Deflated ICCG Method applied on 3-D Multi-Phase Flows

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Received 5 July 2006, accepted 17 July 2006

Key words deflation, conjugate gradient method, preconditioning, Poisson equation, symmetric positive semi-definite matrices, bubbly flow problems.
Subject classification 65F10, 65F50, 65N22.

A deflated ICCG has been used and analyzed in problems with 3-D multi-phase flows. The Poisson solve with discontinuous coefficients leads to numerical difficulties for the original ICCG, while these difficulties do not appear anymore for the deflated ICCG.

1 Introduction

Simulating bubbly flows is a very popular topic in CFD. These bubbly flows are governed by the Navier-Stokes equations. In many popular operator splitting formulations for these equations, solving the singular SPSD linear system

$$Ax = b, \quad A \in \mathbb{R}^{n \times n},$$

coming from the Poisson equation with discontinuous coefficients

$$-\text{div}\left(\frac{1}{\rho}\nabla p\right) = f$$

with Neumann boundary conditions takes the most computational time, despite of its elliptic origins, see e.g. [3, 4, 5]. ICCG is widely used for this purpose, where

$$M^{-1}Ax = M^{-1}b, \quad M \text{ is the IC preconditioner,}$$

should be solved. For complex bubbly flows this method shows slow convergence due to the presence of small eigenvalues in the spectrum of $M^{-1}A$. We show that applying the deflation technique, which leads to the DICCG method, remedies the worse condition number and the worse convergence of ICCG.

2 DICCG Method

In DICCG we solve

$$M^{-1}PA\tilde{x} = M^{-1}Pb, \quad P \text{ is the deflation matrix,}$$

where

$$P = I - AZE^{-1}Z^T, \quad E = Z^TAZ, \quad Z \in \mathbb{R}^{n \times r}, \quad r \ll n. \tag{1}$$

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Piecewise-constant deflation vectors are used to approximate the eigenmodes corresponding to the components which caused the slow convergence of ICCG. More technically, deflation subspace matrix $Z = [z_1 \; z_2 \; \cdots \; z_r]$ consists of

$$z_j(\mathbf{x}) = \begin{cases} 0, & \mathbf{x} \in \Omega \setminus \bar{\Omega}_j; \\ 1, & \mathbf{x} \in \Omega_j, \end{cases}$$

where the domain Ω is divided into non-overlapping subdomains Ω_j which are chosen to be cubes, assuming that the number of grid points in each spatial direction is the same.

3 Application of DICCG in Bubbly Flow Problems

It is known that the deflation technique works well for invertible systems and when the deflation vectors are based on the geometry of the problem, see also References [1, 2]. Main questions in our work are:

- is the deflation method also applicable to singular problems?
- is the deflation method with fixed deflation vectors also applicable to problems where the position and radius of the bubbles changes in every time step?

The second question will be dealt in the next section, where numerical experiments will be applied to show that the fixed deflation vectors are indeed applicable to time-dependent bubbly flow problems.

First we show that DICCG is indeed applicable on systems which are singular. Several new variants of deflation methods are proposed for these singular systems: deflation methods where

- invertibility of A is forced resulting in a deflation matrix P_1, i.e., we adapt the last element of A such that the new matrix, denoted as $\widetilde{A}$, is invertible;
- a column of Z is deleted resulting in a deflation matrix P_2, i.e., instead of Z we take $[z_1 \; z_2 \; \cdots \; z_{r-1}]$ as deflation subspace matrix;
- systems with a singular E are solved iteratively resulting in a deflation matrix P_3, i.e., matrix E^{-1} as in Eq. (1) is considered to be a pseudo-inverse.

We can prove that the three DICCG variants are identical in exact arithmetics, see Theorem 3.1.

Theorem 3.1 $P_1\widetilde{A} = P_2A = P_3A$.

Proof. See [7]. □

Since the variants are equal theoretically, we choose for the first variant with P_1 in the numerical experiments. Results of these experiments are presented below to show the efficiency of this variant.

4 Numerical Experiments

We test the efficiency of the DICCG method for two kind of test problems.

4.1 Test Case 1

First, we take a 3-D bubbly flow applications with eight air-bubbles in a domain of water, see Figure 1 for the geometry. We apply finite differences on a uniform Cartesian grid with $n_x = n_y = n_z = 100$ resulting in a very large but sparse linear system $Ax = b$.

Then the results of ICCG and DICCG can be found in Table 1, where ϕ denotes the relative exact residual and DICCG$-r$ denotes DICCG with r deflation vectors.

From Table 1 one observes that the larger the number of deflation vectors the less iterations DICCG requires. Considering the CPU time, there is an optimum, namely for $k = 10^3$. Hence, in the optimal case, DICCG is more than five times as fast as the original ICCG method while the accuracy of both methods are comparable! The benefit of the deflation method will be larger when we increase the number of grid points in the test problem.

