Delft University of Technology

Faculty of Electrical Engineering, Mathematics and Computer Science

Applied Finite Elements
2008/2009 Take Home Exams, Third Series

All exercises and references are taken from the book *Numerical Methods in Scientific Computing*.

1. Consider the Burgers equation

$$-\frac{d}{dx}\left(\kappa\frac{du}{dx}\right) + u\frac{du}{dx} = f \qquad x \in (0,1) \tag{1}$$

with boundary conditions $u(0) = 0$, $u(1) = 1$.
κ and f are functions of x and u is the unknown.

   (a) Derive the weak formulation for Equation (1), such that the final expression contains derivatives of lowest order.
   (b) Apply Galerkin's method to derive a set of non-linear equations.
   (c) In order to linearize the equations we apply a Picard iteration (successive substitution). Describe two possible ways to apply Picard in this case.
   Compute the element matrix and vector for an arbitrary element for both options. Use linear basis functions and Newton Cotes quadrature.
   (d) An alternative is to use Newton's method for the set of non-linear equations.
   Derive the system of Newton equations.
   Compute the corresponding element matrix and vector.

2. Let Ω be some region in $\mathbf{R}^2$ with boundary $\Gamma = \Gamma_1 \bigcup \Gamma_2$. On Ω we solve the convection-diffusion equation

$$-\text{div}\,(\kappa\nabla\, T) + \mathbf{u}\cdot\nabla\, T = f \tag{2}$$

with boundary conditions

$$T = 0 \qquad \text{on } \Gamma_1 \tag{3}$$

$$\sigma T + (\mathbf{n}, \kappa\nabla T) = g_1 \qquad \text{on } \Gamma_2 \tag{4}$$

The coefficients κ and σ are constant, the functions f, $\mathbf{u}$ and g_1 depend on the space coordinates.
In order to solve this equation we apply the finite element using straight edged triangles with piecewise quadratic interpolations.

   (a) Derive the weak formulation for this convection-diffusion equation, such that only lowest order derivatives remain.
   (b) Derive the corresponding Galerkin equations.
   (c) Let $\mathbf{x}_i$ be a vertex of the quadratic triangle and
   $\mathbf{x}_{ij}$ the mid-side points on the edge connecting points $\mathbf{x}_i$ and $\mathbf{x}_j$.
   Show that the Newton Cotes rule applied to the quadratic triangle is given by:

$$\int_e Int(\mathbf{x})de = \frac{|\Delta|}{6}[Int(\mathbf{x}_{12}) + Int(\mathbf{x}_{23}) + Int(\mathbf{x}_{13})]. \tag{5}$$

(d) The local numbering of the nodes is counterclockwise. The first node is a vertex. Compute the matrix elements s_{11} and s_{12} of the element matrix for an arbitrary internal element.

(e) Compute the element vector corresponding to a quadratic triangle.

(f) Give the Newton-Cotes formula for the (quadratic) boundary element.

(g) Compute the element vector corresponding to this boundary element.

(h) What is the element matrix for this boundary element?

To be submitted at or before May 1, 2009