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Expanding the Applicability of the Competitive Modes Conjecture

Sudipto Choudhury, Huibert Reijm, and Cornelis Vuik

Abstract The Competitive Modes Conjecture is a relatively new approach in the field of Dynamical Systems, aiming to understand chaos in strange attractors using Resonance Theory. Up till now, the Conjecture has only been used to study multipolynomial systems because of their simplicity. As such, the study of non-multipolynomial systems is sparse, filled with ambiguity, and lacks mathematical structure. This paper strives to rectify this dilemma, providing the mathematical background needed to rigorously apply the Competitive Modes Conjecture to a certain set of non-multipolynomial systems. Afterwards, we provide an example of this new theory in the non-multipolynomial Wimol-Banlue Attractor, something that up to this point has not been possible as far as the authors know.

3.1 The Competitive Modes Conjecture

This section is to serve as background knowledge, all obtained from sources [1–6].

We take a general n-dimensional autonomous system of differential equations $\dot{x}_i = F_i(\mathbf{x})$ with $\mathbf{x} \in \mathbb{R}^n$ and $i \in \{1, 2, \ldots, n\}$. We can easily transform this system into a system of interconnected oscillators as follows

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$$\begin{aligned}\ddot{x}_i &= \dot{F}_i(\mathbf{x}) \\ &= \sum_{j=1}^{n} \frac{\partial F_i}{\partial x_j}(\mathbf{x})\dot{x}_j \\ &= \sum_{j=1}^{n} \frac{\partial F_i}{\partial x_j}(\mathbf{x})F_j(\mathbf{x}) \\ &\equiv f_i(\mathbf{x})\end{aligned} \tag{3.1}$$

This of course only works if F_i is x_j-differentiable for all $i, j \in \{1, 2, \ldots, n\}$.

Definition 3.1 (*Splitting of a Function*) In previous literature, function $f_i : \mathbb{R}^n \to \mathbb{R}$ can be split with respect to x_i if it can be rewritten as

$$f_i(\mathbf{x}) = h_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) - x_i g_i(\mathbf{x}) \ \forall i \in \{1, 2, \ldots, n\} \tag{3.2}$$

We name function $h_i : \mathbb{R}^{n-1} \to \mathbb{R}$ the ith forcing function. We name function $g_i : \mathbb{R}^n \to \mathbb{R}$ the i^{th} squared frequency function.

For simplicity, let us define $\mathbf{x}_i^* = [x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]^T \in \mathbb{R}^{n-1}$. If Eq. (3.1) holds and the resulting functions f_i can be split, then we can rewrite our original system of differential equations into the form given below.

$$\begin{cases} \ddot{x}_1 + g_1(\mathbf{x})x_1 = h_1(\mathbf{x}_1^*) \\ \ddot{x}_2 + g_2(\mathbf{x})x_2 = h_2(\mathbf{x}_2^*) \\ \cdots \\ \ddot{x}_i + g_i(\mathbf{x})x_i = h_i(\mathbf{x}_i^*) \\ \cdots \\ \ddot{x}_n + g_n(\mathbf{x})x_n = h_n(\mathbf{x}_n^*) \end{cases} \tag{3.3}$$

In a sense, we have turned our system into a system of interconnected, nonlinear oscillators.

Definition 3.2 (*Competitive Modes*) Say we have the n-dimensional autonomous system of differential equations $\mathbf{x} = \mathbf{F}(\mathbf{x})$. If Eq. (3.1) holds for this system and the resulting functions f_i can be split, then the system can be transformed as shown in Eq. (3.3). The solutions x_i for Eq. (3.3) are then known as the competitive modes of the system, with g_i and h_i being the corresponding squared frequency functions and forcing functions, respectively.

Currently, there is an open conjecture connecting chaos and competitive modes together, and it is presented as follows.

Conjecture 3.1 (Competitive Modes Conjecture) *The conditions for a dynamical system to be chaotic are given below* (*assuming* Eq. (3.1) *holds and the resulting function* f_i's *can be split:*)

- *the dimension n of the dynamical system is greater than 2;*
- *at least two distinct squared frequency functions g_i and g_j are competitive or nearly competitive; that is, there exists $t \in \mathbb{R}$ so that $g_i(t) \approx g_j(t)$ and $g_i(t), g_j(t) > 0$;*
- *at least squared frequency function g_i is not constant with respect to time;*
- *at least one forcing function h_i is not constant with respect to some system variable x_j.*

3.2 Proper Splittings

Notice that the process of splitting as defined in Definition 3.1 is rather ambiguous. Therefore, we now provide a new definition for the splitting of a function. Throughout this paper, we refer to domain D, which is a uncountably infinite, open set in $\mathbb{R}^n$.

