begin{cases} 
 1 - r/R & \text{if node } i \text{ or } j \text{ is marked} \\
 \alpha_{ij}^{(r-1)} & \text{otherwise}
\end{cases}
\]

Low-order solution is recovered in the worst case: \( U_i = U_i^L \)
Zalesak’s flux limiter

Consider positive/negative anti-diffusive contributions separately

\[ P_i^+ + P_i^- \]

\[ R_i^+ = \min \{ 1, Q_i^+ / P_i^+ \} \]

\[ R_i^- = \min \{ 1, Q_i^- / P_i^- \} \]

Limit antidiffusive flux for edge \( ij \) by the minimum of \( R_i \) and \( R_j \).
Zalesak’s flux limiter

- Consider positive/negative antidiffusive contributions separately
- Limit antidiffusion if it exceeds the distance to local maximum/minimum
Zalesak’s flux limiter

- Consider positive/negative antidiffusive contributions separately
- Limit antidiffusion if it exceeds the distance to local maximum/minimum

Nodal correction factors

\[ R_i^+ = \min\{1, \frac{Q_i^+}{P_i^+}\} \quad \text{for positive fluxes into node } i \]
\[ R_i^- = \min\{1, \frac{Q_i^-}{P_i^-}\} \quad \text{for negative fluxes into node } i \]
Zalesak’s flux limiter

- Consider positive/negative anti-diffusive contributions separately
- Limit antidiffusion if it exceeds the distance to local maximum/minimum

Nodal correction factors

\[ R_i^+ = \min\{1, \frac{Q_i^+}{P_i^+}\} \quad \text{for positive fluxes into node } i \]
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- Limit antidiffusive flux for edge \(ij\) by the minimum of \(R_i\) and \(R_j\)
Flux limiting for systems

Apply flux limiter to a set of control variables, e.g.,
the primitive variables density, pressure and velocity.
Flux limiting for systems

Apply flux limiter to a **set of control variables**, e.g.,
the primitive variables density, pressure and velocity.

\[
\begin{align*}
 f_{ij}^\rho & \quad \rightarrow & \quad \text{FCT} \\
 f_{ij}^p & \quad \rightarrow & \quad \text{FCT} \\
 f_{ij}^v & \quad \rightarrow & \quad \text{FCT} \\
 \min\{\alpha_{ij}^\rho, \alpha_{ij}^p, \alpha_{ij}^v\} & \quad \rightarrow & \quad \text{FCT}
\end{align*}
\]
Flux limiting for systems

Apply flux limiter to a set of control variables, e.g.,
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Flux limiting for systems

Apply flux limiter to a set of control variables, e.g.,
the primitive variables density, pressure and velocity.

Nodal transformation of variables $V_i = \mathcal{T}(U_i)U_i$ and $G_{ij} = \mathcal{T}(U_i)F_{ij}$
Failsafe flux correction algorithm

1. Compute low-order solution at time $t^{n+1}$

$$M_L \frac{U_L^n - U^n}{\Delta t} = \theta LU_L^n + (1 - \theta) LU^n$$

2. Perform flux correction by Zalesak’s limiter

$$m_i U_i^{(0)} = m_i U_i^L + \sum_{j \neq i} \alpha_{ij} F_{ij}$$

3. Eliminate spurious undershoots/overshoots

$$m_i U_i^{(r)} = m_i U_i^L + \sum_{j \neq i} \alpha_{ij}^{(r)} [\alpha_{ij} F_{ij}]$$
Double Mach reflection

From: P.R. Woodward and P. Colella, JCP 54, 115 (1984)

\[ U_L = \begin{bmatrix} 8.0 \\ 8.25 \cos 30^\circ \\ -8.25 \sin 30^\circ \\ 116.5 \end{bmatrix} \]

\[ U_R = \begin{bmatrix} 1.4 \\ 0.0 \\ 0.0 \\ 1.0 \end{bmatrix} \]
Double Mach reflection, cont’d

Solution at $T = 0.2$ computed by low-order scheme ($\alpha_{ij} \equiv 0$)

-密度分布

$\Delta t = 1 \cdot 10^{-4}$

$h = 1/64$

$\Delta t = 5 \cdot 10^{-5}$

$h = 1/128$
Double Mach reflection, cont’d

Solution at $T = 0.2$ computed by fixed fraction flux limiter

density distribution

$h = \frac{1}{64}$
$\Delta t = 1 \cdot 10^{-4}$

density distribution

$h = \frac{1}{128}$
$\Delta t = 5 \cdot 10^{-5}$
Double Mach reflection, cont’d

Solution at $T = 0.2$ computed by Zalesak’s flux limiter ($\alpha_{ij} = \alpha^p_{ij} \alpha^p_{ij}$)

