A robust iterative solver for the two-way wave equation based on a complex shifted-Laplace preconditioner

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Summary

An iterative numerical method for solving the wave equation in an inhomogeneous medium with constant density is presented. The method is based on a Krylov iterative method and enhanced by a powerful preconditioner. For the preconditioner, a complex Shifted-Laplace operator is proposed, designed specifically for the wave equation. A multigrid method is used to approximately compute the inverse of the preconditioner. Numerical examples on 2D problems show that the combined method is robust and applicable for a wide range of frequencies. Extension to 3D is straightforward.

Introduction

Wave equation migrations are becoming more and more popular; they are currently based on a one-way scheme because in 3D the solution of the full wave equation is still too expensive. However, it has been shown in 2D that one-way wave equations cannot image steep reflectors and do not predict the correct amplitudes of the reflections (see [4]).

In 2D, it is possible to perform a full wave equation migration in the frequency domain, because a direct solver can be used to solve the large linear system obtained after the discretization of the two-way wave equation. In 3D, the linear system is too large to be solved with a direct solver. An iterative solver is then required. To accelerate the convergence a preconditioner is needed. Due to the indefiniteness of this linear system, most of the classic preconditioners, such as the incomplete LU decomposition, do not work or work only for a limited range of frequencies.

Here, a preconditioner based on a complex shifted-Laplace operator is proposed. With this preconditioner, a generalization of the conjugate gradient method, namely the BICGSTAB method, is used to iteratively solve the linear system. This approach seems to be robust when the frequency increases, even though the number of iterations increases.

The paper is organized as follows. In the first section, the theory is briefly explained. In the second section, two 2D examples are described and the results are compared with the solution obtained by the direct solver. Finally, some conclusions are drawn.

Theory

The time-harmonic wave problem is represented by the Helmholtz equation:

$$\left(\sum_{1}^{d}\frac{\partial^2}{\partial x_d^2}+k^2(x_1,\cdots,x_d)\right)u=f\,, \qquad (1)$$

with d=2 or 3 for the 2D or the 3D case. Here $k(x_1,\cdots,x_d)=2\pi f/c(x_1,\cdots,x_d)$ is the wave-number, c the speed of sound, and f the frequency. Absorbing boundary conditions have been used in the implementation. We have only considered the first-order absorbing condition [2], as the topic of boundary conditions is at this moment not our main concern.

The discretization of Eq.(1) leads to a set of linear equations of the form

$$Ax=b\,,\ A\in C^{n\times n}\,,\ x,b\in C^n \quad (2)$$

A is typically a complex matrix due to the inclusion of the radiation condition. A is sparse, but can be large because of resolution requirements of the solution. In 2D for example, with a 5-point finite-difference stencil, the matrix contains only 4 non-zero sub-diagonals, but has a maximum band width of n, with n the number of discretization points in one direction. In 3D, the maximum bandwidth is n^2; this is the reason why a direct solver cannot be used to perform the LU decomposition.

In the 2D case, direct methods with nested dissection may be used to effectively solve Eq. (2). As we aim at solutions of Eq. (2) in the 3D case, direct methods become impractical due to excessive fill-in during the LU decomposition. We choose iterative methods within the class of Krylov subspace methods. In particular we prefer BiCGSTAB [6] because this method can be applied to general matrices and retains the same amount of work per iteration.

It is well known, especially for ill-conditioned linear systems that without pre- (and post-) conditioning the Krylov iteration converges slowly. By pre-(and post-) conditioning we solve

$$M_1^{-1}AM_2^{-1}\tilde{x}=M_1^{-1}b\,,\ \tilde{x}=M_2^{1}x\,, \qquad (3)$$

where $M=M_1M_2$ is the preconditioning matrix. With preconditioning in the BiCGSTAB algorithm, four operations with M_1^{-1} and M_2^{-1} must be performed.

Several types of preconditioners have been proposed. However, it is known that existing preconditioners are still susceptible to breakdown or stagnation for sufficiently high frequencies. The convergence also deteriorates with sharp and large contrasts in k .
The preconditioning matrix M should approximate the matrix A in order to reduce the number of iterations, but at the same time, the linear preconditioner systems, $Mx=y$, should be easy to solve to obtain an efficient method.

Recently, [3], we propose a preconditioner of the form

$$\left(\sum_{1}^{d}\frac{\partial^2}{\partial x_d^2}-ik^2\right),\ d=1,2,\ldots \quad (4)$$

where $i=\sqrt{-1}$. We call this operator the "complex Shifted-Laplace (CSL)" operator. The preconditioning matrix, M , is then obtained by discretizing Eq. (4), e.g. with the same discretization than the one used for Eq. (1). Note that one can also implement different discretizations for (1) and (4).