Definition 3.3 (*Splitting of a Function*) We now say that function $f : D \to \mathbb{R}$ can be split with respect to $x_i \in \mathbb{R}$ and $\mathbf{c} \in D$ if over D, it can be rewritten as

$$f(\mathbf{x}) = h(\mathbf{x}_i^*) - (x_i - c_i)g(\mathbf{x}) \tag{3.4}$$

where $\mathbf{x}_i^* = [x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]^T$ and

- f is continuous in x_i for all $\mathbf{x} \in D$;
- the subset $D_i^*(\mathbf{c}) = \{\mathbf{x} \in D : x_i = c_i\}$ is not empty;
- h is constant and finite in x_i, given $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$;
- g is continuous with respect to x_i, given $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$

Here, h is again called the forcing function and g is the squared frequency function.

We then have the following results, lemmas, and theorems.

Lemma 3.1 *Say function $f : D \to \mathbb{R}$ can be split with respect to $x_i \in \mathbb{R}$ and $\mathbf{c} \in D$ into forcing function h and squared frequency function g. Then $h(\mathbf{x}_i^*) = f(\mathbf{x})|_{x_i=c_i}$.*

Proof Say function $f : D \to \mathbb{R}$ can be split with respect to $x_i \in \mathbb{R}$ into forcing function h and squared frequency function g. Then for all $\mathbf{x} \in D$, since g is continuous in x_i,

$$g(\mathbf{x})|_{x_i=\alpha} = \lim_{x_i \to \alpha} \left(\frac{h(\mathbf{x}_i^*) - f(\mathbf{x})}{x_i - c_i} \right)$$

Thus, we can conclude that

$$g(\mathbf{x})|_{x_i=c_i} = \lim_{x_i \to c_i} \left(\frac{h(\mathbf{x}_i^*) - f(\mathbf{x})}{x_i - c_i} \right) \in \mathbb{R}$$

Because of this, $\lim_{x_i \to c_i} \big(h(\mathbf{x}_i^*) - f(\mathbf{x})\big) = 0$. Otherwise, $\lim_{x_i \to c_i} g(\mathbf{x})$ would surely be infinite or undefined. Thus, we can conclude that, since f is continuous in x_i,

$$0 = \lim_{x_i \to c_i} \big(h(\mathbf{x}_i^*) - f(\mathbf{x})\big) = h(\mathbf{x}_i^*) - \lim_{x_i \to c_i} f(\mathbf{x}) = h(\mathbf{x}_i^*) - f(\mathbf{x})|_{x_i=c_i}$$

This lemma is important, as it symbolizes the ideology behind Definition 3.3. Our research started by trying to rigorously define the forcing function h, and then defining the squared frequency function g as a direct result. We noticed that in multipolynomial systems, Lemma 3.1 was always true. In fact, it seemed that previous literature had specifically defined h so that the lemma would always hold when $\mathbf{c} = \mathbf{0}$ [1–6]. We decided to expand this idea to Taylor Series, Laurent Series, and finally to general continuous functions. It is on this idea that we can build the rest of our theory.

Lemma 3.2 (Uniqueness Lemma) *Say function $f : D \to \mathbb{R}$ can be split with respect to $x_i \in \mathbb{R}$ and $\mathbf{c} \in D$ into forcing function h and squared frequency function g. Then h and g are uniquely defined.*

Proof Say function $f : D \to \mathbb{R}$ can be split with respect to $x_i \in \mathbb{R}$ and $\mathbf{c} \in D$ into forcing function h_1 and squared frequency function g_1, and also into forcing function h_2 and squared frequency function g_2. Then for all $\mathbf{x} \in D$,

$$f(\mathbf{x}) = h_1(\mathbf{x}_i^*) - (x_i - c_i)g_1(\mathbf{x}) = h_2(\mathbf{x}_i^*) - (x_i - c_i)g_2(\mathbf{x})$$

Recall that $D_i^*(\mathbf{c}) = \{\mathbf{x} \in D : x_i = c_i\}$.
Since we know from Lemma 3.1 that $h_1(\mathbf{x}_i^*) = h_2(\mathbf{x}_i^*) = f(\mathbf{x})|_{x_i=c_i}$, we can immediately conclude that $h_1 = h_2$.
As a result, for all $\mathbf{x} \in D$,

$$(x_i - c_i)(g_1(\mathbf{x}) - g_2(\mathbf{x})) = h_1(\mathbf{x}_i^*) - h_2(\mathbf{x}_i^*) = 0$$

