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Eigenvalue analysis of the SIMPLE preconditioning for incompressible flow

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SUMMARY

In this paper, an eigenvalue analysis of the SIMPLE preconditioning for incompressible flow is presented. Some formulations have been set up to characterize the spectrum of the preconditioned matrix. This leads to a generalized eigenvalue problem. The generalized eigenvalue problem is investigated. Some eigenvalue bounds and the estimation for the spectral condition number in the symmetric case are given. Numerical tests are reported to illustrate the theoretical discussions. Copyright © 2004 John Wiley & Sons, Ltd.

KEY WORDS: SIMPLE preconditioner; spectral analysis; incompressible Navier Stokes

1. INTRODUCTION

The steady state incompressible Navier–Stokes equations

$$-\nu\Delta u + u\cdot \operatorname{grad} u + \operatorname{grad} p = f$$
$$-\operatorname{div} u = 0$$

combined with appropriate boundary conditions, are widely used to simulate the incompressible flow of a fluid. The vector field u represents the velocity, p represents the pressure and ν is the viscosity. Discretization and linearization of the equations leads to the following large sparse linear algebraic system:

$$\begin{pmatrix} Q & G \\ G^{\mathrm{T}} & O \end{pmatrix}\begin{pmatrix} u \\ p \end{pmatrix}=\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \tag{1}$$

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where $Q \in \mathbb{R}^{n \times n}, G \in \mathbb{R}^{n \times m}, m \leqslant n, \det(Q) \neq 0, \operatorname{rank}(G) = m$; $u \in \mathbb{R}^n$ and $p \in \mathbb{R}^m$ are the velocity vector and the pressure vector, respectively. For problems with three space dimensions, iterative solvers are required. Preconditioning often determines the numerical performance of the Krylov subspace solvers [1].

In References [2, 3], Vuik *et al.* proposed the GCR–SIMPLE(R) algorithm for solving the large linear system (1). The algorithm can be considered as a combination of the Krylov subspace method GCR [4] with the SIMPLE(R) algorithm [5]. In this combined algorithm, the SIMPLE(R) iteration is used as a preconditioner in the GCR method. Numerical tests indicate that the SIMPLE(R) preconditioning is effective and competitive for practical use.

Other methods to solve the incompressible Navier–Stokes equations are: the (approximate) Uzawa method [6–11], SIMPLE-type methods [5, 12], penalty method [13], pressure correction method [14], PISO method [15], preconditioners for indefinite systems [16–19], and multigrid methods [20–25]. For an overview of these methods we refer to Reference [26, Section 9.6].

In this paper, we focus on the eigenvalue analysis of the SIMPLE preconditioned matrix $\tilde{A}$. Two related formulations are derived to describe the spectrum of $\tilde{A}$. The spectrum has some connection with that of the Schur complement of the matrix A. The relationship between the two different formulations has been investigated by using the theory of matrix singular value decomposition. Some useful eigenvalue bounds are obtained for a symmetric matrix A. A diagonal scaling [2] is studied. Numerical tests are used to illustrate the theoretical bounds.

In the remaining parts of this paper, the linear system (1) is abbreviated as $Ax = b$, where $A \in \mathbb{R}^{(n+m)\times(n+m)}$, $b \in \mathbb{R}^{n+m}$. The set of all eigenvalues of matrix A is denoted as $\sigma(A)$. Besides, we assume that the matrix Q is non-singular and the diagonal entries of $D := \operatorname{diag}(Q)$ are positive.

