Deflated PCG Method for the Poisson Solver

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Introduction

The Preconditioned Conjugate Gradient (PCG) method is applied to a linear system $Ax = b$, coming from the 3D heterogeneous Poisson equation:

$$\text{div}\left(\frac{1}{\rho(\mathbf{x})}\nabla p(\mathbf{x})\right) = f(\mathbf{x}), \quad \mathbf{x} = (x, y, z) \in \Omega, \qquad (1)$$

Due to large jumps in ρ, matrix A is ill-conditioned, so PCG suffers from slow convergence.

Figure 1: Our application of the Poisson equation: multi-phase flows, where the phases air and water interact.

Deflated PCG method

In Deflated PCG, we solve

$$M^{-1}PA\tilde{x} = M^{-1}Pb, \qquad (2)$$

where M is the preconditioner and P is the deflation matrix,

$$P := I - AZE^{-1}Z^T, \quad E := Z^TAZ, \qquad (3)$$

with $Z = [z_1\ z_2\ \cdots\ z_r]$ consisting of sparse deflation vectors:

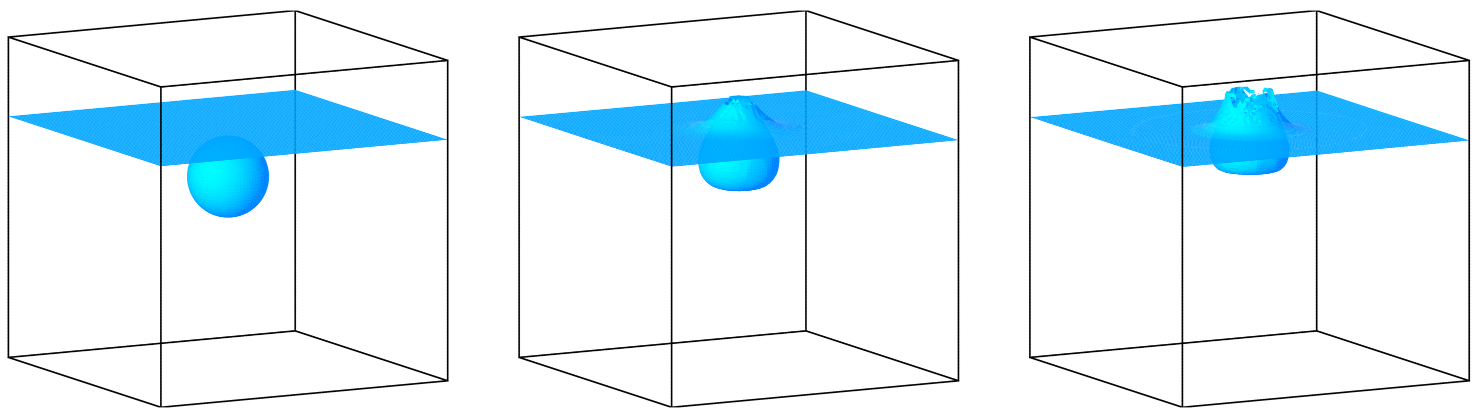
$$(z_j)_i := \begin{cases} 0, & x_i \in \Omega \setminus \Omega_j, \\ 1, & x_i \in \Omega_j. \end{cases} \qquad (4)$$

Main advantages of Deflated PCG:

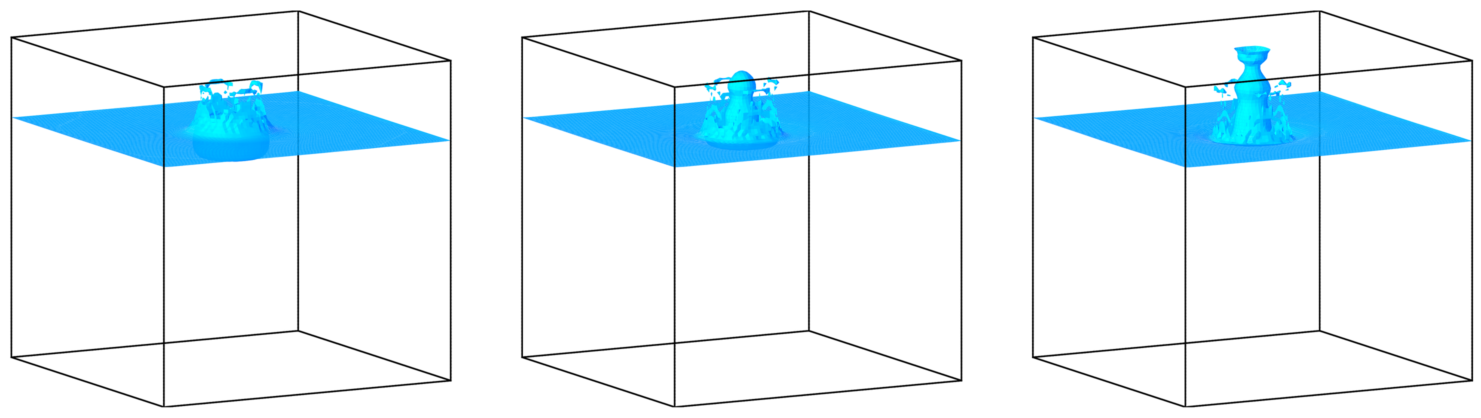
- it converges faster than PCG;
- the extra cost is low;
- it is easy to implement in an existing code.

