Nonlinear model reduction via invariant manifolds for high-dimensional IgA models

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Summary

The equations governing physical phenomena may involve several million unknowns that renders a reliable prediction of the system's nonlinear response computationally challenging. At the same time, this nonlinear response is often a function of only a few variables that evolve over low-dimensional attracting invariant manifolds in the full phase space of the dynamical system. In this project, we will apply dynamical systems theory to reduce the dimensionality of models created via Isogeometric Analysis (IgA) and achieve rigorous results with improved computational performance.

Background

Isogeometric Analysis (IgA) [4] is a generalization of the finite element method that is particularly suited for solving (initial-) boundary value problems on complex geometries and manifolds. IgA overcomes the distinction between a computational mesh that approximates the computational domain and the approximation of the solution and uses a unified representation of both in terms of spline basis functions. This approach not only simplifies the treatment of complex shapes, which can be much better described by splines than by discrete meshes, but also provides a higher accuracy per degree of freedom and enables novel solution approaches of higher-order differential equations. The latter means that, say, fourth order differential equations do not have to be treated by introducing auxiliary variables but can be handled with cubic splines directly.

Invariant manifolds are low-dimensional surfaces in the phase space of a dynamical system that constitute organizing centers of nonlinear dynamics. While the theory of invariant manifolds has been applied to numerous fields for the qualitative understanding of nonlinear systems for over a century of research, the computation of invariant manifolds has only recently become accessible to high-dimensional problems of engineering significance [1].

Project description

The project will be accomplished in two broad stages:

1. **Nonlinear modelling based on IgA**: The student will use IgA to discretize PDEs governing a nonlinear physical process, e.g., motion of a two-dimensional, elastic, beam or vortex-shedding behind a cylinder. To this end, the open-source FE package, yetAnotherFEcode [2] may be a useful starting point to implement IgA. The student will benchmark their IgA-based model against an established finite-element software e.g., COMSOL.

2. **Solving invariance equations and prediction of nonlinear response**: Here, the student will compute a low-dimensional invariant manifold in the spatially discretized system from the previous step. To this end, the MATLAB-based open-source package, SSMTool [3], will be useful. The reduced dynamics on the invariant manifold will then be used to make fast predictions for the relevant nonlinear phenomena (e.g., nonlinear forced response in a mechanical system). Finally, these predictions will be compared to the full system simulations in terms of speed and accuracy.

Novel aspects: IgA is a modern generalization of the finite element method and computation of invariant manifolds constitutes a modern view on model reduction based in dynamical systems theory. This projects offers the opportunity to combine these modern tools for modelling and analysis of nonlinear problems for the first time.

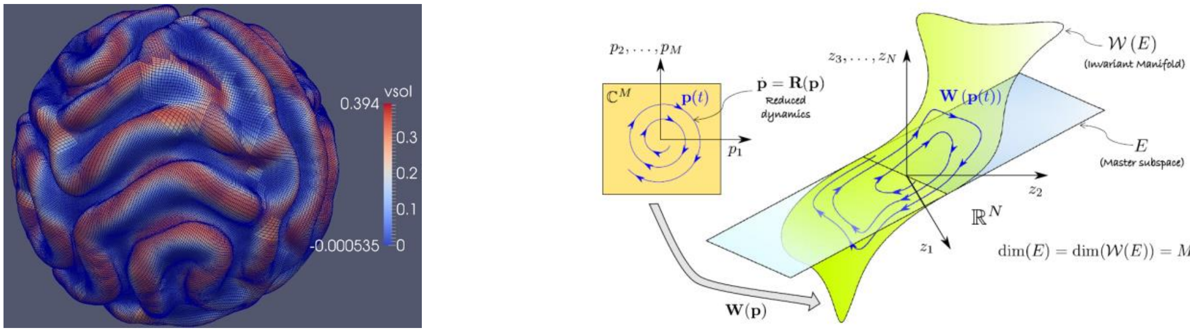

Figure 1: (left) IgA-based discretization of the Gray-Scott reaction-diffusion equations on evolving surfaces [5]. (right) Parametrization of an *M*-dimensional invariant manifold in the *N*-dimensional phase space of variables **z**, and its reduced dynamics in the reduced variables **p** [1].

Learning objectives

At the end of this project, the student will be able to

1. Explain the importance of invariant manifolds in analysing nonlinear dynamical systems.
2. Numerically discretize the partial differential equations governing nonlinear mechanical systems via IgA.
3. Compute low-dimensional invariant manifolds in spatially discretized PDEs.
4. Analyze low-dimensional dynamics to predict nonlinear phenomena (e.g., forced response) in physical (mechanical) systems.
5. Critically evaluate their results in comparison to the available literature.

To achieve the above objectives, the student can expect guidance and support in terms of weekly update meetings, where they receive detailed and in-depth feedback on their work.

Learning activities

1. Literature study.
2. Modelling and simulation of mechanical systems via available software packages.
3. Programming IgA in MATLAB to model simple nonlinear mechanical (dynamical) systems like beams, plates etc.
4. Solving invariance equations on the full system to compute low-dimensional invariant manifolds via SSMTool.
5. Comparing the low-dimensional dynamics against full system simulations in terms of speed and accuracy.

For this project, the student should have an affinity to programming (e.g., in MATLAB) and enthusiasm about the application of numerical methods to model and analyze real-world problems.

Familiarity with MATLAB is desirable and prior experience with dynamical systems (e.g., WI:4019, WI4660 etc.) and IgA/Finite elements (WI:4205, AM3530) is an advantage.

References

1. Jain & Haller (2022). How to compute invariant manifolds and their reduced dynamics in high-dimensional finite element models. *Nonlinear Dynamics*, *107*(2), 1417–1450.
2. Jain, Marconi & Tiso (2022). YetAnotherFEcode (v1.3.0). Zenodo. https://doi.org/10.5281/zenodo.4011281

3. Jain, Li, Thurnher & Haller (2023). SSMTool 2.3: Computation of invariant manifolds in high-dimensional mechanics problems. Zenodo. https://doi.org/10.5281/zenodo.4614201
4. Hughes, Cottrell & Bazilevs (2005). Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement. *Computer Methods in Applied Mechanics and Engineering, Elsevier*, 194 (39-41), 4135–4195.
5. Hinz, van Zwieten, Möller & Vermolen (2019). Isogeometric Analysis of the Gray-Scott Reaction-Diffusion Equations for Pattern Formation on Evolving Surfaces and Applications to Human Gyrification. arXiv: 1910.12588