

# A conservative pressure-correction method for the Euler and ideal MHD equations at all speeds

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## SUMMARY

To efficiently compute weakly compressible magnetohydrodynamic flows in astrophysical applications, approximate low Mach number reduced forms of the compressible MHD equations are frequently used. This is because standard characteristic-based schemes for the full compressible MHD equations are inefficient and inaccurate for computing low Mach number magnetohydrodynamic flow, as a result of the increasing stiffness and weakening pressure/density coupling of the equations when  $M \downarrow 0$ . Furthermore, these schemes are colocated, so that additional and artificial measures have to be taken to ensure solenoidality of the magnetic field. We present a new method with a staggered spatial discretization and a pressure-correction solution algorithm that is more suitable for computing weakly compressible MHD flow, because of its Mach-uniformity and efficient and accurate handling of the solenoidality constraint on the magnetic field. Copyright © 2002 John Wiley & Sons, Ltd.

KEY WORDS: staggered; Mach-uniform; magnetohydrodynamics

## 1. INTRODUCTION

We will discuss a staggered scheme and a conservative pressure-correction solution algorithm for the computation of magnetohydrodynamic and gasdynamic flow at all speeds, with accuracy and efficiency uniform in the Mach number, which we refer to as a *Mach-uniform method*.

The need for Mach-uniform methods for both the Euler and MHD equations is briefly discussed. For the MHD equations, the staggered discretization that is an integral part of the proposed Mach-uniform method has an additional advantage: efficient and accurate preservation of the solenoidality of the magnetic field. This is discussed in Section 4. In Section 5 we present the spatial discretization, followed by an explanation of an efficient sequential solution procedure in Section 6. In Section 7 we discuss results of numerical experiments that illustrate the accuracy of the method and support our claim of Mach-uniformity.

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## 2. GOVERNING EQUATIONS

The equations of ideal magnetohydrodynamics describe the dynamics of an inviscid conducting plasma under the influence of a magnetic field, and are given by

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad (1)$$

$$\frac{\partial \mathbf{m}}{\partial t} + \left( \nabla \cdot \left( \frac{\mathbf{m} \mathbf{m}^T}{\rho} + \frac{1}{2} |\mathbf{B}|^2 \mathbf{I} - \mathbf{B} \mathbf{B}^T \right) \right)^T = -\nabla p \quad (2)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left( \frac{\mathbf{m}}{\rho} \times \mathbf{B} \right) = -\nabla \times \mathbf{E} \quad (3)$$

$$\frac{\partial \rho E}{\partial t} + \nabla \cdot \left( \left( \frac{\rho E + p}{\rho} \right) \mathbf{m} + \left( \frac{|\mathbf{B}|^2 \mathbf{I} - 2 \mathbf{B} \mathbf{B}^T}{\rho} \right) \mathbf{m} \right) = 0 \quad (4)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (5)$$

$$p = (\gamma - 1) \rho e \quad (6)$$

Equations (1), (2) and (4) are the conservation equations for density  $\rho$ , momentum  $\mathbf{m}$  and total energy  $E$ , respectively, while (3) is the *induction* equation, relating the evolution of the magnetic field  $\mathbf{B}$  to the electric field  $\mathbf{E}$ . The magnetic field has to respect the solenoidality constraint (5). The system is closed with the equation of state (6), relating the thermal pressure  $p$  to the density and the internal energy  $e$ . The ideal MHD equations reduce to the Euler equations if initial and boundary conditions  $\mathbf{B} \equiv \mathbf{0}$  are prescribed.

## 3. FLOW AT ALL SPEEDS: MACH-UNIFORM METHODS

Mach-uniform methods are required when in one flow domain regions with weakly compressible and compressible flow occur simultaneously, as for instance, in flow around aircraft in take-off or landing conditions, or when the small density variations in weakly compressible flow are of physical importance, as for instance in many astrophysical applications. In both cases a compressible formulation has to be used, but this becomes increasingly difficult to solve efficiently and accurately with a standard characteristic-based scheme, when  $M \downarrow 0$ .

