Robust Algorithms for Discrete Tomography

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2 Algebraic Reconstruction Methods (ARM’s)
   - Model Description
   - ART, SIRT and SART
   - ARM Experiments

3 Discrete Tomography
   - DART
   - DART Experiments

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2. **Algebraic Reconstruction Methods (ARM’s)**
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   - ART, SIRT and SART
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Introduction

τόμος (tomos) + γράφειν (graphein) = Tomography

- tomos: slice/part
- graphein: to write
- Invention X-ray 1895 by Wilhelm Röntgen
- Non-invasive way to see inside of an object
Applications:

- Medical
- Geophysics
- Astrophysics
- Material Science
- Many others...
Introduction
Algebraic Reconstruction Methods (ARM's)
Discrete Tomography
Research Goals

$f(x, y)$

$P_\theta(t)$
Introduction

Algebraic Reconstruction Methods (ARM's)

Discrete Tomography

Research Goals

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Robust Algorithms for DT
Introduction
Algebraic Reconstruction Methods (ARM's)
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Original Image: $x$

Projection data: $t$

Reconstruction: $x$

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Robust Algorithms for DT
Roughly two ways to reconstruct an object from projections:

- *Analytical*: Using Fourier transforms
- *Algebraic*: Formulating problem as system of linear equations
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Model Description
Pixels / cells: $f_j, j = 1, 2, \ldots, N$
Rays: $p_i, i = 1, 2, \ldots, M$
Contribution (weight) cell $j$ to ray $i$: $w_{ij}$, assume $w_{ij} \geq 0$

\[
\begin{align*}
    w_{11} f_1 + w_{12} f_2 + \cdots + w_{1N} f_N &= p_1 \\
    w_{21} f_1 + w_{22} f_2 + \cdots + w_{2N} f_N &= p_2 \\
    &\vdots \\
    w_{M1} f_1 + w_{M2} f_2 + \cdots + w_{MN} f_N &= p_M.
\end{align*}
\]

$Wf = p$
Pixels / cells: \( f_j, j = 1, 2, \ldots, N \)
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\[ Wf = p \]
Kaczmarz’s Method / ART

By Stefan Kaczmarz (1937). Rediscovered (1970) as Algebraic Reconstruction Technique (ART) by Gordon, Bender and Herman.

Idea: Subsequently project approximation onto hyperplanes
Let \( \mathbf{w}_i = (w_{i1}, w_{i2}, \ldots, w_{iN})^T \), the \( i \)-th row of \( \mathbf{W} \)

And \( \mathbf{r}^k = \mathbf{p} - \mathbf{W} \mathbf{f}^k \), the \( k \)-th residual

The \( k \)-th approximation is found as\(^1\)

\[
\mathbf{f}^k = \mathbf{f}^{k-1} + \frac{\langle \mathbf{r}^{k-1}, \mathbf{w}_i \rangle}{\langle \mathbf{w}_i, \mathbf{w}_i \rangle} \mathbf{w}_i, \quad i = (k - 1) \mod (M) + 1.
\]

\(^1\) Avinash C. Kak and Malcolm Slaney; *Principles of Computerized Tomographic Imaging* (IEEE Press, 1987).
Let $\mathbf{w}_i = (w_{i1}, w_{i2}, \ldots, w_{iN})^T$, $i$-th row of $\mathbf{W}$
And $\mathbf{r}^k = \mathbf{p} - \mathbf{Wf}^k$, $k$-th residual
The $k$-th approximation is found as

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f^k = f^{k-1} + \frac{\langle r^{k-1}, w_i \rangle}{\langle w_i, w_i \rangle} w_i, \quad i = (k - 1) \mod (M) + 1.
\]
Simultaneous Iterative Reconstruction Technique (1979) by Dines and Lyttle.

ART: Project successively onto hyperplanes.

SIRT\(^1\):

i. First compute correction for all rows using current approximation.

ii. Average over all corrections.

\[
f_j^k = f_j^{k-1} + \frac{1}{\sum_{i=1}^{M} w_{ij}} \sum_{i=1}^{M} \frac{w_{ij} r_i^{k-1}}{\sum_{h=1}^{N} w_{ih}}.
\]

\(^1\)Jens Gregor and Thomas Benson, "Computational analysis and improvement of SIRT", *IEEE Transactions on Medical Imaging* 27(7) (July 2008):918–924.
SIRT

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SART


SART\(^2\): Update per projection angle
i. Compute correction for all rays with angle \(\theta_i\).
ii. Average over these corrections.