Example

A rising air bubble in water is considered, see Figure 2.



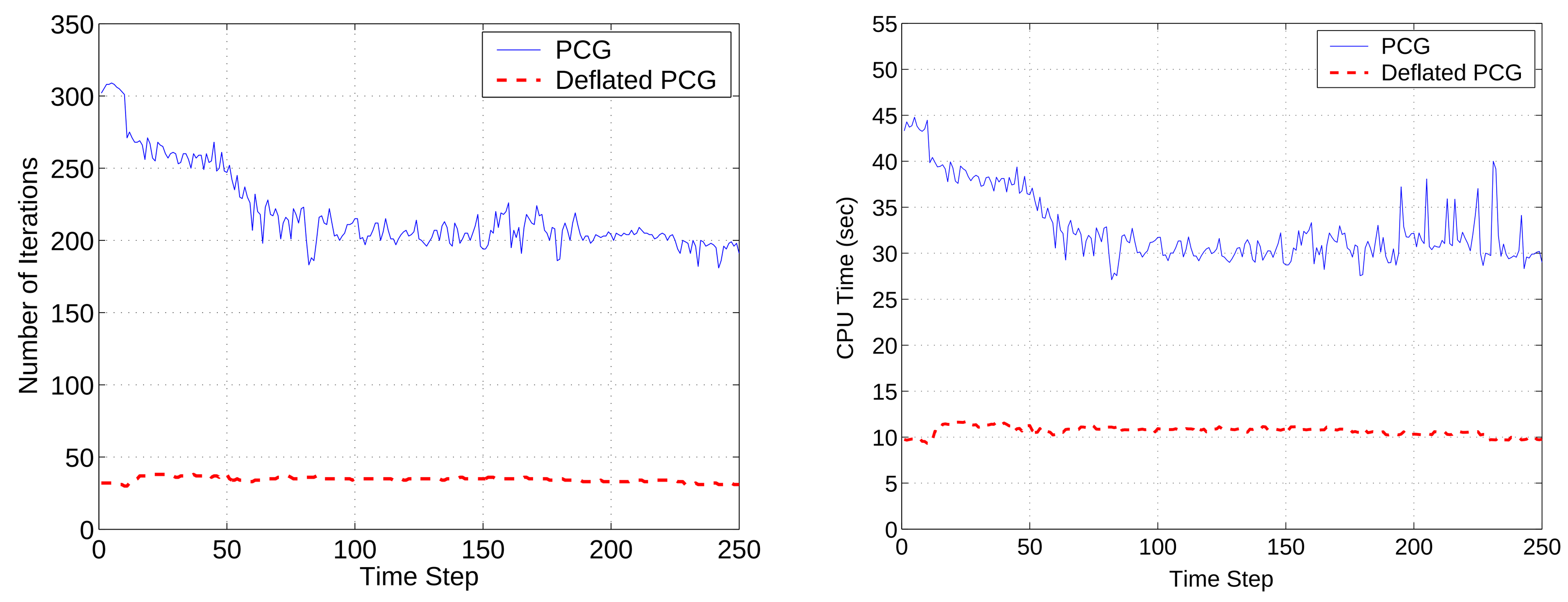
(a) $t = 0$. (b) $t = 50$. (c) $t = 100$.



(d) $t = 150$. (e) $t = 200$. (f) $t = 250$.

Figure 2: Evolution of the rising bubble in water with $\rho_{\text{water}} = 1$ and $\rho_{\text{air}} = 10^{-3}$.

We use the Incomplete Cholesky preconditioner and $r = 10^3$ deflation vectors to solve $Ax = b$ with $n = 100^3$ unknowns. The results for PCG and Deflated PCG can be found in Figure 3.



(a) Time step versus number of iterations. (b) Time step versus computing time.

Figure 3: Results for the comparison of PCG and Deflated PCG during the simulation.

It can be observed that Deflated PCG requires approximately 4–8 times fewer iterations compared to PCG. More importantly, the method converges 2–4 times faster at all time steps!

Conclusion

A simple subdomain deflation technique clearly accelerates the convergence of PCG, so that computations for the simulation of multi-phase flows can be performed significantly faster.

References

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