MASTER THESIS PROJECT PROPOSAL

High-order accurate and stable material point method

Introduction

Solid and fluid dynamics problems of practical interest are typically modelled by coupled systems of nonlinear partial differential equations, which cannot be solved analytically. Therefore, one has to apply a suitable numerical method to compute an approximation to the exact solution. Most of these numerical schemes can be classified as either Lagrangian or Eulerian-type methods with further distinction into mesh-less schemes and those that introduce a mesh that connects individual data points.

In mesh-less Lagrangian-type methods, the unknowns are fixed to individual material points (hereafter referred to as particles), which travel with the velocity of the fluid or the displacement of the solid over time. In essence, the particle position is updated by solving Newton's law of motion for the internal force due to particle interaction. This makes it rather simple to simulate multi-material problems since each particle has its own characteristic data (position, velocity, mass, volume, stress and application specific quantities) and can be displaced individually. However, a huge number of particles is necessary to obtain accurate approximations, whereas the individual movement of particles becomes a major difficulty once data points are connected in a mesh. In contrast, Eulerian-type methods monitor the rate of change of quantities at a fixed position in space (or a finite-dimensional control cell) but additional effort is required when it comes to the numerical simulation of multi-phase or multi-material problems.

The *Material Point Method* (MPM) [Du94, Du95] tries to combine the advantages of both worlds by using dedicated particles to reflect multi-materials or multiple phases of the same material, whereby the equations of motion are solved on a background mesh. It is an extension of the Fluid-Implicit-Particle method (FLIP [Br86]), which was developed in the field to fluid dynamical problems, to solid mechanical problems. Recently, MPM has gained interest for solving multi-phase and fluid-structure interaction problems.

Problem description and challenges

This master thesis will focus on enhancing the stability of the *Spline-based Material Point Method* or *SMPM* [Ko19, Ti19]. SMPM has been proposed to counter the problems associated with the classical MPM where discrete particles are coupled with the finite element method (FEM) to solve the equations of motion on a background mesh. For instance, the solutions computed using C^0-finite elements are known to display unwanted artefacts when particles cross mesh cell boundaries because the gradients of C^0-finite elements are discontinuous at cell boundaries. SMPM remedies this by employing $C^{\geq 1}$-smooth B-spline functions instead of C^0-finite elements, thus ensuring continuous gradients at element boundaries and artefact-free solutions. Note that B-splines and their generalizations (e.g., Non-Uniform Rational B-Splines or NURBS) form the main building block of *Isogeometric Analysis* [Hu05], which is a natural extension of FEM for solving real-world engineering problems on complex geometries.

The aim of this thesis is to enhance the stability and robustness of SMPM while retaining its high-order accuracy. When the particle distribution is highly inhomogeneous over the background mesh, it may happen that several mesh cells are almost empty, i.e., they contain very few particles. While B-splines that are non-zero over almost-empty elements still contribute to the computation of the solution, their contributions are usually much smaller than of those B-splines that are non-zero over elements with a higher fill-ratio. Such differences in contributions can lead to highly undesirable numerical instabilities. The objective of this thesis is to develop efficient stabilization techniques that do not deteriorate the high-order accuracy that B-splines offer. A rigorous treatment of partially-filled and even nearly-empty cells will be a major focus. This would greatly enhance the applicability of the SMPM to challenging physical problems. The specific problem (e.g., solid mechanics, fluid-structure interactions, two-phase flows) that the resulting method is tested upon can be chosen depending on the scientific background and the interest of the student.

Note that open-source libraries (C++, MATLAB, Python) containing implementations of SMPM will be provided so that the focus can be placed on the development of novel theoretical and algorithmic components (stabilization of the B-spline basis, optimal numerical quadrature, effective preconditioning techniques) rather than on the implementation of an isogeometric finite element code.

Time schedule

The following tasks are foreseen:

- Literature study on MPM and a brief introduction to advanced spline-constructions and their use in numerical methods;
- Selection of concrete test problems based on the findings of the literature study;
- Selection of the open-source codebase to be used depending on the student's experience;
- Implementation of the novel techniques for stabilization, quadrature etc., on top of the chosen SMPM codebase;
- Analysis and numerical investigation of the proposed techniques;
- Writing the thesis.

Contact

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Literature

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