For all $\mathbf{x} \in D \backslash D_i^*(\mathbf{c})$, $g_1(\mathbf{x}) - g_2(\mathbf{x}) = 0$.
Furthermore, since g_1 and g_2 are both continuous in $D_i^*(\mathbf{c})$, we can conclude that

$$g_1(\mathbf{x})|_{x_i=c_i} = \lim_{x_i \to c_i} g_1(\mathbf{x}) = \lim_{x_i \to c_i} g_2(\mathbf{x}) = g_2(\mathbf{x})|_{x_i=c_i}$$

Thus, we have proven that $g_1(\mathbf{x}) = g_2(\mathbf{x})$ for all $\mathbf{x} \in D$.

Lemma 3.3 (Combination Lemma) *Say function $f_1 : D \to \mathbb{R}$ can be split with respect to $x_i \in \mathbb{R}$ and $\mathbf{c} \in D$ into forcing function h_1 and squared frequency function g_1. Say function $f_2 : D \to \mathbb{R}$ can be split with respect to x_i and $\mathbf{c}$ into forcing function h_2 and squared frequency function g_2.*

- *For arbitrary $\alpha, \beta \in \mathbb{R}$, the sum $(\alpha f_1 + \beta f_2) : D \to \mathbb{R}$ can be split with respect to x_i and $\mathbf{c}$ into forcing function $(\alpha h_1 + \beta h_2)$ and squared frequency function $(\alpha g_1 + \beta g_2)$.*
- *The product $(f_1 f_2) : D \to \mathbb{R}$ can be split with respect to x_i into forcing function $(h_1 h_2)$ and squared frequency function $(h_1 g_2 + h_2 g_1 - (x_i - c_i) g_1 g_2)$.*
- *The quotient $(f_1/f_2) : D \to \mathbb{R}$ can be split with respect to x_i and $\mathbf{c}$ into forcing function (h_1/h_2) and squared frequency function $((h_2 g_1 - h_1 g_2)/(h_2 f_2))$, provided both $f_2(\mathbf{x})$ and $h_2(\mathbf{x}_i^*)$ are nonzero for all $\mathbf{x} \in D$.*

Proof Say function $f_1 : D \to \mathbb{R}$ can be split with respect to $x_i \in \mathbb{R}$ and $\mathbf{c} \in D$ into forcing function h_1 and squared frequency function g_1. Then for all $\mathbf{x} \in D$,

$$f_1(\mathbf{x}) = h_1(\mathbf{x}_i^*) - (x_i - c_i)g_1(\mathbf{x})$$

Say function $f_1 : D \to \mathbb{R}$ can be split with respect to x_i and $\mathbf{c}$ into forcing function h_2 and squared frequency function g_2. Then for all $\mathbf{x} \in D$,

$$f_2(\mathbf{x}) = h_2(\mathbf{x}_i^*) - (x_i - c_i)g_2(\mathbf{x})$$

First of all, notice that $D_i^*(\mathbf{c}) = \{\mathbf{x} \in D : x_i = c_i\}$ is automatically not empty since both f_1 and f_2 can be split on D.

Take $\alpha, \beta \in \mathbb{R}$ arbitrarily.

$$\begin{aligned}\alpha f_1(\mathbf{x}) + \beta f_2(\mathbf{x}) &= \alpha \left(h_1(\mathbf{x}_i^*) - (x_i - c_i)g_1(\mathbf{x})\right) + \beta \left(h_2(\mathbf{x}_i^*) - (x_i - c_i)g_2(\mathbf{x})\right) \\ &= \left(\alpha h_1(\mathbf{x}_i^*) + \beta h_2(\mathbf{x}_i^*)\right) - (x_i - c_i)\left(\alpha g_1(\mathbf{x}) - \beta g_2(\mathbf{x})\right)\end{aligned}$$

Notice that

- the linear combination $\alpha f_1 + \beta f_2$ is continuous over D in x_i since f_1 and f_2 are continuous over D in x_i;
- the linear combination $\alpha h_1 + \beta h_2$ is constant and finite over D in x_i since h_1 and h_2 are constant and finite over D in x_i;
- the linear combination $\alpha g_1 + \beta g_2$ is continuous over D in x_i since g_1 and g_2 are continuous over D in x_i.