2. SPECTRAL ANALYSIS OF THE SIMPLE PRECONDITIONED MATRIX

In order to estimate the convergence of SIMPLE preconditioned Krylov solvers we study the spectrum of the SIMPLE preconditioned matrix. The relation between the spectrum of the iteration matrix and the convergence of non-symmetric Krylov solvers (GMRES, GCR) is less straightforward than for the symmetric Krylov solvers (CG, Minres). The following result is given in Reference [27]:

Theorem 1

Suppose that A is diagonalizable, so $A = X\Lambda X^{-1}$ and let

$$\varepsilon^{(i)} = \min_{\substack{p \in P_i \\ p(0)=1}} \max_{\lambda \in \sigma(A)} |p(\lambda)|$$

where P_i is the space of polynomials of degree less than or equal to i. Then the residual norm of the ith GMRES or GCR iterate satisfies:

$$\|r_i\|_2 \leqslant K(X)\varepsilon^{(i)}\|r_0\|_2$$

where $K(X) = \|X\|_2\|X^{-1}\|_2$. If furthermore all eigenvalues are enclosed in a circle centered at $C \in \mathbb{R}$ with $C > 0$ and having radius R with $C > R$, then $\varepsilon^{(i)} \leqslant (R/C)^i$.

Furthermore under the same conditions a superlinear convergence result is proven in Reference [28].

Note that in these results it is important that the matrix A is diagonalizable. If A is not diagonalizable the relation between the spectrum and the convergence can be more complicated (see Reference [29]). This can already be seen by the following example: take

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad x = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \quad b = e_4 := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Note that for this example $\varepsilon^{(i)}$ is 'zero' starting from the first iteration, but the above Theorem 1 is not applicable due to the non-diagonalizability of the matrix. It is easy to see for this example that $K_1\{A;b\} = \text{span}\{e_4\}$, $K_2\{A;b\} = \text{span}\{e_3, e_4\}$, and $K_3\{A;b\} = \text{span}\{e_2, e_3, e_4\}$. This implies that for full GMRES or GCR, n iterations are required before convergence sets in, for $A \in \mathbb{R}^{n\times n}$.

2.1. Two formulations of the spectrum

Consider the right preconditioning to the linear system (1)

$$AP^{-1}y = b, \quad x = P^{-1}y \tag{2}$$

If the SIMPLE algorithm is used as preconditioning, it is equivalent to choose the preconditioner P^{-1} as [3, 22]

$$P^{-1} = BM^{-1}, \qquad P = MB^{-1} \tag{3}$$

where

$$B = \begin{pmatrix} I & -D^{-1}G \\ O & I \end{pmatrix}, \quad M = \begin{pmatrix} Q & O \\ G^{\mathrm{T}} & R \end{pmatrix}, \quad D = \text{diag}(Q), \quad R = -G^{\mathrm{T}}D^{-1}G$$

We call this preconditioning a SIMPLE preconditioning, and the preconditioner P^{-1} as SIMPLE preconditioner. For SIMPLE preconditioning, we have the following result:

Proposition 2

If the right preconditioner P^{-1} is taken to be the matrix defined by (3), then the preconditioned matrix is

$$\tilde{A} := AP^{-1} = \begin{pmatrix} I - (I - QD^{-1})GR^{-1}G^{\mathrm{T}}Q^{-1} & (I - QD^{-1})GR^{-1} \\ O & I \end{pmatrix} \tag{4}$$

And, therefore, the spectrum of the SIMPLE preconditioned matrix $\tilde{A}$ is

$$\sigma(\tilde{A}) = \{1\} \cup \sigma(I - (I - QD^{-1})GR^{-1}G^{\mathrm{T}}Q^{-1}) \tag{5}$$

Proof
It is easy to verify that

$$M^{-1} = \begin{pmatrix} Q^{-1} & O \\ -R^{-1}G^{\mathrm{T}}Q^{-1} & R^{-1} \end{pmatrix} \tag{6}$$

and

$$\begin{aligned}\tilde{A} &= \begin{pmatrix} Q & G \\ G^{\mathrm{T}} & O \end{pmatrix} \begin{pmatrix} I & -D^{-1}G \\ O & I \end{pmatrix} \begin{pmatrix} Q^{-1} & O \\ -R^{-1}G^{\mathrm{T}}Q^{-1} & R^{-1} \end{pmatrix} \\ &= \begin{pmatrix} I-(I-QD^{-1})GR^{-1}G^{\mathrm{T}}Q^{-1} & (I-QD^{-1})GR^{-1} \\ O & I \end{pmatrix}\end{aligned}$$