\(R\): No. rays per angle

\[
f_j^{k} = f_j^{k-1} + \frac{1}{\sum_{i=R \cdot (l-1)+1}^{R \cdot l} W_{ij}} \sum_{i=R \cdot (l-1)+1}^{R \cdot l} \frac{r_i^{k-1} W_{ij}}{\sum_{h=1}^{N} W_{ih}}.
\]

SART


SART\textsuperscript{2}: Update per projection angle

i. Compute correction for all rays with angle $\theta_i$.

ii. Average over these corrections.

$R$: No. rays per angle

\[
    f_j^k = f_j^{k-1} + \frac{1}{\sum_{i=R\cdot(l-1)+1}^{R\cdot l} W_{ij}} \sum_{i=R\cdot(l-1)+1}^{R\cdot l} \frac{r_i^{k-1} W_{ij}}{\sum_{h=1}^{N} W_{ih}}.
\]

SART


SART\(^2\): Update per projection angle

i. Compute correction for all rays with angle \(\theta_l\).

ii. Avarage over these corrections.

\(R\): No. rays per angle

\[
f_j^k = f_j^{k-1} + \frac{1}{\sum_{i=R \cdot (l-1)+1}^{R \cdot l} W_{ij}} \sum_{i=R \cdot (l-1)+1}^{R \cdot l} \frac{r_i^{k-1} W_{ij}}{\sum_{h=1}^{N} W_{ih}}.
\]

Convergence of SIRT

Recall SIRT

\[ f_j^k = f_j^{k-1} + \frac{1}{\sum_{i=1}^{M} w_{ij}} \sum_{i=1}^{M} \frac{w_{ij} r_i^{k-1}}{\sum_{h=1}^{N} w_{ih}}. \]

Let \( C, R \) be diagonal matrices containing inverse column (\( C \)) and row (\( R \)) sums.

\[
C = \begin{pmatrix} \ldots & \frac{1}{\sum_{i=1}^{M} w_{ij}} & \ldots \\ & \ldots & \ldots \end{pmatrix}, \quad R = \begin{pmatrix} \ldots & \frac{1}{\sum_{j=1}^{N} w_{ij}} & \ldots \\ & \ldots & \ldots \end{pmatrix},
\]

Then SIRT can be written as

\[ f^k = f^{k-1} + CW^T R r^{k-1} \]
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\[ f^k = f^{k-1} + CW^T Rr^{k-1}. \]
Rewrite

\[ f^k = f^{k-1} + CW^T Rr^{k-1} \]
\[ = f^{k-1} + CW^T R(p - Wf^{k-1}) \]
\[ = (I - CW^T RW)f^{k-1} + CW^T Rp. \]

\((I - CW^T RW)\) is iteration matrix.

**Definition**

The spectral radius of \( A \in \mathbb{R}^{n \times n} \), denoted \( \rho(A) \), is defined as

\[ \rho(A) = \max_{\lambda_i, i=1,\ldots,n} |\lambda_i| \]

where \( \lambda_i \) are the eigenvalues of \( A \).

If \( \rho(I - CW^T RW) < 1 \), then convergence is guaranteed\(^3\).

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Let $\lambda$ be an eigenvalue of $CW^T RW \Rightarrow 1 - \lambda$ eigenvalue of $(I - CW^T RW)$

To prove

$$\rho \left( I - CW^T RW \right) = \max_{\lambda} |1 - \lambda| < 1 \Leftrightarrow 0 < \lambda < 2$$

Unfortunately, if $W$ is not of full rank, one can only show $\lambda \geq 0$. Then stagnation may occur: error does not change.
Let $\lambda$ be an eigenvalue of $CW^TRW \Rightarrow 1 - \lambda$ eigenvalue of $(I - CW^TRW)$

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Unfortunately, if $W$ is not of full rank, one can only show $\lambda \geq 0$. Then stagnation may occur: error does not change.
Recall: $R, C > 0$ diagonal matrices.