Apart from the expected loss of efficiency because of the stiffness of the equations, standard density-based compressible formulations are not directly applicable to weakly compressible flow, because of loss of accuracy due to weak coupling between density and pressure. Both effects can, at considerable computing cost, be neutralized to some degree by preconditioning/pseudo-compressibility techniques.

To avoid the solution of the compressible MHD equations in the weakly compressible case, a number of approximate low Mach number reduced models have been developed [1–3] to

try to incorporate the effects of variable density in an incompressible formulation. However, the assumptions for these approaches may be restrictive.

On the other hand, incompressible methods can be extended to incorporate compressibility following the approach of Harlow and Amsden [4]. This is the approach we will pursue here to formulate a Mach-uniform method.

Below, we formulate the demands we put on the Mach-uniform method and present a Mach-uniform dimensionless pressure-based formulation of the governing equations. For brevity, we assume temporarily  $\mathbf{B} \equiv \mathbf{0}$ . In Reference [5] a more elaborate discussion of the motivation of the following Mach-uniform formulation is given.

### 3.1. Requirements for Mach-uniform methods

We distinguish between three regimes for the Mach number: incompressible flow:  $M = 0$ ; weakly compressible flow:  $0 < M \leq 0.3$ ; and compressible flow:  $M > 0.3$ .

- For  $M = 0$ , we require that the scheme reduces to the classical MAC-scheme [6]. Therefore, we adopt a pressure-based formulation. The MAC-scheme is the natural discretization for incompressible flow, because it does not require regularization to avoid odd-even decoupling.
- Based on an asymptotic expansion of the variables we find that in the limit  $M \downarrow 0$  the energy equation reduces to the (incompressible) solenoidality constraint, and the mass conservation equation to a convection equation for the density. We require our scheme to show the same behaviour.
- The limit  $M \downarrow 0$  must be regular. This can be accomplished by choosing the dimensionless pressure in the following way:

$$p = \frac{p^* - p_r}{\rho_r w_r^2} \tag{7}$$

where  $p_r$ ,  $\rho_r$  and  $w_r$  are the reference pressure, density and velocity, respectively, to be specified later.

- For  $M > 0.3$ , we require that the scheme is conservative, to ensure convergence to physical weak solutions.

### 3.2. Mach-uniform formulation

With (7) and  $E = e + \frac{1}{2}|u|^2$  the dimensionless conservative Mach-uniform pressure-based Euler equations in Cartesian co-ordinates are given by

$$\frac{\partial \rho}{\partial t} + (\rho u_\alpha)_{,\alpha} = 0 \tag{8}$$

$$\frac{\partial m_\alpha}{\partial t} + (m_\alpha u_\beta)_{,\beta} = -p_{,\alpha} \tag{9}$$

$$M_r^2 \left[ \frac{\partial (p + [(\gamma - 1)/2]\rho|u|^2)}{\partial t} + \left( \gamma p u_\alpha + \frac{\gamma - 1}{2} \rho |u|^2 u_\alpha \right)_{,\alpha} \right] + u_{\alpha,\alpha} = 0 \tag{10}$$

where  $m_x = \rho u_x$  and  $M_r$  is the reference Mach number defined as

$$M_r = \frac{w_r}{\sqrt{\gamma p_r / \rho_r}} \quad (11)$$

The dimensionless equation of state is given by

$$\rho(p, T) = \gamma M_r^2 \frac{p}{T} + \frac{1}{T} \quad (12)$$

We see that density becomes independent of pressure in the limit  $M_r \downarrow 0$ , as expected for incompressible flow without acoustics.