So, the fact about the spectrum of $\tilde{A}$, described by (5), follows. □

Now, we study the spectrum defined by (5) in more detail. By multiplying with matrices Q^{-1} and Q from the left- and right-hand side of the matrix $I-(I-QD^{-1})GR^{-1}G^{\mathrm{T}}Q^{-1}$, respectively, we get

$$\begin{aligned}\sigma(I-(I-QD^{-1})GR^{-1}G^{\mathrm{T}}Q^{-1}) &= \sigma(I-(Q^{-1}-D^{-1})GR^{-1}G^{\mathrm{T}}) \\ &= \sigma(I-D^{-1}(D-Q)Q^{-1}GR^{-1}G^{\mathrm{T}}) \\ &= \sigma(I-JQ^{-1}GR^{-1}G^{\mathrm{T}})\end{aligned}$$

in which, the matrix J ($J := D^{-1}(D-Q)$) is the *Jacobi* iteration matrix for the matrix Q. This observation leads to the following proposition:

Proposition 3
For the SIMPLE preconditioned matrix $\tilde{A}$,

(1) 1 is an eigenvalue with multiplicity at least of m, and
(2) the remaining eigenvalues are $1-\mu_i$, $i=1,2,\ldots,n$, where μ_i is the ith eigenvalue of

$$ZEx = \mu x \tag{7}$$

where

$$E = GR^{-1}G^{\mathrm{T}} \in \mathbb{R}^{n\times n}, \quad Z = JQ^{-1} \in \mathbb{R}^{n\times n}$$

If J is non-singular (7) is identical to the generalized eigenvalue problem

$$Ex = \mu Z^{-1}x \tag{8}$$

Next, to investigate the spectrum of $\tilde{A}$ more accurately, we derive another formulation of it. Consider the eigenvalue problem

$$\tilde{A}x = AP^{-1}x = \lambda x \tag{9}$$

Note that AP^{-1} has the same spectrum as $P^{-1}A$. So, the eigenvalue problem (9) is equivalent to the generalized eigenvalue problem

$$Ax = \lambda Px \tag{10}$$

Here,

$$A = \begin{pmatrix} Q & G \\ G^{\mathrm{T}} & O \end{pmatrix} \quad \text{and} \quad P = MB^{-1} = \begin{pmatrix} Q & QD^{-1}G \\ G^{\mathrm{T}} & O \end{pmatrix}$$

The generalized eigenvalue problem (10) can be written as

$$\begin{pmatrix} Q & G \\ G^{\mathrm{T}} & O \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \lambda \begin{pmatrix} Q & QD^{-1}G \\ G^{\mathrm{T}} & O \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} \tag{11}$$

that is

$$\begin{aligned} Qu + Gp &= \lambda(Qu + QD^{-1}Gp) \\ G^{\mathrm{T}}u &= \lambda G^{\mathrm{T}}u \end{aligned}$$

Multiplying by Q^{-1} from the left to the first equation, and re-arranging of the terms yields

$$\begin{aligned} (1-\lambda)u &= (\lambda D^{-1} - Q^{-1})Gp \\ G^{\mathrm{T}}(1-\lambda)u &= 0 \end{aligned} \tag{12}$$

From (12), we see that 1 is an eigenvalue of (11). If the matrix $D^{-1} - Q^{-1}$ is non-singular it follows from the right-hand side of the first equation of (12), with $\lambda = 1$, that the eigenvectors corresponding to eigenvalue 1 are

$$v_i = \begin{pmatrix} u_i \\ 0 \end{pmatrix} \in \mathbb{R}^{(n+m)}, \qquad u_i \in \mathbb{R}^n, \quad i = 1, 2, \ldots, n$$

where, $\{u_i\}_{i=1}^n$ is a basis of $\mathbb{R}^n$.