- $h = 1/64$
- $\Delta t = 1 \cdot 10^{-4}$

- $h = 1/128$
- $\Delta t = 5 \cdot 10^{-5}$
Double Mach reflection, cont’d

Solution at $T = 0.2$ computed by Zalesak’s flux limiter ($\alpha_{ij} = \alpha_{ij}^p \alpha_{ij}^\rho$)

Generalized Euler system coupled with scalar tracer equation

\[
\frac{\partial}{\partial t} \begin{bmatrix}
\rho \\
\rho \mathbf{v} \\
\rho E \\
\rho \lambda
\end{bmatrix} + \nabla \cdot \begin{bmatrix}
\rho \mathbf{v} \\
\rho \mathbf{v} \otimes \mathbf{v} + p \mathbf{I} \\
\rho E \mathbf{v} + p \mathbf{v} \\
\rho \lambda \mathbf{v}
\end{bmatrix} = \begin{bmatrix}
0 \\
f \\
f \cdot \mathbf{v} \\
0
\end{bmatrix}
\]

Equation of state

\[p = (\gamma - 1) \rho (E - 0.5|\mathbf{v}|^2)\]

Non-dimensional Lorentz force

\[\mathbf{f} = (\rho \lambda) \left( \frac{I(t)}{I_{\text{max}}} \right)^2 \frac{\hat{e}_r}{r_{\text{eff}}}\]
Coupled solution algorithm

Given: \((U, \rho \lambda)^n\)

Low-order scheme

\[ M_L \frac{dU}{dt} = L(U)U + S(v, \rho \lambda) \]

\[ M_L \frac{d(\rho \lambda)}{dt} = L(v)(\rho \lambda) \]

Zalesak’s limiter \(\alpha_{ij} = \alpha_{ij}^p \alpha_{ij}^L\) [+ failsafe correction]

\( (U, \rho \lambda)^{n+1} \)
Idealized Z-pinch implosion

From: J.W. Banks and J.N. Shadid, JCP 61, 725 (2009)

\[ \rho = 10^6 \]
\[ \lambda = 1.0 \]
\[ \rho = 1.0 \]
\[ \lambda = 0.0 \]
\[ \rho = 0.5 \]
\[ \lambda = 0.0 \]

\( v = 0.0, \ p = 1.0 \) everywhere
Idealized Z-pincho implosion, cont’d
Idealized Z-pinch implosion, cont’d
Idealized Z-pinch implosion, cont’d
Idealized Z-pinuch implosion, cont’d

- Time: 0.8
- Time: 0.9
- Time: 1.1

density distribution
Conclusions and outlook

- Linearized flux correction algorithm for time-dependent flows
  mass conservation, boundedness of physical quantities,
  failsafe strategy if density/pressure becomes negative

- Coupled solution algorithm for idealized Z-pinch implosions
  positivity and symmetry preservation on unstructured grids

- Todo: Extension to more ’realistic’ scenarios
  current drive, r-z plane, RT-instabilities, AMR
Thank you!
Appendix
Extended version of Zalesak’s FCT limiter

Input: auxiliary solution $u^L$ and antidiffusive fluxes $f_{ij}^u$, where $f_{ji}^u \neq f_{ij}^u$

1. Sums of positive/negative antidiffusive fluxes into node $i$
   \[
P_i^+ = \sum_{j \neq i} \max\{0, f_{ij}^u\}, \quad P_i^- = \sum_{j \neq i} \min\{0, f_{ij}^u\}
   \]

2. Upper/lower bounds based on the local extrema of $u^L$
   \[
   Q_i^+ = m_i (u_i^{\max} - u_i^L), \quad Q_i^- = m_i (u_i^{\min} - u_i^L)
   \]

3. Correction factors $\alpha_{ij}^u = \alpha_{ji}^u$ to satisfy the FCT constraints
   \[
   \alpha_{ij}^u = \min\{R_{ij}, R_{ji}\}, \quad R_{ij} = \begin{cases} 
   \min\{1, Q_i^+ / P_i^+\} & \text{if } f_{ij}^u \geq 0 \\
   \min\{1, Q_i^- / P_i^-\} & \text{if } f_{ij}^u < 0
   \end{cases}
   \]
Node-based transformation of control variables

- Conservative variables: density, momentum, total energy
  \[ U_i = [\rho_i, (\rho v)_i, (\rho E)_i], \quad F_{ij} = \left[ f_{ij}^\rho, f_{ij}^{\rho v}, f_{ij}^{\rho E} \right], \quad F_{ji} = -F_{ij} \]