$W^T W$ is symmetric positive semidefinite (SPSD) and since $R > 0$, $W^T R W$ is SPSD.

$C$ is positive definite thus $C W^T R W$ has eigenvalues $\lambda \geq 0$. 
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Recall: $R, C > 0$ diagonal matrices. $W^T W$ is symmetric positive semidefinite (SPSD) and since $R > 0$ $W^T RW$ is SPSD. $C$ is positive definite thus $CW^T RW$ has eigenvalues $\lambda \geq 0$. 
Remains to show that $\lambda < 2$.

The spectral radius of a matrix is less or equal to any operator norm\textsuperscript{4}. Thus:

$$\rho(CW^T RW) \leq \|CW^T RW\|_{\infty} \leq \|CW^T\|_{\infty} \|RW\|_{\infty}.$$

Recall: $c_{jj}$ are inverse column sums of $W \rightarrow$ inverse rows sums of $W^T \Rightarrow \|CW^T\|_{\infty} = 1$. Equivalently $\|RW\|_{\infty} = 1$

Thus $0 \leq \lambda \leq 1 < 2$

Hence SIRT either converges or stagnates.

\textsuperscript{4} James W. Demmel. \textit{Applied numerical linear algebra}. (S.I.A.M., 1997)
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ARM Experiments

Used image was the *Shepp-Logan head phantom*. The image was 128 by 128 pixels and scanned using 32 projection angles with 192 rays per projection.
Without Noise

ART: 5 iter.
SIRT: 200 iter.
SART: 200 iter.
Without Noise

\[
\frac{\|\Delta f\|_2}{\|f\|_2}
\]

- ART
- SIRT
- SART

Iterations

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With Noise

![Graph showing iterations vs. \( \| \Delta f \|_2 \) and \( \| f \|_2 \)]
With Noise

ART: 7 iterations  SIRT: 200 iterations  SART: 3 iterations

ART: 200 iterations  SIRT: 200 iterations  SART: 200 iterations
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Discrete Tomography

Discrete tomography

- Object consists of some finite set of densities $\{\rho_1, \rho_2, \ldots, \rho_l\}$.
- In general very few projections angles ($< 15$) resulting from a small angular range;
- Different strategies for solving:
  - Combinatorial
  - Statistical
  - Continuous optimisation
  - Continuous with discretisation step $\Rightarrow$ DART\textsuperscript{5}

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  - Combinatorial
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  - Continuous with discretisation step \( \Rightarrow \text{DART}^5 \)

---

Discrete Algebraic Reconstruction Technique (DART)

Initial ARM reconstruction

Identify fixed pixels $F$ and free pixels $U$

Apply ARM to free pixels $U$

Smooth reconstruction

Stop criterion met?

Final reconstruction

Segment the reconstruction

no

yes
Discrete Algebraic Reconstruction Technique (DART)

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Smooth reconstruction

Stop criterion met?

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Final reconstruction

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Discrete Algebraic Reconstruction Technique (DART)

**Initial ARM reconstruction**
- Identify fixed pixels $F$ and free pixels $U$
- Apply ARM to free pixels $U$

**Segment the reconstruction**

**Stop criterion met?**
- yes: Final reconstruction
- no: Segment the reconstruction

**Final reconstruction**

**Identify fixed pixels $F$ and free pixels $U$**

**Smooth reconstruction**
**Segmentation**

*Segmentation* is setting the values of the pixels to one of the admitted grey values $\rho \in \{\rho_1, \rho_2, \ldots, \rho_l\}$.

Most intuitive segmentation, rounding values to nearest grey value:

$$\tau_i = \frac{\rho_i + \rho_{i+1}}{2},$$

$$r(v) = \begin{cases} 
\rho_1, & (v < \tau_1) \\
\rho_2, & (\tau_1 \leq v < \tau_2) \\
\vdots \\
\rho_l, & (\tau_{l-1} \leq v)
\end{cases}.$$
Segmentation

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Discrete Algebraic Reconstruction Technique (DART)

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Apply ARM to free pixels $U$

Smooth reconstruction

Segment the reconstruction

Stop criterion met?

yes

no

Final reconstruction
Fixed and Free Pixels

Set of fixed pixels $F$: Pixels surrounded by pixels with the same grey value.
Free (boundary) pixels $U$: At least one neighbour with a different grey value.