For $\lambda \neq 1$, it follows from the second equation in (12) that $G^{\mathrm{T}}u = 0$. Multiplying the first equation in (12) with G^{T} shows that

$$\begin{aligned} 0 &= \lambda G^{\mathrm{T}}D^{-1}Gp - G^{\mathrm{T}}Q^{-1}Gp \\ -G^{\mathrm{T}}Q^{-1}Gp &= -\lambda G^{\mathrm{T}}D^{-1}Gp \end{aligned}$$

This generalized eigenvalue problem is notated as

$$Sp = \lambda Rp$$

in which, $S = -G^{\mathrm{T}}Q^{-1}G \in \mathbb{R}^{m\times m}$ is the *Schur* complement of the matrix A, and $R = -G^{\mathrm{T}}D^{-1}G \in \mathbb{R}^{m\times m}$.

To conclude the above analysis, the following proposition is derived.

Proposition 4
For the SIMPLE preconditioned matrix $\tilde{A}$,
(1) 1 is an eigenvalue with (algebraic and geometric) multiplicity of n, and
(2) the remaining eigenvalues are defined by the generalized eigenvalue problem

$$Sp = \lambda Rp \tag{13}$$

In the following section, we investigate the generalized eigenvalue problems (7) and (13) in more detail.

2.2. The relation between both spectral formulations

In Section 2.1, two different generalized eigenvalue problems (7) and (13) have been derived to describe the spectrum of $\tilde{A}$. In this section, we shall show that the two generalized eigenvalue problems are closely related.

Firstly, we investigate the generalized eigenvalue problem (13). Re-write matrix R as

$$R = -G^{\mathrm{T}} D^{-1} G = -(D^{-1/2}G)^{\mathrm{T}}(D^{-1/2}G)$$

Making the singular value decomposition of the matrix $D^{-1/2}G \in \mathbb{R}^{n\times m}$, we have

$$D^{-1/2}G = U\Sigma V^{\mathrm{T}} \tag{14}$$

in which, $U \in \mathbb{R}^{n\times n}, V \in \mathbb{R}^{m\times m}$ are unitary matrices and σ_i, $i = 1, 2, \ldots, m$, are the singular values of the matrix $D^{-1/2}G$, which are all positive numbers since $\mathrm{rank}(D^{-1/2}G) = m$. So,

$$\begin{aligned}
G &= D^{1/2} U\Sigma V^{\mathrm{T}} \\
R &= -(U\Sigma V^{\mathrm{T}})^{\mathrm{T}}(U\Sigma V^{\mathrm{T}}) = -V\Sigma^{\mathrm{T}}\Sigma V^{\mathrm{T}} \\
S &= -G^{\mathrm{T}} Q^{-1} G \\
&= -(D^{1/2}U\Sigma V^{\mathrm{T}})^{\mathrm{T}} Q^{-1} (D^{1/2} U\Sigma V^{\mathrm{T}}) \\
&= -V\Sigma^{\mathrm{T}} U^{\mathrm{T}} D^{1/2} Q^{-1} D^{1/2} U\Sigma V^{\mathrm{T}}
\end{aligned}$$

It follows that

$$R^{-1}S = V(\Sigma^{\mathrm{T}}\Sigma)^{-1}\Sigma^{\mathrm{T}} U^{\mathrm{T}} D^{1/2} Q^{-1} D^{1/2} U\Sigma V^{\mathrm{T}} \tag{15}$$

To study the generalized eigenvalue problem (7), by using the same singular value decomposition for matrix $D^{-1/2}G$, we have

$$\begin{aligned}
E &= GR^{-1}G^{\mathrm{T}} \\
&= (D^{1/2}U\Sigma V^{\mathrm{T}})(-V(\Sigma^{\mathrm{T}}\Sigma)^{-1}V^{\mathrm{T}})(D^{1/2}U\Sigma V^{\mathrm{T}})^{\mathrm{T}} \\
&= -D^{1/2}U\Sigma(\Sigma^{\mathrm{T}}\Sigma)^{-1}\Sigma^{\mathrm{T}} U^{\mathrm{T}} D^{1/2}
\end{aligned}$$