As the matrix M is complex and symmetric definite, several efficient methods are available to solve the linear system $Mx=y$. In our code, a multigrid method was used. Furthermore, only post preconditioning is performed, reducing the work to only two preconditioner systems per iteration. To further reduce the amount of computational work, M^{-1} is not computed very accurately. We perform one multigrid iteration to approximate M^{-1} and this appears to be sufficient.

The resulting method is a combination of inner and outer iterations with BiCGSTAB acting as the outer (or main) iteration and multigrid as the inner iteration.

Examples

We have performed computations on two test problems. The first example is based on the Marmousi model [1]. In this paper we only consider a portion of this model, shown in Figure 1a and having a size of 6000 × 1600 m^2. The second example is a multi-layered rectangular domain of 130 × 150 m^2 (Figure 2a). In this case, a crosswell situation is modeled and this model will be referred as the wave guide model because of the main propagation is parallel to the layering. The computations were performed on a single 2.6 GHz Pentium 4 processor with 512 Mb of RAM. The method was coded in Fortran77 and compiled under the LINUX operating system.

The results of the simulation at 10 Hz on the Marmousi model are shown in Figures 1b and 1c. In these figures, the real part of the wave field at a given frequency is displayed. We notice that the result with the iterative solver is similar to the result with the direct solver except close to the boundaries. The discrepancy at the boundaries may be due to different implementations of the boundary conditions between the two solvers (the direct solver used the second order boundary condition instead of the first order). Nevertheless, this comparison shows that the iterative solver has converged.
Table 1 lists the numerical performance of BiCGSTAB accelerated by the CSL preconditioner we proposed, for various frequencies. In the Marmousi problem, a unit wave source is placed just below the center of the surface. In Table 1, we also include the numerical performance of BiCGSTAB+CSL where the CSL preconditioning matrix is approximated with the incomplete LU decomposition, ILU(0).

Table 1. Performance of BiCGSTAB+CSL. The iteration number and the CPU-time are listed

	f (Hz)	1	10	20	30
ILU(0)	# iter	2950	1519	3465	-
	time (s)	351	178	1671	-
Multigrid	# iter	16	177	311	485
	time (s)	9	75	537	1445

We observe that the number of iterations depends more or less linearly on the frequency when the preconditioner system is solved with a multigrid method. However, the required cpu-time increases more rapidly.

Previously the Marmousi model has also been used to test the Separation of Variables (SV) preconditioner [5]. With this type of preconditioner, BiCGSTAB converges after 648 iterations for f = 20 Hz, but fails to converge after 2000 iterations for f = 30 Hz. The method based on the CSL preconditioner, therefore, outperforms the SV preconditioner and does not show any breakdown. With respect to the CPU-time, the method, however, is still slower than direct method based on the nested-dissection ordering. For f = 30 Hz with 2001 × 534 grid points, on a Xeon 3.0 GHz machine, the linear system is solved in 340 seconds. For low frequencies, the CPU-time is more or less comparable.

The main problem in the current implementation is that the solution of the preconditioner system is still expensive. In fact it may take about 80% of the total cpu-time. Nevertheless, since we target 3-D problems, a combination of BiCGSTAB and multigrid may become more efficient compared to direct methods. Furthermore, an improvement of the efficiency of the multigrid method seems feasible and will be the subject of future research.

The result with the wave guide model is shown Figure 2. In this case the source is at the depth of 60 m inside a low velocity zone. The example mimics a crosswell situation where the wave propagation is mainly parallel to the layering. In Figure 2b, we can see that most of the energy is trapped inside the low velocity layer due to guided-wave effects. The wave field obtained with the direct solver is plotted Figure 2c. Once again, the main differences occur at the boundaries.

Table 2 summarizes the numerical performance of BiCSTAB+CSL for the waveguide problem. For various frequencies the method still converges to the specified accuracy. Furthermore, no indication of breakdown is observed. With ILU(0) and $f = 10$ Hz, BiCGSTAB does not converge after 500 iterations. In this example, the number of iterations and the cpu-time increase more or less linearly with frequency.

Table 2. Numerical performance of BiCGSTAB+CSL for wave guide problem

f (Hz)	10	100	200	300
# iter	7	49	86	130
time (s)	18	77	128	191

Conclusions

We have presented an iterative method to solve the two-way wave equation. The key ingredient is the use of a complex Shifted-Laplace preconditioner inside the BiCGSTAB iterative method. Combined with a multigrid method to solve the preconditioner system, this results in a robust and efficient method. The two numerical examples show that the iterative solver converges when the model contains relative large wave-numbers and sharp contrasts.

The iterative solver still provides a solver slower than the direct solver in 2D. However the results are encouraging. The next step is to study the efficiency of this approach for 3D models.

