Final Master Thesis Project proposal: Positivity-preserving Runge-Kutta Algebraic Flux Correction Schemes for Scalar Conservation Laws with Discontinuous Fluxes

Introduction

Conservation laws play an important role in the mathematical description of physical processes such fluid flow, bodies in motion or rotation, and electric charging. In essence, conservation laws state that a particular physical property of a measurable quantity within an isolated physical system does not change with time. As an example, consider the total amount of energy present in an isolated system. Even though energy can take different forms (kinetic, chemical, mechanical, etc.) and a conversion between one form and the other is possible, the sum of all energies in an isolated system does not change.
The most general form of a scalar one-dimensional conservation law is as follows

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$$

where $u(x,t)$ is the quantity to be conserved over time and $f(x,u)$ denotes the flux function. This master project will focus on fluxes of the form

$$f(x,u) = c(x)g(u)$$

or

$$f(x,u) = c(x)g(u) + \big(1 - c(x)\big)h(u),$$

where the coefficient function $c(x)$ is allowed to have jump discontinuities, e.g.,

$$c(x) = \begin{cases} c_L, & x < x_0, \\ c_R, & x \geq x_0, \end{cases}$$

and the flux functions $g(u)$ and $h(u)$ are typically non-linear and may or may not be convex. Fluxes of this type may occur for instance in car traffic flow models [To99], in continuous sedimentation of solid particles in liquids, or in two-phase flow [Di94]. Their numerical treatment is quite complicated since many amenable properties of standard fluxes such as the strict hyperbolicity or the convexity may be lost [LV02].

Problem description and challenges

This master thesis will focus both on theoretical aspects such as the existence and uniqueness of weak solutions to the aforementioned one-dimensional conservation law and its numerical solution. Standard solution techniques have severe problems in producing sufficiently accurate and physically correct solutions with crisp resolution of steep gradients and discontinuities without generating unphysical wiggles.
In this thesis work the algebraic flux correction (AFC) approach described in [K12a] for scalar conservation laws and generalised to hyperbolic systems in [K12b] should be applied and extended to conservation laws with discontinuous fluxes. In essence, it constitutes a modern high-resolution method that combines a Galerkin finite element method and a non-oscillatory first-order scheme - derived on the discrete level - to be used in the vicinity of discontinuities in a non-linear manner. During this thesis work, the AFC methodology should be combined with strong stability preserving Runge-Kutta time-stepping methods to achieve higher accuracies in time. As part of this master thesis, the design criteria underlying the AFC approach have to be considered and revised in the context of the multi-stage nature of Runge Kutta methods.

Time schedule

The following tasks are foreseen:

- Literature study in order to understand the basic theory of scalar conservation laws and the concept of algebraic flux correction and SSP-Runge Kutta methods;
- Selection of concrete model problems and test cases (definition of fluxes and Cauchy problems based on literature study) and formulation of a brief description of the theoretical properties;
- Development of the numerical method based on the existing AFC approach;
- Extension of AFC schemes to strongly stability preserving Runge-Kutta methods;
- Investigation and numerical comparison of the SSP-RK-AFC schemes;
- Writing the thesis.

Contact

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Literature

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[K12b] D. Kuzmin; M. Möller; M. Gurris: Algebraic Flux Correction II: Compressible Flow Problems. In: D. Kuzmin; R. Löhner; S. Turek (eds.) Flux-Corrected Transport: Principles, Algorithms, and Applications. Second Edition. Springer, 2012.

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