Delft University of Technology

Faculty of Electrical Engineering, Mathematics and Computer Science

Applied Finite Elements
2008/2009 Take Home Exams, First Series

1. Derive the Euler-Lagrange equations (including boundary conditions) for the following problem

$$\min J[u] = \int_{r_0}^{r_1} r e^u \sqrt{1 + \left(\frac{du}{dr}\right)^2}\, dr \,, \tag{1}$$

$$u(r_0) = y_0 \,. \tag{2}$$

   Use the method described in Section 5.1 of Numerical methods in Scientific Computing (NMSC).
   It is not allowed to apply Theorem 5.2.1.

2. Find the minimization problem corresponding to the differential equation

$$\frac{d^4u}{dx^4} + \lambda u = f \,, \qquad (\lambda \geq 0),$$

   with boundary conditions

$$u(0) = \frac{d^2u}{dx^2}(0) = 0 \,, \frac{d^2u}{dx^2}(1) = 2 \,, \quad \frac{d^3u}{dx^3}(1) = 1 \,.$$

   Is the number of boundary conditions sufficient for a unique solution? Motivate your answer.

3. Consider the differential equation

$$\frac{d^4u}{dx^4} = f \,, \tag{3}$$

   with boundary conditions

$$u(0) = \frac{du}{dx}(0) = 0 \,, \quad \frac{du}{dx}(1) = 0 \,, \quad \frac{d^3u}{dx^3}(1) = 1 \,. \tag{4}$$

   Derive the weak formulation for Equation (3) with boundary conditions (4), using derivatives of the lowest order that is possible.

4. Apply Galerkin's method and derive a set of linear equations for the coefficients. What are the continuity requirements for the basis functions?

5. To solve these equations we use the Finite Element Method. We use a two noded line element with u and $\frac{du}{dx}$ as unknowns in both points.
What is the degree of the interpolation polynomial?
Do these polynomials satisfy the continuity requirements? Motivate your answer.
Show that the interpolation polynomial per element can be written as:

$$\tilde{u}(x) = \sum_{i=1}^{2} u_i \phi_{i0}(x) + \sum_{i=1}^{2} (u_x)_i \phi_{i1}(x),$$

with

$$\begin{aligned} \phi_{i0}(x^j) = \delta_{ij}, \quad & \tfrac{d\phi_{i0}}{dx}(x^j) = 0, \\ \phi_{i1}(x^j) = 0, \quad & \tfrac{d\phi_{i1}}{dx}(x^j) = \delta_{ij}. \end{aligned}$$

Express the basis functions ϕ_{i0} and ϕ_{i1} in the linear basis functions $\phi_i(x)$ defined at page 96 of NMSC.
Hint: Compute the sum of the linear basis functions and use this relation to simplify the expressions.

6. Derive the element matrix for an arbitrary element with step size h.
Derive the element vector in case f is constant.

7. Give the complete system of equations.

To be submitted before April 6, 2009, 12:00