Thus we constructed the splitting of $(\alpha f_1 + \beta f_2)$ with respect to x_i and $\mathbf{c}$.

We can also split the product of f_1 and f_2.

$$\begin{aligned}f_1(\mathbf{x})f_2(\mathbf{x}) &= \left(h_1(\mathbf{x}_i^*) - (x_i - c_i)g_1(\mathbf{x})\right)\left(h_2(\mathbf{x}_i^*) - (x_i - c_i)g_2(\mathbf{x})\right) \\ &= \left(h_1(\mathbf{x}_i^*)h_2(\mathbf{x}_i^*)\right) - (x_i - c_i)\left(h_1(\mathbf{x}_i^*)g_2(\mathbf{x}) + h_2(\mathbf{x}_i^*)g_2(\mathbf{x}) - (x_i - c_i)g_1(\mathbf{x})g_2(\mathbf{x})\right)\end{aligned}$$

Notice that

- the product $f_1 f_2$ is continuous over D in x_i since f_1 and f_2 are continuous over D in x_i;
- the product $h_1 h_2$ is constant and finite over D in x_i since h_1 and h_2 are constant and finite over D in x_i;
- the function $h_1(\mathbf{x}_i^*)g_2(\mathbf{x}) + h_2(\mathbf{x}_i^*)g_2(\mathbf{x}) - (x_i - c_i)g_1(\mathbf{x})g_2(\mathbf{x})$ is continuous over D in x_i since g_1 and g_2 are continuous and h_1 and h_2 are constant and finite over D in x_i.

Thus we constructed the splitting of $f_1 f_2$ with respect to x_i and $\mathbf{c}$.

We can also split the quotient of f_1 and f_2, provided $h_2(\mathbf{x}_i^*) \neq 0$ and $f_2(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in D$.

$$\frac{f_1(\mathbf{x})}{f_2(\mathbf{x})} = \frac{h_1(\mathbf{x}_i^*) - (x_i - c_i)g_1(\mathbf{x})}{h_2(\mathbf{x}_i^*) - (x_i - c_i)g_2(\mathbf{x})}$$
$$= \left(\frac{h_1(\mathbf{x}_i^*)}{h_2(\mathbf{x}_i^*)}\right) - (x_i - c_i)\left(\frac{h_2(\mathbf{x}_i^*)g_1(\mathbf{x}) - h_1(\mathbf{x}_i^*)g_2(\mathbf{x})}{h_2(\mathbf{x}_i^*)f_2(\mathbf{x})}\right)$$

Notice that

- the quotient f_1/f_2 is continuous over D in x_i since f_1 and f_2 are continuous over D in x_i and $f_2(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in D$;
- the quotient h_1/h_2 is constant and finite over D in x_i since h_1 and h_2 are constant and finite over D in x_i and $h_2(\mathbf{x}_i^*) \neq 0$ for all $\mathbf{x} \in D$;
- the function $(h_2 g_1 - h_1 g_2) / (h_2 f_2)$ is continuous over D in x_i since g_1 and g_2 are continuous over D in x_i, h_1 and h_2 are constant and finite over D in x_i, and $h_2(\mathbf{x}_i^*) \neq 0$ and $f_2(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in D$.

Thus we have constructed the splitting of f_1/f_2 with respect to x_i and $\mathbf{c}$.

The following theorem is perhaps the most useful theorem concerning splittable functions.

Theorem 3.1 (Existence of Splittings for Differentiable Functions) *Say function $f : D \to \mathbb{R}$ is differentiable over D with respect to $x_i \in \mathbb{R}$. Take $\mathbf{c} \in D$. If the partial derivative $\partial f/\partial x_i$ is continuous with respect to x_i in c_i, then f can be split into proper forcing function h and proper squared frequency function g, defined as*

$$h(\mathbf{x}_i^*) = f(\mathbf{x})|_{x_i=c_i}$$
$$g(\mathbf{x}) = \begin{cases} \dfrac{f(\mathbf{x})|_{x_i=c_i} - f(\mathbf{x})}{x_i - c_i} & x_i \neq c_i \\ -\left.\dfrac{\partial f(\mathbf{x})}{\partial x_i}\right|_{x_i=c_i} & x_i = c_i \end{cases}$$

Proof Say function $f : D \to \mathbb{R}$ is differentiable over D with respect to x_i. Lets define functions h and g as above.