Every pixel in $F$ is freed with probability $1 - p$.

$p$: The fix probability

Needed to find overlooked holes in the image.
Fixed and Free Pixels

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Needed to find overlooked holes in the image.
Discrete Algebraic Reconstruction Technique (DART)

- Initial ARM reconstruction
- Identify fixed pixels $F$ and free pixels $U$
- Apply ARM to free pixels $U$
- Segment the reconstruction
- Smooth reconstruction
- Stop criterion met?
- Final reconstruction
Apply ARM to Free Pixels

Original system:

\[
\begin{pmatrix}
  w_{:,1} & \cdots & w_{:,N}
\end{pmatrix}
\begin{pmatrix}
  f_1 \\
  \vdots \\
  f_N
\end{pmatrix}
= \begin{pmatrix}
  p_1 \\
  \vdots \\
  p_M
\end{pmatrix}.
\]

Suppose pixel \( j \) is fixed, new system:

\[
\begin{pmatrix}
  w_{:,1} & \cdots & w_{:,j-1} & w_{:,j+1} & \cdots & w_{:,N}
\end{pmatrix}
\begin{pmatrix}
  f_1 \\
  \vdots \\
  f_{j-1} \\
  f_{j+1} \\
  f_N
\end{pmatrix}
= \begin{pmatrix}
  p_1 \\
  \vdots \\
  p_M
\end{pmatrix} - w_{:,j} f_j.
Apply ARM to Free Pixels

Original system:

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\]
Discrete Algebraic Reconstruction Technique (DART)

1. Initial ARM reconstruction
2. Identify fixed pixels $F$ and free pixels $U$
3. Apply ARM to free pixels $U$
4. Smooth reconstruction
5. Stop criterion met?
   - yes: Final reconstruction
   - no: Segment the reconstruction and repeat from step 2.
Discrete Algebraic Reconstruction Technique (DART)

Initial ARM reconstruction

- Identify fixed pixels $F$ and free pixels $U$

- Apply ARM to free pixels $U$

- Smooth reconstruction

- Segment the reconstruction

- Stop criterion met?
  - yes
  - no

Final reconstruction
**Discrete Algebraic Reconstruction Technique (DART)**

1. **Initial ARM reconstruction**
2. Identify fixed pixels $F$ and free pixels $U$
3. Apply ARM to free pixels $U$
4. Smooth reconstruction
5. Stop criterion met?
6. If no, go back to Segment the reconstruction.
7. Yes, Final reconstruction.

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**Research Goals**

- **Discrete Tomography**
- **Discrete Algebraic Reconstruction Methods (ARM's)**

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**DART Experiments**

- **D ART**
- **D ART Experiments**

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**Robust Algorithms for DT**
DART

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Robust Algorithms for DT
Introduction
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DART
DART Experiments

Initial ARM Reconstruction
Segmentation
Free Pixels

ARM on Free Pixels
Smoothed Image
Segmentation,
First DART reconstruction
DART Experiments
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The approach of DART is rather heuristic at the moment:

- The smoothing operation;
- The random subset construct.

*Can the DART algorithm be improved?*

- Which algorithm should be used as ARM in DART and does it matter?
- Can better results be obtained by introducing *regularization* directly onto the set of free pixels $U$?
- Are there alternatives for the random subset construct?
Research Questions

The approach of DART is rather heuristic at the moment:

- The smoothing operation;
- The random subset construct.

*Can the DART algorithm be improved?*

- Which algorithm should be used as ARM in DART and does it matter?
- Can better results be obtained by introducing *regularization* directly onto the set of free pixels $U$?
- Are there alternatives for the random subset construct?
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_Can the DART algorithm be improved?_

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Regularization is the use of additional information to make an ill-posed problem well-posed. The segmentation is a form of regularization.
To answer the questions one could solve the system

\[
\begin{pmatrix}
W \\
D
\end{pmatrix}
\mathbf{f} = \begin{pmatrix}
p \\
D\mathbf{v}
\end{pmatrix},
\]

\(D\) diagonal matrix,
\(\mathbf{v}\) vector.
Questions