#### 4. SOLENOIDALITY CONSTRAINT $\nabla \cdot \mathbf{B} = 0$

Although the use of a staggered discretization is crucial for Mach-uniformity, staggered placement of the components of the magnetic field has an extra advantage, namely accurate and efficient treatment of the solenoidality constraint on the magnetic field  $\mathbf{B}$ . If solenoidality of the magnetic field is not guaranteed, unphysical effects, e.g. magnetic monopoles [7], will arise. The *constrained transport* method of Reference [8] uses a discrete version of the Stokes theorem to discretize the induction equation on a staggered grid in a way that guarantees solenoidality of a staggered magnetic field to machine precision. For simplicity, we illustrate the approach for a coplanar problem ( $\mathbf{E} = (0, 0, E_z)$ ) discretized on a uniform Cartesian mesh (Figure 1). Under the condition that  $\nabla \cdot \mathbf{B}^n = 0$ , straightforward finite difference discretization of (3) and summation leads to

$$\begin{aligned} -(B_{xij-1/2}^{n+1} - B_{xij-1/2}^n) &= \frac{\tau}{h}(E_{zij} - E_{zij-1}) \\ (B_{xi+1j-1/2}^{n+1} - B_{xi+1j-1/2}^n) &= -\frac{\tau}{h}(E_{zi+1j} - E_{zi+1j-1}) \\ -(B_{yi+(1/2)j-1}^{n+1} - B_{yi+(1/2)j-1}^n) &= -\frac{\tau}{h}(E_{zi+1j-1} - E_{zij-1}) \\ \underline{(B_{yi+(1/2)j}^{n+1} - B_{yi+(1/2)j}^n)} &= \frac{\tau}{h}(E_{zi+1j} - E_{zij}) \quad + \\ \nabla \cdot \mathbf{B}^{n+1} &= 0 \end{aligned} \quad (13)$$

To impose the solenoidality constraint two other ways can be followed: An advanced form of the Hodge projection [9] or the non-conservative approach of Powell [10]. The first approach will substantially increase the computational cost, because an additional Poisson equation has to be solved, while the second approach is reported to give unsatisfactory results for strong shocks [11].

Because the constrained transport method is so attractive, this has led to the development of algorithms that combine extensions of popular colocated schemes for the Euler equations as base schemes with a constrained transport method to update a staggered magnetic field  $\mathbf{b}$ , using the discrete Stokes theorem [7, 12]. To compute the fluxes for the base scheme  $\mathbf{b}$  is interpolated to the cell centres. In the review paper [11], a variation to this approach is described that circumvents the explicit use of the staggered help variable  $\mathbf{b}$  altogether.

The aim of these methods is clearly to mimic the advantageous behaviour of staggered schemes, while using the proven technology of standard characteristic-based colocated

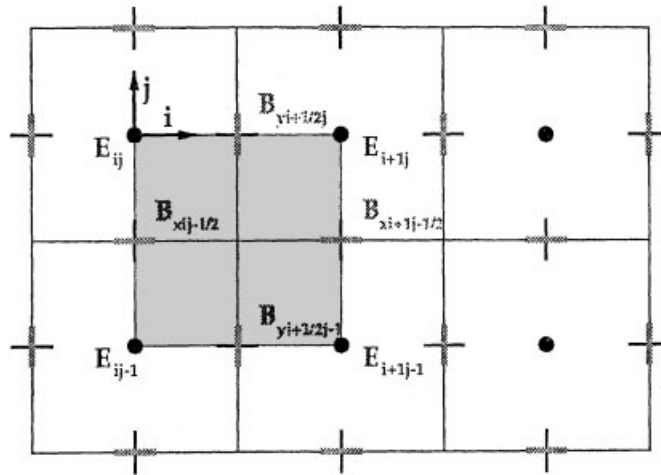


Figure 1. Illustration of the constrained transport approach.

schemes. This is based on the general (mis)conception that staggered schemes are inherently less accurate and robust for fully compressible flow than collocated schemes and that staggered discretization in general co-ordinates is inaccurate. However, in References [13–16] a staggered scheme for the Euler and Navier–Stokes equations is presented that is accurate on non-smooth non-orthogonal grids.