Since f is differentiable and thus continuous over D with respect to x_i, we know immediately from Lemma 3.1 that h is constant and finite in terms of x_i, given $x_1, \ldots x_{i-1}, x_{i+1}, \ldots x_n$.

Investigating the properties of g takes a bit more work. Lets take $x_i \neq c_i$, then g is continuous over D in x_i because f is differentiable and thus also continuous over D in x_i.
Lets take $x_i = c_i$, then we can conclude the following, using L'Hopital's Theorem and the prerequisite that the derivative $\partial f/\partial x_i$ must be continuous in c_i.

$$\begin{aligned}\lim_{x_i\to c_i} g(\mathbf{x}) &= \lim_{x_i\to c_i}\left(\frac{f(\mathbf{x})|_{x_i=c_i} - f(\mathbf{x})}{x_i - c_i}\right)\\ &= -\lim_{x_i\to c_i}\frac{\partial f(\mathbf{x})}{\partial x_i}\\ &= -\frac{\partial f(\mathbf{x})}{\partial x_i}\bigg|_{x_i=c_i}\\ &= g(\mathbf{x})|_{x_i=c_i}\end{aligned}$$

Thus, we have proven that g is continuous in D with respect to x_i.

Finally, we must prove that the equation

$$f(\mathbf{x}) = h(\mathbf{x}_i^*) - (x_i - c_i)g(\mathbf{x})$$

is valid in the first place. Take $\mathbf{x} \in D$ arbitrarily. We then have to consider two mutually exclusive cases.
Say $x_i \neq c_i$. Then

$$\begin{aligned}h(\mathbf{x}_i^*) - (x_i - c_i)g(\mathbf{x}) &= f(\mathbf{x})|_{x_i=c_i} - (x_i - c_i)\left(\frac{f(\mathbf{x})|_{x_i=c_i} - f(\mathbf{x})}{x_i - c_i}\right)\\ &= f(\mathbf{x})|_{x_i=c_i} - \left(f(\mathbf{x})|_{x_i=c_i} - f(\mathbf{x})\right)\\ &= f(\mathbf{x})\end{aligned}$$

Say instead $x_i = c_i$. Then we know that $((x_i - c_i)g(\mathbf{x}))\,|_{x_i=c_i} = 0$ since $g(\mathbf{x})|_{x_i=c_i}$ is continuous and therefore finite. Thus

$$\begin{aligned}h(\mathbf{x}_i^*) - (x_i - c_i)g(\mathbf{x}) &= f(\mathbf{x})|_{x_i=c_i} - 0\\ &= f(\mathbf{x})\end{aligned}$$

Thus, for any $\mathbf{x} \in D$, $h(\mathbf{x}_i^*) - (x_i - c_i)g(\mathbf{x}) = f(\mathbf{x})$. Thus, h is the forcing function and g is the squared frequency function of f.

Of course, a splitting of f can not be achieved without defining $\mathbf{c} \in D$ first. The constant $\mathbf{c}$ can of course be arbitrary, but we will primarily focus on one particular scenario. When a function f is split with respect to $\mathbf{c} = \mathbf{0}$, then we define this to be the proper splitting of f, with h defined to be the proper forcing function and g defined to be the proper squared frequency function.The reason for this is made clear with an example.

Lets say we have a multipolynomial second order ODE $\ddot{x}_i = f(\mathbf{x})$, where $f{:}D \to \mathbb{R}$. Previous literature (as far as the authors are aware) has strictly focused on gathering evidence for the Competitive Modes Conjecture from dynamical systems whose set of differential equations consist of these sorts of ODEs. It can be easily proven[1] that the proper splitting of f always exists, and that the resulting proper forcing function and proper squared frequency function are defined identically to the forcing functions and squared frequency functions defined in previous literature [1–6]. As a result, the theory of proper splittings is a direct expansion of Definition 3.1.

3.3 Example: The Wimol-Banlue Attractor

To show the applicability of this new theory of proper splittings, we will apply it to a modification of the system mentioned in [7], which we will call the Wimol-Banlue System. The original Wimol Banlue Dynamical System is given by

$$\begin{cases} \dot{x} &= y - x \\ \dot{y} &= -z\tanh(x) \\ \dot{z} &= -\alpha + xy + |y| \end{cases} \tag{3.5}$$

where $\alpha \in \mathbb{R}$. The reason we chose to work with the Wimol-Banlue System is because it is the most accessible non-multipolynomial system which has been proven to exhibit a chaotic attractor. An unfortunate property of this system is that $\dot{z}$ is not differentiable with respect to y at $y = 0$. To counterattack this, we introduce function ϕ, dependent on parameter $\beta > 0$, defined as

