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Numerical challenges in parallel scientific computing
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Krylov subspace solvers and preconditioners

Practical exercises

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Experiments with iterative methods

In this exercise we consider various iterative methods to solve linear systems. We start with systems where the coefficient matrix is symmetric and positive definite. These systems are solved by basic iterative methods and the Conjugate Gradient method. Finally we investigate if these methods can also be applied to non-symmetric systems.

a Construct the matrix a and the right-hand-side vector f using the call `[a,f] = poisson(10,11,0,0,'central');` Compute the solution of the linear system with the Gauss-Jacobi method:

```
x = jacobi(a,f,10^-10);
```

Write down the number of iterations for comparison with other methods.

Exercise aims:

- Gauss-Jacobi is a slowly converging method.
- A linear converging iteration method.

b First try to modify the program `jacobi.m` to obtain the Gauss-Seidel method. To check your implementation you also use:

```
x = seidel(a,f,10^-10);
```

Compare the results with those obtained in exercise a.

Exercise aims:

- Gauss-Seidel converges two times as fast as Gauss-Jacobi.

c In this exercise we use the SOR method. The estimate of the optimal ω is not straightforward. Therefore we do some experiments with various choices of ω. Type

```
x = sor(a,f,10^-10,omega);
```

where `omega` is a real number between 0 and 2. Try to approximate the optimal value. Using the call

```
x = sor(a,f,10^-10,0);
```

an approximation of the optimal ω is calculated from theory. Compare the results.

Exercise aims:

- SOR converges much faster than the previous methods.
- The convergence behavior can be non-linear.
- There are values of ω which leads to somewhat less iterations than the optimal ω calculated from theory.

d In this part we consider the Conjugate Gradient method.

```
x = cg(a,f,10^-10);
```

Compare the results with the results obtained with the basic iterative methods.

Exercise aims:

- CG converges very fast.
- It is not necessary to estimate an optimal parameter.
- The convergence behavior is super linear.

e In the theory used to analyze CG three quantities are used: $\|x - x_i\|_2$, $\|x - x_i\|_A$, and $\|b - Ax_i\|_2$. These quantities are plotted by the call

```
x = cganal(a,f,10^-10);
```

Compute the condition of a by the Matlab command `cond(a)` and estimate the number of required CG iterations. Compare this with your experiments.

Exercise aims:

- All quantities decrease.
- The number of required estimations obtained from the theory is an upper bound.

f There are matrices where the theory for the CG method no longer holds, due to rounding errors. One of them is a discretization of the *bending beam equation*:

$$\frac{d^4c}{dx^4} = f.$$

The matrix can be constructed by `beam;` Take for the dimension 40 (thereafter also try 80 and 160). Solve the system by

```
x = cganal(a,f,10^-10);
```

and observe the results.

Exercise aims:

- $\|x - x_i\|_A$ forms a monotone decreasing sequence.
- $\|b - Ax_i\|_2$ increases at some iterations.
- CG has not been converged when the number of iterations is equal to the dimension of a.

g In the final exercises we investigate if the methods given in Chapter 2 and 3 can be used when the matrices are non-symmetric. Construct the matrix as follows: `[a,f] = poisson(5,5,0.1,0,'central');` and apply Gauss-Jacobi, Gauss-Seidel, SOR, and CG.

Exercise aims:

- All methods converge although the theory for CG is not valid.

h Construct the matrix as follows: `[a,f] = poisson(5,5,1,0,'central');` and apply Gauss-Jacobi, Gauss-Seidel, SOR, and CG.

Exercise aims:

- The basic iterative methods converge but CG does no longer converge.

i Construct the matrix as follows: `[a,f] = poisson(5,5,100,0,'central');` and apply Gauss-Jacobi, Gauss-Seidel, SOR, and CG.

Exercise aims:

- The iterative methods are divergent.

j Construct the matrix as follows: `[a,f] = poisson(5,5,100,0,'upwind');` and apply Gauss-Jacobi, Gauss-Seidel, SOR, and CG.

Exercise aims:

- The basic iterative methods converge.
- The choice of the discretization influences the convergence behavior of an iterative method.