### 5. STAGGERED DISCRETIZATION

Although there is general consensus that in a staggered scheme the unknowns  $u_\alpha, m_\alpha, \rho, p$  should be positioned according to the arrangement of Reference [6] (Marker-and-Cell or ARAKAWA-C grid) and the constrained transport method dictates the relative positions of  $B_\alpha$ , there is still freedom in the mutual positioning of the magnetic grid (coloured blue in Figure 2) with respect to the hydrodynamic grid (coloured black in Figure 2).

We employ the staggered arrangement of unknowns of Figure 2(a), with collocation of  $B_\alpha, m_\beta, \alpha \neq \beta$ . This arrangement corresponds to the original constrained transport discretization [8]. However, Stone *et al.* [17, 18] changed the original staggering, in a way that enables them to use a collocated characteristic-based scheme to solve the coupled momentum and induction equation. Our choice for the first approach is motivated in Section 6. A key feature of the staggered scheme is that no use is made of the eigensystem of the Jacobian of the flux function to project the unknowns on characteristic variables. This results in very simple expressions for the flux, even for a rather complicated hyperbolic system like the equations of magnetohydrodynamics. The convective terms are upwind interpolated in the direction of the fluid velocity, while non-convective terms are centrally discretized. Because the time-integration method we use has strong A-stability, the central discretization does not produce wiggles in the vicinity of discontinuities. Higher order spatial discretization of the convective

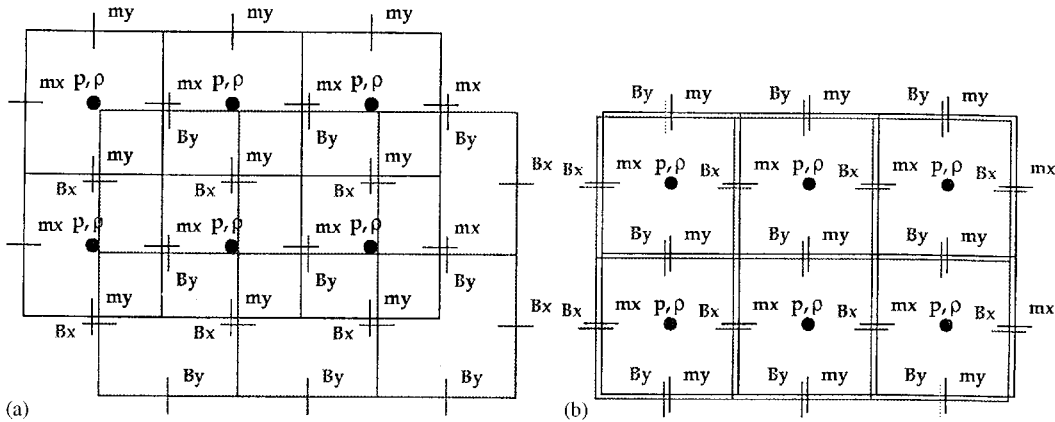


Figure 2. Staggered placement of unknowns: (a) Arrangement of Reference [8] and current approach. (b) Arrangement of References [17, 18].

terms can be obtained with the MUSCL strategy [19, 20]. Unlike the staggered scheme of Stone *et al.* [17, 18] there is no need to introduce artificial viscosity.