$$\phi(y;\beta) = \sqrt{y^2 + \beta^2} \tag{3.6}$$

First, notice that ϕ is a well-defined, positive, differentiable function over all $\mathbf{R}$, with its derivative being

$$\phi'(y;\beta) = \frac{y}{\sqrt{y^2 + \beta^2}}$$

We want to compare $\phi(y;\beta)$ to $|y|$; to that end, we construct the difference function $\varphi(y;\beta) = \phi(y;\beta) - |y|$. It is easy to prove that φ is a positive, continuous function for $y \in \mathbb{R}$. Furthermore φ is differentiable for $y \neq 0$, with its derivative being

$$\varphi'(y;\beta) = \begin{cases} \dfrac{\sqrt{y^2} - \sqrt{y^2 + \beta^2}}{\sqrt{y^2 + \beta^2}} & y > 0 \\ \dfrac{-\sqrt{y^2} + \sqrt{y^2 + \beta^2}}{\sqrt{y^2 + \beta^2}} & y < 0 \end{cases}$$

[1] The calculations needed to prove this are straightforward but cumbersome. For the sake of space, we chose to omit them.

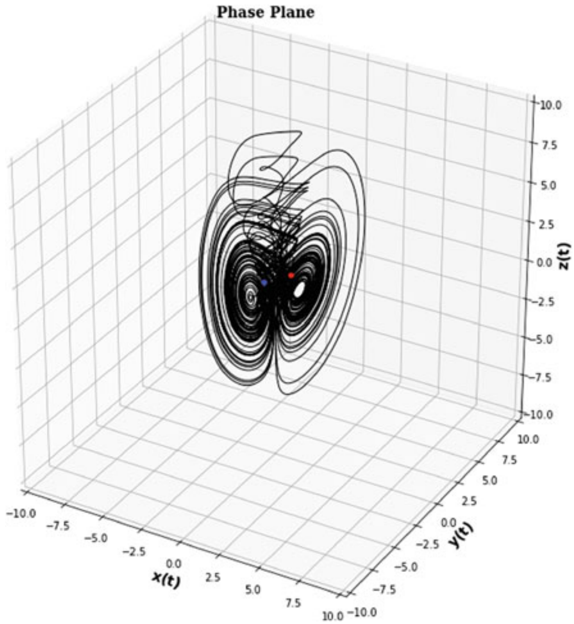

Fig. 3.1 The trajectory of our modified Wimol-Banlue Attractor as defined in Eq. (3.7) with initial condition $\mathbf{x}_0 = [1.32, -0.63, 1.91]^T$. The trajectory was approximating using 70,000 iterations of an adaptive RK4 method, using a time step of 0.01. Notice the presence of an attractor

Because of this, $\varphi'(y; \beta) < 0$ for $y > 0$ and $\varphi'(y; \beta) > 0$ for $y < 0$; we can then make the following inequality

$$|\varphi(y; \beta)| \leq |\varphi(0; \beta)| = \beta$$

Thus ϕ converges uniformly to $|y|$ as β goes to 0. Therefore, ϕ is a sufficiently accurate, differentiable approximation of $|y|$ and we can modify the Wimol-Banlue System slightly into

$$\begin{cases} \dot{x} &= y - x \\ \dot{y} &= -z \tanh(x) \\ \dot{z} &= -\alpha + xy + \sqrt{y^2 + \beta^2} \end{cases} \tag{3.7}$$

Let us first prove that this modified system still has a chaotic attractor. For the continuation of this example, lets say $\alpha = 2$ and $\beta = 0.001$. With arbitrary initial vector $\mathbf{x}_0 = [1.32, -0.63, 1.91]^T$, the resulting trajectory is presented in Fig. 3.1. As one can see, an attractor is still present in this system.

Through this trajectory, the Lyapunov Exponent is approximately equal to 0.228483. As further evidence of the attractor's chaotic nature, we provide the plot of the convergence of the Lyapunov Exponent in Fig. 3.2.

We consider this sufficient evidence to safely proven the presence of a chaotic attractor in our system.