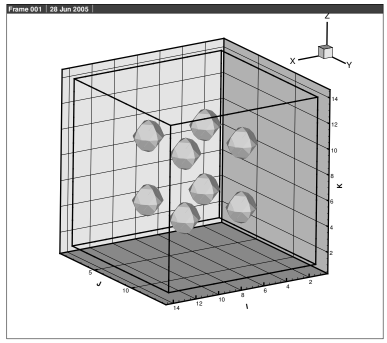


Fig. 1 An example of a bubbly flow problem: eight air-bubbles in a unit domain filled with water.

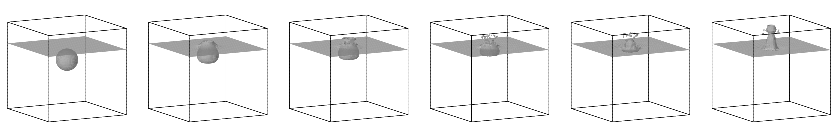
Method	# Iterations	CPU Time (sec)	$\phi\ (\times 10^{-9})$
ICCG	291	43.0	1.1
DICCG-2^3	160	29.1	1.1
DICCG-5^3	72	14.2	1.2
DICCG-10^3	**36**	**8.2**	**0.7**
DICCG-20^3	22	27.2	0.9

Table 1 Convergence Results of ICCG and DICCG$-k$ solving $Ax = b$ with $n = 100^3$ for the test problem as given in Figure 1.

4.2 Test Case 2

Next, we present some results from 3-D simulations of a rising air bubble in water in order to show that the deflation method is also applicable in real-life problems. We adopt the mass-conserving level-set method [4] for the simulations, but it could be replaced by any operator-splitting method in general. At each time step a Poisson equation has to be solved which is the most time-consuming part of the whole simulation. Therefore, during this section we only concentrate on this part at each time step. We investigate whether DICCG is still efficient for all those time steps.

We consider a test problem with a rising air bubble in water without surface tension. The exact material constants and other relevant conditions can be found in [4, Sect. 8.3.2]. The starting position of the bubble in the domain and the evolution of the movement during the 250 time steps can be seen in Figure 2.

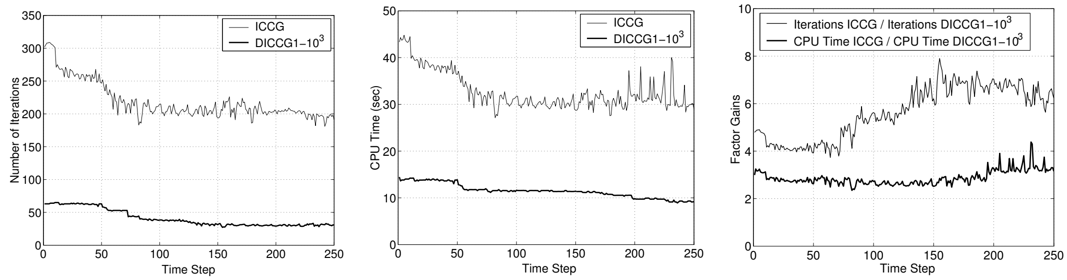


(a) $t = 0$. (b) $t = 50$. (c) $t = 100$. (d) $t = 150$. (e) $t = 200$. (f) $t = 250$.

Fig. 2 Evolution of the rising bubble in water without surface tension in the first 250 time steps.

In [4] the Poisson solver is based on ICCG. Here we will compare this method to DICCG with $r = 10^3$ deflation vectors in the case for $n = 100^3$. The results are presented in Figure 3. In future, r can be adapted to obtain a more efficient method.

From Subfigure 3(a), we notice that the number of iterations is strongly reduced by the deflation method. DICCG with $r = 10^3$ requires more or less 60 iterations, while ICCG converges between 200 and 300 iterations at the most time steps. Moreover, we observe the erratic behavior of ICCG, whereas DICCG seems to be less sensitive of the geometries during the evolution of the simulation. Also considering the CPU time, DICCG shows very good performance, see Subfigure 3(b). In the most time steps, ICCG requires 25–45 seconds to converge, whereas DICCG needs only around 11–14 seconds which is a relatively large benefit. Moreover, in Figure 3(c) one can find the gain factors considering both the ratios of the iterations and the CPU time between ICCG and



(a) Number of iterations versus time step. (b) CPU time versus time step. (c) Gain factors with respect to ICCG and DICCG.

Fig. 3 Results of ICCG and DICCG with $r = 10^3$ for the simulation with a rising air bubble in water.

DICCG. From the figure, we notice that DICCG needs approximately 4–8 times less iterations, depending on the time step. More important, at all time steps DICCG converges more or less 2–4 times faster to the solution compared to ICCG.

In general, we can conclude that, compared to ICCG, DICCG decreases significantly the number of iterations and the computational time as well, which are required for solving Poisson equation in applications of 3-D bubbly flows.

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