Automated parameters for troubled-cell indicators using outlier detection

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June 24, 2015
Introduction: Nonlinear hyperbolic PDE’s

Limiters:
- Too few elements: oscillatory approximation
- Too many elements: too diffusive, computationally expensive

Which elements need limiting? Troubled-cell indicator
Outline

1. Building blocks: DG and multiwavelets
2. Multiwavelet troubled-cell indicator (with parameter)
3. Outlier detection
4. Results
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4 Results
Discontinuous Galerkin method

\[
\begin{cases}
  u_t + f(u)_x = 0, & x \in [-1, 1], \ t > 0, \\
  u(x, 0) = u_0(x), & x \in [-1, 1].
\end{cases}
\]

- Discretize $[-1, 1]$ into $2^n$ elements
- Approximation space $V^k_h$: $k$th-degree piecewise polynomials
- Approximate $u$ by $u_h \in V^k_h$
- Multiply PDE by $v_h \in V^k_h$, integrate over $I_j$
- Integrate by parts
DG approximations and multiwavelets

Global DG approximation, $2^n$ elements on $[-1, 1]$: 

$$u_h(x) = \sum_{j=0}^{2^n-1} \sum_{\ell=0}^{k} u_j^{(\ell)} \phi_\ell(x)$$

Corresponding multiwavelet expansion:

$$u_h(x) = \sum_{\ell=0}^{k} s_{0,0}^{\ell} \phi_\ell(x) + \sum_{m=0}^{n-1} \sum_{j=0}^{2^m-1} \sum_{\ell=0}^{k} d_{\ell,j}^m \psi_{\ell,j}^m(x)$$

- Global average
- Finer details
Multiresolution idea

\[ V_{n}^{k} = \{ f : f \in \mathbb{P}^{k}(I_{j}^{n}), j = 0, \ldots, 2^{n} - 1 \} \]

\[ V_{0}^{k} \subset V_{1}^{k} \subset \cdots \subset V_{n}^{k} \subset \cdots \]

Scaling functions and DG basis

DG basis functions:

- Orthonormal Legendre polynomials
- Basis for $V_0^k$: scaling function basis
- Basis functions for $V_n^k$: dilation and translation

$$\phi_{\ell j}^n(x) = 2^{n/2} \phi_\ell(2^n(x + 1) - 2j - 1),$$

$$\ell = 0, \ldots, k, \ j = 0, \ldots, 2^n - 1, \ x \in I_j^n$$

Multiwavelets

Multiwavelet space $W^k_m$:

- Orthogonal complement of $V^k_m$ in $V^k_{m+1}$:

$$V^k_m \oplus W^k_m = V^k_{m+1}, \quad W^k_m \perp V^k_m, \quad W^k_m \subset V^k_{m+1}$$

- $V^k_n$ can be split into $n + 1$ orthogonal subspaces:

$$V^k_n = V^k_0 \oplus W^k_0 \oplus W^k_1 \oplus \cdots \oplus W^k_{n-1}$$

Split up $f \in V^k_n$ into different levels:

$$u_h(x) = \sum_{\ell=0}^{k} s^0_{\ell} \phi_{\ell}(x) + \sum_{m=0}^{n-1} \sum_{j=0}^{2^m-1} \sum_{\ell=0}^{k} d^m_{\ell j} \psi^m_{\ell j}(x)$$
Jumps in DG approximations

\[ u_h(x) = \sum_{\ell=0}^{k} s_{\ell 0}^0 \phi_{\ell}(x) + \sum_{m=0}^{n-1} \sum_{j=0}^{2^m-1} \sum_{\ell=0}^{k} d_{\ell j}^m \psi_{\ell j}^m(x) \]

Coefficient \( d_{\ell j}^{n-1} \): measures jump in (derivatives) approximation

\[ d_{\ell j}^{n-1} = \sum_{m=0}^{k} c_{m\ell}^n \left( u_h^{(m)}(x_{j+1/2}^+) - u_h^{(m)}(x_{j+1/2}^-) \right) , \]

where

\[ c_{m\ell}^n = \frac{(-n+1)^m}{m!} \cdot \int_0^1 x^m \psi_\ell(x) \, dx. \]
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Original approach

Detect elements $l_j$ and $l_{j+1}$ if

$$|d_{kj}^{n-1}| > C \cdot \max_j |d_{kj}^{n-1}|, \quad C \in [0, 1].$$

64 elements, $k = 1$
Approximation

How to choose $C$?
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Outlier detection

\( d_{kj}^{n-1} \):
- vector containing jumps over element boundaries
- coefficient big compared to neighbors: detect

⇒ Boxplot approach

(Tukey, 1977)
Boxplot

\[
d = \begin{pmatrix}
-1 \\
-1 \\
-1 \\
0 \\
1 \\
20 \\
2 \\
3 \\
1 \\
0 \\
-3 \\
-2 \\
0
\end{pmatrix}, \quad d^s = \begin{pmatrix}
-3 \\
-2 \\
-1 \\
-1 \\
0 \\
0 \\
0 \\
1 \\
1 \\
2 \\
3 \\
20
\end{pmatrix}
\]

- 25th and 75th percentiles:
  \( Q_1 = -1, \quad Q_3 = 1 \)

- Lower bound:
  \( Q_1 - 3(Q_3 - Q_1) = -7 \)

- Upper bound:
  \( Q_3 + 3(Q_3 - Q_1) = 7 \)
Whisker length

\[ d_j < Q_1 - W \cdot (Q_3 - Q_1) \text{ or } d_j > Q_3 + W \cdot (Q_3 - Q_1) \]

Whisker length 3:
- Coverage of 99.9998%
- Normally distributed: 0.0002% detected asymptotically
- Few false positives if data well behaved
- Continuous function: no elements are detected!

(Hoaglin et al., J. Amer. Statist. Assoc. (1986))
Local information

- Divide global vector in local vectors
- Apply boxplot approach for each local vector
- Ignore 'outliers' near split boundaries
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Applications

- Apply original indicator with optimal parameter $C$
- Compare with outlier-detected results (no parameter)
- Euler equations: Sod, sine-entropy wave
Minmod-based TVB indicator

\[ u_{j+\frac{1}{2}}^- = \bar{u}_j + \tilde{u}_j, \quad \tilde{u}_j = \sum_{\ell=1}^{k} u_{j}^{(\ell)} \phi_{\ell}(1) \]

\[ u_{j-\frac{1}{2}}^+ = \bar{u}_j - \tilde{u}_j, \quad \tilde{u}_j = -\sum_{\ell=1}^{k} u_{j}^{(\ell)} \phi_{\ell}(-1) \]

\[ M = 100 \]

Outlier
KXRCF detector

Jump across inflow edge:

\[ I_j = \left| \int_{\partial I_j^-} (u_h|_{I_j} - u_h|_{I_{nj}}) \, ds \right| \]

Threshold equal to 1

Outlier
Conclusion and future research

- Original troubled-cell indicator: problem-dependent parameter
- Outlier-detection technique using boxplots
- Local-vector approach
- Parameters no longer needed!

- General meshes