The matrix Z is a notation for matrix JQ^{-1}, so

$$Z = JQ^{-1} = D^{-1}(D - Q)Q^{-1} = (Q^{-1} - D^{-1})$$

Finally, we get

$$ZE = -(Q^{-1} - D^{-1})D^{1/2}U\Sigma(\Sigma^{\mathrm{T}}\Sigma)^{-1}\Sigma^{\mathrm{T}}U^{\mathrm{T}}D^{1/2} \tag{16}$$

Multiplying by $U^{\mathrm{T}}D^{1/2}$ and $D^{-1/2}U$ to (16) from the left and right, respectively, a spectrum equivalent matrix is produced as

$$U^{\mathrm{T}}D^{1/2}ZED^{-1/2}U = -U^{\mathrm{T}}D^{1/2}Q^{-1}D^{1/2}U\Sigma(\Sigma^{\mathrm{T}}\Sigma)^{-1}\Sigma^{\mathrm{T}} + \Sigma(\Sigma^{\mathrm{T}}\Sigma)^{-1}\Sigma^{\mathrm{T}}$$

We denote this equation by

$$U^{\mathrm{T}}D^{1/2}ZED^{-1/2}U = -MN + N \tag{17}$$

in which,

$$M = U^{\mathrm{T}}D^{1/2}Q^{-1}D^{1/2}U \in \mathbb{R}^{n\times n} \quad \text{and} \quad N = \Sigma(\Sigma^{\mathrm{T}}\Sigma)^{-1}\Sigma^{\mathrm{T}} = \begin{pmatrix} I_m & O \\ O & O \end{pmatrix} \in \mathbb{R}^{n\times n}$$

Partitioning matrix M according to the structure of N, (17) can be written in a sub-matrix form

$$U^{\mathrm{T}}D^{1/2}ZED^{-1/2}U = -MN + N = \begin{pmatrix} I_m - M_{11} & O \\ -M_{21} & O \end{pmatrix} \tag{18}$$

Its characteristic polynomial is

$$\det(\mu I - U^{\mathrm{T}}D^{1/2}ZED^{-1/2}U) = \mu^{n-m}\det((\mu - 1)I_m + M_{11})$$

So, we get to know that 0 is an eigenvalue of ZE with multiplicity of $n - m$, and the remaining eigenvalues are $\mu_i = 1 - \eta_i,\ i = 1, 2, \ldots, m$, where η_i is the ith non-zero eigenvalue of the sub-matrix M_{11}. From (18), η_i is also an eigenvalue of MN at the same time, since

$$\det(\eta I - MN) = \eta^{n-m}\det(\eta I_m - M_{11})$$

By Proposition 3, we have

$$\sigma(\tilde{A}) = \{1\} \cup \{1 - \mu_i\} = \{1\} \cup \{\eta_i\} \tag{19}$$

in which, the eigenvalue 1 has the multiplicity of $m + (n - m) = n$, and $\eta_i \in \sigma(MN),\ \eta_i \neq 0,\ i = 1, 2, \ldots, m$.

On the other hand, if we denote

$$T_1 := U^{\mathrm{T}}D^{1/2}Q^{-1}D^{1/2}U\Sigma \in \mathbb{R}^{n\times m} \quad \text{and} \quad T_2 := (\Sigma^{\mathrm{T}}\Sigma)^{-1}\Sigma^{\mathrm{T}} \in \mathbb{R}^{m\times n}$$

then $MN = T_1T_2$. We know that $T_1T_2 \in \mathbb{R}^{n\times n}$ and $T_2T_1 \in \mathbb{R}^{m\times m}$ have the same spectrum except for the possible zero eigenvalue [30, pp. 69]. The spectrum of T_2T_1 is

$$\begin{aligned}\sigma(T_2T_1) &= \sigma((\Sigma^{\mathrm{T}}\Sigma)^{-1}\Sigma^{\mathrm{T}}U^{\mathrm{T}}D^{1/2}Q^{-1}D^{1/2}U\Sigma)\\ &= \sigma(V(\Sigma^{\mathrm{T}}\Sigma)^{-1}\Sigma^{\mathrm{T}}U^{\mathrm{T}}D^{1/2}Q^{-1}D^{1/2}U\Sigma V^{\mathrm{T}})\\ &= \sigma(R^{-1}S)\end{aligned}$$

The last equation is based on the fact of Equation (15). This relation motivates the following proposition.