### 6. SOLUTION PROCEDURE

The solution procedure we use is extensively discussed for the Euler equations discretized in general co-ordinates in Reference [5]. For simplicity, we will discuss the one-dimensional case. The solution procedure consists of the following steps:

First, the density at the new time-level is computed:

$$\frac{\rho^{n+1} - \rho^n}{\delta t} + (u_x^n \rho^{n+1})_{,x} = 0 \tag{14}$$

Then the equation for  $m_x$  is advanced in time using the thermal pressure  $p$  and magnetic pressure  $P \equiv B_y^2 + B_z^2 - B_x^2$  at the previous time level:

$$\frac{m_x^* - m_x^n}{\delta t} + (u_x^n m_x^*)_{,x} = -(p^n + P^n)_{,x} \tag{15}$$

Next, the two subsystems for  $(B_y, m_y)$  and  $(B_z, m_z)$  are solved in a coupled manner:

$$\begin{aligned} \frac{B_y^{n+1} - B_y^n}{\delta t} + \left( u_x^n B_y^{n+1} - \frac{B_x^n}{\rho^{n+1}} m_y^* \right)_{,x} &= 0 \\ \frac{m_y^* - m_y^n}{\delta t} + \left( u_x^n m_y^* - \frac{B_x^n}{4\pi} B_y^{n+1} \right)_{,x} &= 0 \end{aligned} \tag{16}$$

$$\begin{aligned} \frac{B_z^{n+1} - B_z^n}{\delta t} + \left( u_x^n B_z^{n+1} - \frac{B_x^n}{\rho^{n+1}} m_z^* \right)_{,x} &= 0 \\ \frac{m_z^* - m_z^n}{\delta t} + \left( u_x^n m_z^* - \frac{B_x^n}{4\pi} B_z^{n+1} \right)_{,x} &= 0 \end{aligned} \tag{17}$$

In the case of one-dimensional flow with  $u_x = 0$  and  $B_{x,x} = 0$ , both subsystems reduce to a system similar to the linearized shallow-water equations. For the latter system staggering of both unknowns is essential if a central discretization is utilized to avoid odd–even decoupling such as occur with the Lax–Friedrichs scheme [21]. This motivates our choice to adopt the staggered discretization of Figure 2(a) [8] instead of the one of Figure 2(b) [17, 18]. We postulate the following pressure correction:

$$u_x^{n+1} = \frac{1}{\rho^{n+1}} (m_x^* - \delta t (p^{n+1} - p^n)_{,x}) = u_x^* - \frac{\delta t}{\rho^{n+1}} \delta p_{,x} \tag{18}$$

The postulate is substituted in the energy equation:

$$\begin{aligned} M_r^2(\gamma - 1) \left\{ \frac{1}{\gamma - 1} \frac{\delta p}{\delta t} + \frac{1}{2} \frac{\delta(B_x B_x)}{\delta t} + \frac{\frac{1}{2} \rho^{n+1} (u_x^* - (\delta t/\rho) \delta p_{,x})^2 - \frac{1}{2} \rho^n u_x^n u_x^n}{\delta t} \right. \\ \left. + \left[ \left( \frac{\gamma}{\gamma - 1} (p^n + \delta p) + P^{n+1} + \frac{1}{2} (B_x B_x)^{n+1} + \frac{1}{2} \rho^{n+1} \left( u_x^* - \frac{\delta t}{\rho} \delta p_{,x} \right)^2 \right) \left( u_x^* - \frac{\delta t}{\rho} \delta p_{,x} \right) \right. \right. \\ \left. \left. + u_y^{n+1} \Lambda_y + u_z^{n+1} \Lambda_z \right]_{,x} \right\} + \left( u_x^* - \frac{\delta t}{\rho} \delta p_{,x} \right)_{,x} = 0 \end{aligned} \tag{19}$$

where  $\Lambda_x = -\frac{1}{2} B_x B_x$ . Linearization of (19) leads to a modified Poisson’s equation for the pressure correction with a stencil identical to the MAC-scheme [5], as required in Section 3.

Finally the pressure and momentum are updated:

$$p^{n+1} = p^n + \delta p \tag{20}$$

$$m_x^{n+1} = m_x^* - \delta t \delta p_{,x} \tag{21}$$

In the one-dimensional case we use a direct solution method to solve the implicit equations, while in the two-dimensional case a preconditioned Krylov method is utilized.