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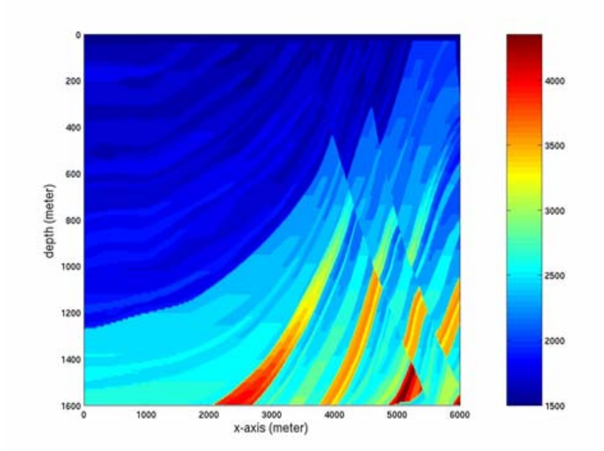
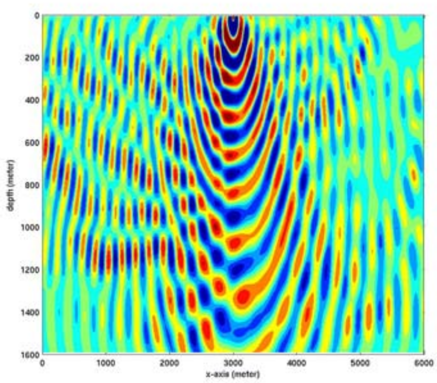
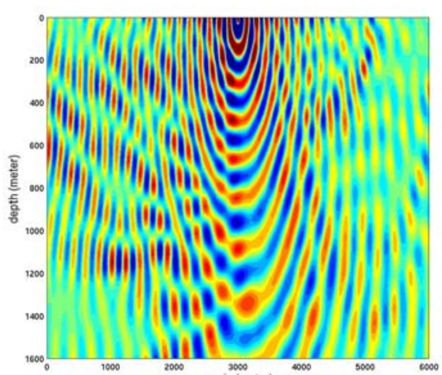
Robust iterative solver for the two-way wave equation



(a) Velocity profile

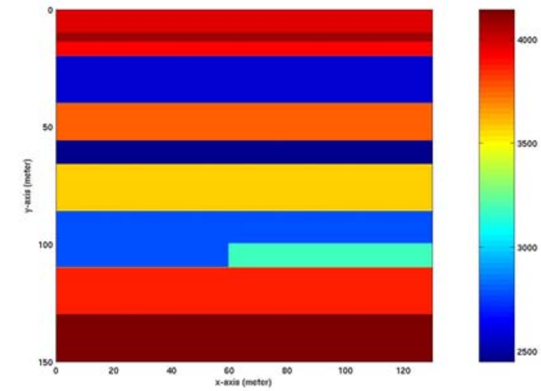


(b) Real part of the wave field computed with the iterative solver

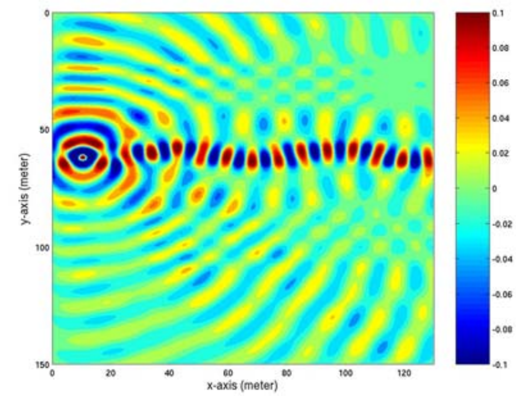


(c) Real part of the wave field computed with a direct solver

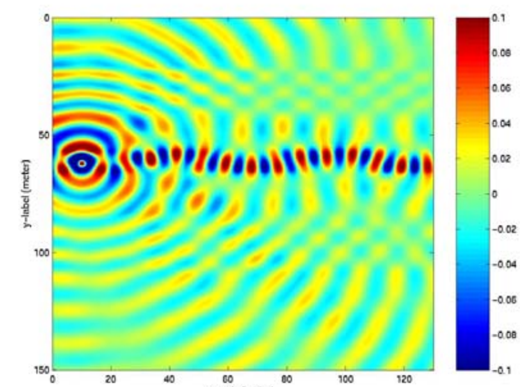
Figure 1: Marmousi problem solved at f = 10 Hz with BiCGSTAB+SCL om a grid of 751×201 points



(a) Velocity profile



(b) Real part of the wave field computed with the iterative solver:



(c) Real part of the wave field computed with the direct solver

Figure 2: Wave guide problem solved at f = 300 hz with BiCGSTAB+CSL on a grid of 651×751 points