Proposition 5

For the two generalized eigenvalue problem (7) and (13), suppose that $\mu_i \in \sigma(ZE), i = 1, 2, \ldots, n$, and $\lambda_i \in \sigma(R^{-1}S),\ i = 1, 2, \ldots, m$, the relationship between the two problems is that $\mu = 0$ is an eigenvalue of (7) with multiplicity of $n - m$, which can be denoted as $\mu_{m+1} = \mu_{m+2} = \cdots = \mu_n = 0$, and that $\lambda_i = 1 - \mu_i,\ i = 1, 2, \ldots, m$, holds for the remaining m eigenvalues.

3. SOME EIGENVALUE BOUNDS FOR THE SYMMETRIC CASE

In this section, we assume that Q is symmetric positive definite, which corresponds to the cases when term $u\,\mathrm{grad}\,u$ is deleted from Navier–Stokes equations leading to the incompressible Stokes equations. In this case, the coefficient matrix A is symmetric and indefinite.

Consider the generalized eigenvalue problem (13)

$$Sp = \lambda Rp \tag{20}$$

It is obvious that the problem $-Sp = -\lambda Rp$ is completely equivalent to the problem $Sp = \lambda Rp$. Since both $-S$ and $-R$ are s.p.d. matrices, we call (20) a s.p.d. generalized eigenvalue problem by neglecting the negative signs on both sides. Note that there are m independent eigenvectors (see Reference [31, Corollary 8.7.2]) and all eigenvalues are positive. For the s.p.d. generalized eigenvalue problem, the extreme eigenvalues ($\lambda_{\max}$ and $\lambda_{\min}$) are the extreme values of [30, p. 379]:

$$\frac{p^{\mathrm{T}}Sp}{p^{\mathrm{T}}Rp} = \frac{p^{\mathrm{T}}G^{\mathrm{T}}Q^{-1}Gp}{p^{\mathrm{T}}G^{\mathrm{T}}D^{-1}Gp}, \quad p \neq 0, \ \ p \in \mathbb{R}^m \tag{21}$$

which is the ratio of the Rayleigh quotients of S and R. So,

$$\lambda_{\max} = \max_{p\neq 0} \frac{p^{\mathrm{T}}G^{\mathrm{T}}Q^{-1}Gp}{p^{\mathrm{T}}G^{\mathrm{T}}D^{-1}Gp} = \max_{p\neq 0} \frac{(Gp)^{\mathrm{T}}Q^{-1}(Gp)}{(Gp)^{\mathrm{T}}D^{-1}(Gp)} \tag{22}$$

Since that the matrix G has column full rank, i.e. $\mathrm{rank}(G) = m$, $Gp = 0$ if and only if $p = 0$. Denoting $y = Gp$, it follows that

$$\lambda_{\max} \leqslant \max_{y\neq 0} \frac{y^{\mathrm{T}}Q^{-1}y}{y^{\mathrm{T}}D^{-1}y} \tag{23}$$

Let μ_1, μ_n be the largest and the smallest eigenvalues of the matrix Q, and d_1, d_n be the largest and the smallest diagonal elements of Q, respectively, then

$$\lambda_{\max} \leqslant \frac{d_1}{\mu_n} \quad \text{and} \quad \lambda_{\min} \geqslant \frac{d_n}{\mu_1} \tag{24}$$

So, combining (24) and Proposition 4, we get the following bounds for the eigenvalues of the preconditioned matrix $\tilde{A}$:

$$\min\left\{1, \frac{d_n}{\mu_1}\right\} \leqslant \lambda \leqslant \max\left\{1, \frac{d_1}{\mu_n}\right\} \quad \forall \lambda \in \sigma(\tilde{A}) \tag{25}$$

If both sides of (25) are taken to be d_n/μ_1 and d_1/μ_n, respectively, then

$$\kappa(\tilde{A}) = \frac{\lambda_{\max}}{\lambda_{\min}} \leqslant \frac{d_1}{d_n} \cdot \frac{\mu_1}{\mu_n} = \frac{d_1}{d_n} \kappa(Q) \tag{26}$$

where $\kappa(\cdot)$ represents the (spectral) condition number.