### 7. NUMERICAL RESULTS

Because staggered schemes are well established for computing incompressible flow, because of their accuracy and robustness, it remains to be shown here that the Mach-uniform method also produces accurate results for the compressible case. In Reference [5] we present results for the Euler equations, for a number of Riemann problems and two-dimensional test cases with  $0 \leq M_r \leq 1.65$ . The new conservative pressure-correction method shows improved accuracy for computing weak solutions with respect to earlier non-conservative methods [13, 22]. In

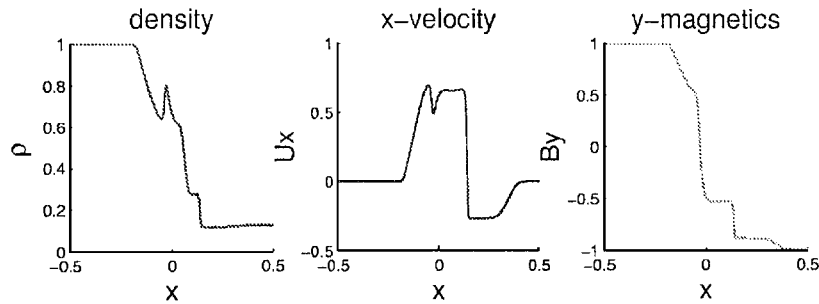


Figure 3. Results for Brio and Wu test case, 400 cells,  $\delta t/\delta x = 0.05$ ,  $t_{\text{end}} = 0.15$ ,  $\gamma = \frac{5}{3}$ .

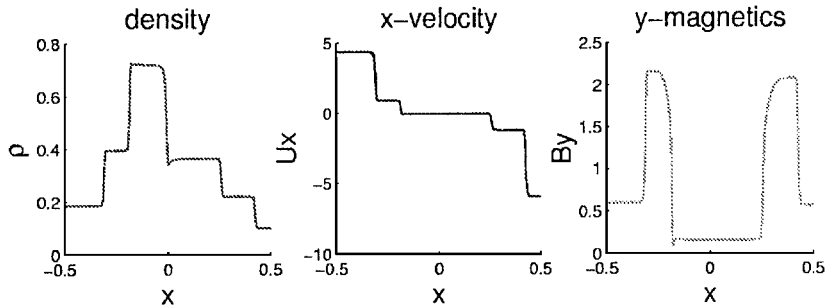


Figure 4. Results for Dai and Woodward test case, 400 cells,  $\delta t/\delta x = 0.05$ ,  $t_{\text{end}} = 0.10$ ,  $\gamma = \frac{5}{3}$ .

Figures 3 and 4, results are shown for two classic Riemann problems for magnetohydrodynamic flow: the Brio and Wu test case [23] and the second test case of Dai and Woodward [24]. We have chosen to show results for these test cases because the situation  $u_x = 0$ , discussed in the previous section occurs locally. Although central discretization is used, no wiggles occur in the vicinity of the discontinuities. Close comparison with the results computed with colocated methods presented in References [23, 24] shows that the numerical solutions converge to the correct weak solutions and have comparable accuracy.

## 8. CONCLUSIONS AND FUTURE EXTENSIONS

We have presented a new conservative pressure-correction method for the Euler and ideal MHD equations that has efficiency and accuracy nearly uniform in the Mach number. The staggered scheme guarantees solenoidality of the magnetic field to machine precision. By numerical experiments, solutions obtained with the new method are shown to converge to the correct weak solutions for a number of Riemann problems for the Euler and ideal MHD equations. The claim of Mach-uniform efficiency is supported by two-dimensional test cases for the Euler equations.



Apart from the extension of the method to second order in space and time, the behaviour of the method for different ranges of the plasma  $\beta = M_A^2/\gamma M^2$ ,  $\beta \ll 1$ ,  $\beta \sim 1$  and  $\beta \gg 1$  has still to be studied.

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