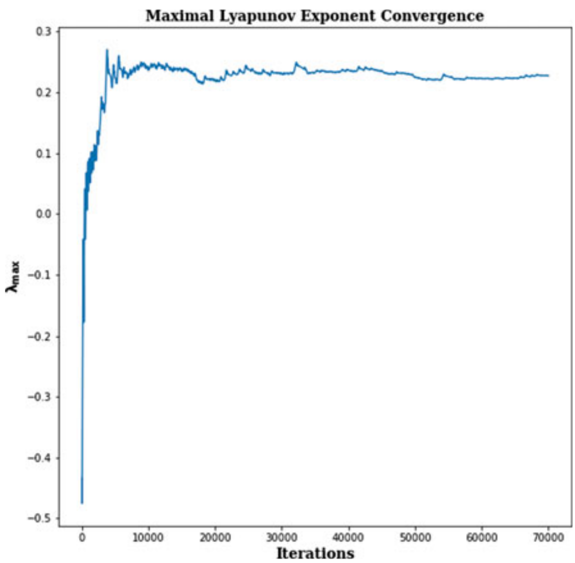

Fig. 3.2 The convergence of the maximal Lyapunov Exponent of our modified Wimol-Banlue Attractor, using a trajectory with initial condition $\mathbf{x}_0 = [1.32, -0.63, 1.91]^T$. The trajectory was approximating using 70,000 iterations of an adaptive RK4 method, using a time step of 0.01

To see if the modified system in Eq. (3.7) can be properly split, the system must first be differentiated in terms of time, which is done as follows.

$$
\begin{aligned}
\ddot{x} &= -\dot{x} + \dot{y} \\
&= -(y - x) + (-z\tanh(x)) \\
&= x - y - z\tanh(x) \\
\ddot{y} &= -z\,\mathrm{sech}^2(x)\dot{x} - \tanh(x)\dot{z} \\
&= -z\,\mathrm{sech}^2(x)(y - x) - \tanh(x)\,(-\alpha + xy + \phi(y;\beta)) \\
&= (x - y)z\,\mathrm{sech}^2(x) + \left(\alpha - xy - \sqrt{y^2 + \beta^2}\right)\tanh(x) \\
\ddot{z} &= y\dot{x} + \left(x + \frac{y}{\sqrt{y^2 + \beta^2}}\right)\dot{y} \\
&= y(y - x) + \left(x + \frac{y}{\sqrt{y^2 + \beta^2}}\right)(-z\tanh(x)) \\
&= y^2 - xy - \left(x + \frac{y}{\sqrt{y^2 + \beta^2}}\right)z\tanh(x)
\end{aligned}
$$

We can differentiate $\ddot{x}$ with respect to x, $\ddot{y}$ with respect to y, and $\ddot{z}$ with respect to z as follows.

$$\frac{\partial \ddot{x}}{\partial x} = 1 - z\,\mathrm{sech}^2(x)$$
$$\frac{\partial \ddot{y}}{\partial y} = -z\,\mathrm{sech}^2(x) - \left(x + \frac{y}{\sqrt{y^2+\beta^2}}\right)\tanh(x)$$
$$\frac{\partial \ddot{z}}{\partial z} = -\left(x + \frac{y}{\sqrt{y^2+\beta^2}}\right)\tanh(x)$$

Since sech and tanh are continuous and bounded over all $\mathbb{R}$, $\partial\ddot{x}/\partial x$, $\partial\ddot{y}/\partial y$, and $\partial\ddot{z}/\partial z$ exist and are continuous over all $\mathbb{R}^3$. Thus, we can use Theorem 3.1 to define the following proper forcing functions and proper squared frequency functions.

$$\ddot{x}(x,y,z) = h_x(y,z) - x g_x(x,y,z) \tag{3.8}$$
$$\ddot{y}(x,y,z) = h_y(x,z) - y g_y(x,y,z) \tag{3.9}$$
$$\ddot{z}(x,y,z) = h_z(x,y) - z g_y(x,y,z) \tag{3.10}$$
$$h_x(y,z) = -y \tag{3.11}$$
$$g_x(x,y,z) = \begin{cases} \dfrac{z\tanh(x)}{x} - 1 & x \neq 0 \\ z-1 & x = 0 \end{cases} \tag{3.12}$$
$$h_y(x,z) = xz\,\mathrm{sech}^2(x) + (\alpha-\beta)\tanh(x) \tag{3.13}$$
$$g_y(x,y,z) = \begin{cases} z\,\mathrm{sech}^2(x) + x\tanh(x) + \dfrac{\left(\sqrt{y^2+\beta^2}-\beta\right)\tanh(x)}{y} & y \neq 0 \\ z\,\mathrm{sech}^2(x) + x\tanh(x) & y = 0 \end{cases} \tag{3.14}$$
$$h_z(x,y) = y^2 - xy \tag{3.15}$$
$$g_z(x,y,z) = \left(x + \frac{y}{\sqrt{y^2+\beta^2}}\right)\tanh(x) \tag{3.16}$$