- Primitive variables \( V = TU \): density, velocity, pressure
  \[ V_i = [\rho_i, v_i, p_i], \quad v_i = \frac{(\rho v)_i}{\rho_i}, \quad p_i = (\gamma - 1) \left[ (\rho E)_i - \frac{|(\rho v)_i|^2}{2\rho_i} \right] \]
  \[ G_{ij} = \left[ f_{ij}^\rho, f_{ij}^{\rho v}, f_{ij}^p \right] = T(U_i)F_{ij}, \quad T(U_j)F_{ji} = G_{ji} \neq -G_{ij} \]

- Raw antidiffusive fluxes for the velocity and pressure
  \[ f_{ij}^{\rho v} = \frac{f_{ij}^{\rho v} - v_i f_{ij}^\rho}{\rho_i}, \quad f_{ij}^\rho = (\gamma - 1) \left[ \frac{|v_i|^2}{2} f_{ij}^\rho - v_i \cdot f_{ij}^{\rho v} + f_{ij}^{\rho E} \right] \]
Constrained initialization

- Pointwise initialization

\[ U(x_i) = U_0(x_i) \]

\[ \rho = \begin{cases} 
1.0 & \text{in } \Omega_1 \\
0.01 & \text{in } \Omega_2 
\end{cases} \]

\[ u = v = 0.0, \ p = 1.0 \]
Constrained initialization

Pointwise initialization

\[ U(x_i) = U_0(x_i) \]

Conservative initialization

\[ \int_{\Omega} wU_h \, dx = \int_{\Omega} wU_0 \, dx \]

\[ \rho = \begin{cases} 
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  \[ U(x_i) = U_0(x_i) \]

- **Conservative initialization**
  \[ \int_{\Omega} wU_h \, dx = \int_{\Omega} wU_0 \, dx \]

- **Consistent \( L_2 \)-projection**
  \[ \sum_j m_{ij} U_j^H = \int_{\Omega} \varphi_i U_0 \, dx \]

- **Mass-lumped \( L_2 \)-projection**
  \[ m_i U_i = m_i U_{Li} + \sum_{j \neq i} \alpha_{ij} m_{ij} (U_{Li} - U_{Lj}) \]

- **Limited \( L_2 \)-projection (\( 0 \leq \alpha_{ij} \leq 1 \))**
  \[ m_i U_i = m_i U_{Li} + \sum_{j \neq i} \alpha_{ij} m_{ij} (U_{Li} - U_{Lj}) \]

\[ \rho = \begin{cases} 
  1.0 & \text{in } \Omega_1 \\
  0.01 & \text{in } \Omega_2 
\end{cases} \]

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  \[ m_i U_i^L = \int_\Omega \varphi_i U_0 \, dx \]

- \( \rho = \begin{cases} 1.0 & \text{in } \Omega_1 \\ 0.01 & \text{in } \Omega_2 \end{cases} \)

- \( u = v = 0.0, \ p = 1.0 \)
Constrained initialization

- Pointwise initialization
  \[ U(x_i) = U_0(x_i) \]

- Conservative initialization
  \[ \int_{\Omega} wU_h \, dx = \int_{\Omega} wU_0 \, dx \]

- Consistent \( L_2 \)-projection
  \[ \sum_j m_{ij} U_j^H = \int_{\Omega} \varphi_i U_0 \, dx \]

- Mass-lumped \( L_2 \)-projection
  \[ m_i U_i^L = \int_{\Omega} \varphi_i U_0 \, dx \]

- Limited \( L_2 \)-projection (0 \( \leq \alpha_{ij} \leq 1 \))
  \[ m_i U_i = m_i U_i^L + \sum_{j \neq i} \alpha_{ij} m_{ij} (U_i^L - U_j^L) \]

\[ \rho = \begin{cases} 
  1.0 & \text{in } \Omega_1 \\
  0.01 & \text{in } \Omega_2 
\end{cases} \]

\[ u = v = 0.0, \ p = 1.0 \]
Initialization for bilinear elements

(a) consistent $L_2$-projection

(b) lumped $L_2$-projection

(c) $L_2$-projection, $\alpha_{ij} = \alpha_{ij}^\rho$

<table>
<thead>
<tr>
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<th>$|\rho - \rho_h|_2$</th>
<th>min($\rho_h$)</th>
<th>max($\rho_h$)</th>
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<tr>
<td>(a)</td>
<td>1.048e-1</td>
<td>-1.031e-1</td>
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<td>(b)</td>
<td>1.168e-1</td>
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<td>(c)</td>
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</table>

computed by adaptive cubature formulae
Initialization for linear elements

(a) consistent $L_2$-projection

(b) lumped $L_2$-projection

(c) $L_2$-projection, $\alpha_{ij} = \alpha_{ij}^\rho$

<table>
<thead>
<tr>
<th></th>
<th>linear elements, 3-point Gauss rule</th>
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<tbody>
<tr>
<td></td>
<td>$|\rho - \rho_h|_2$</td>
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<td>(a)</td>
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<td>(c)</td>
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computed by adaptive cubature formulae