The forcing functions and the squared frequency functions over our trajectory plotted in Figs. 3.1 are shown in Figs. 3.3 and 3.4, respectively. Notice that the squared frequency functions are most definitely competitive. All in all, our theory of properly splittable functions concludes that the Competitive Modes Conjecture (Conjecture 3.1) is valid for our modified Wimol-Banlue Attractor, which is what we expected. This is significant since, as far as the authors know, this sort of Competitive Modes analysis has never been applied to these sorts of systems before.

3.4 Further Research: Improper Splittings

Notice the requisite in Definition 3.3 stating that $D_i^*(\mathbf{0}) = \{\mathbf{x} \in D : x_i = 0\} \neq$ for a proper splitting. In other words, for a function f to have a proper splitting in terms of x_i, it must be defined on $x_i = 0$. Obviously this is not the case for all functions, such as the logarithm and reciprocal functions.

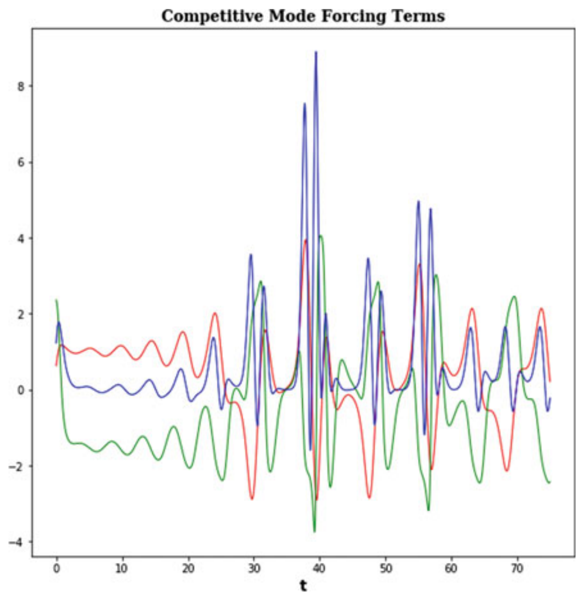

Fig. 3.3 The functions h_x (in red), h_y (in green), and h_z (in blue) of our modified Wimol-Banlue Attractor as defined in Eq. (3.7), using a trajectory with initial condition $\mathbf{x}_0 = [1.32, -0.63, 1.91]^T$. The trajectory was approximating using 7500 iterations of an adaptive RK4 method, using a time step of 0.01

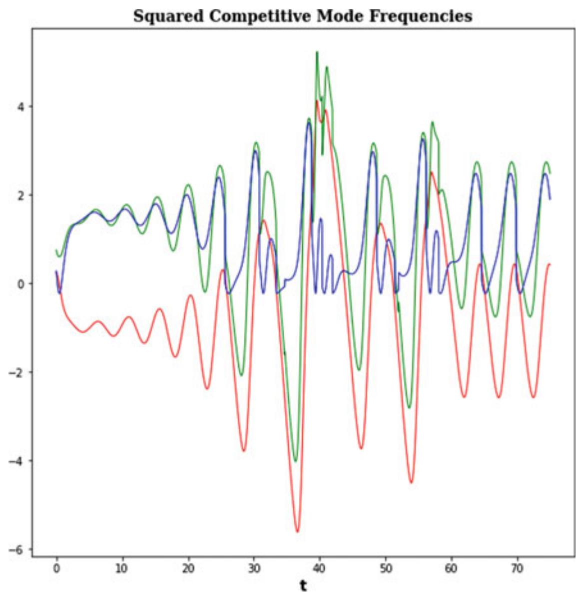

Fig. 3.4 The functions g_x (in red), g_y (in green), and g_z (in blue) based on the trajectory of our modified Wimol-Banlue Attractor as defined in Eq. (3.7), using a trajectory with initial condition $\mathbf{x}_0 = [1.32, -0.63, 1.91]^T$. The trajectory was approximating using 7500 iterations of an adaptive RK4 method, using a time step of 0.01

A work-around to this problem is the introduction of of an improper splitting, which is simply the splitting of a function with respect to $\mathbf{c} \in D \backslash D_i^*(0)$. How this will affect the resulting improper forcing function and improper squared frequency function is yet unclear and requires much more in-depth research to fully understand.

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