4. THE INFLUENCE OF THE DIAGONAL SCALING

In Reference [2] a diagonal scaling strategy is proposed for a practical implementation of the SIMPLE preconditioning. Scale the coefficient matrix A by (left) multiplying with the diagonal matrix

$$\hat{D} := \begin{pmatrix} D^{-1} & O \\ O & D_R^{-1} \end{pmatrix} \tag{27}$$

where

$$D = \operatorname{diag}(Q) \quad \text{and} \quad D_R = \operatorname{diag}(R)$$

After this scaling, the coefficient matrix becomes

$$\mathscr{A} := \hat{D}A = \begin{pmatrix} D^{-1}Q & D^{-1}G \\ D_R^{-1}G^{\mathrm{T}} & O \end{pmatrix} \tag{28}$$

At this moment,

$$\mathscr{D} = \operatorname{diag}(D^{-1}Q) = I \in \mathbb{R}^{n\times n}, \qquad \mathscr{R} = -(D_R^{-1}G^{\mathrm{T}})\mathscr{D}^{-1}(D^{-1}G) = D_R^{-1}R \in \mathbb{R}^{m\times m}$$

and

$$\mathscr{B} = \begin{pmatrix} I & -D^{-1}G \\ O & I \end{pmatrix}, \quad \mathscr{M} = \begin{pmatrix} D^{-1}Q & O \\ D_R^{-1}G^{\mathrm{T}} & \mathscr{R} \end{pmatrix}, \quad \mathscr{M}^{-1} = \begin{pmatrix} Q^{-1}D & O \\ -\mathscr{R}^{-1}D_R^{-1}G^{\mathrm{T}}Q^{-1}D & \mathscr{R}^{-1} \end{pmatrix}$$

The SIMPLE preconditioned matrix now is

$$\begin{aligned}\tilde{\mathscr{A}} &= \mathscr{A}\mathscr{B}\mathscr{M}^{-1}\\ &= \begin{pmatrix} D^{-1}Q & D^{-1}G \\ D_R^{-1}G^{\mathrm{T}} & O \end{pmatrix} \begin{pmatrix} I & -D^{-1}G \\ O & I \end{pmatrix} \begin{pmatrix} Q^{-1}D & O \\ -\mathscr{R}^{-1}D_R^{-1}G^{\mathrm{T}}Q^{-1}D & \mathscr{R}^{-1} \end{pmatrix}\\ &= \begin{pmatrix} \tilde{\mathscr{A}}_{11} & \tilde{\mathscr{A}}_{12} \\ \tilde{\mathscr{A}}_{21} & \tilde{\mathscr{A}}_{22} \end{pmatrix}\end{aligned}$$

in which, by doing some elementary matrix calculation, these sub-matrices are:

$$\begin{aligned}\tilde{\mathscr{A}}_{11} &= I + D^{-1}[QD^{-1}G\mathscr{R}^{-1}D_R^{-1}G^{\mathrm{T}}Q^{-1} - G\mathscr{R}^{-1}D_R^{-1}G^{\mathrm{T}}Q^{-1}]D\\ &= I - D^{-1}Q(Q^{-1} - D^{-1})GR^{-1}G^{\mathrm{T}}Q^{-1}D\\ \tilde{\mathscr{A}}_{12} &= -D^{-1}QD^{-1}G\mathscr{R}^{-1} + D^{-1}G\mathscr{R}^{-1} = D^{-1}(I - QD^{-1})G\mathscr{R}^{-1}\\ \tilde{\mathscr{A}}_{21} &= D_R^{-1}G^{\mathrm{T}}Q^{-1}D + D_R^{-1}G^{\mathrm{T}}D^{-1}G\mathscr{R}^{-1}D_R^{-1}G^{\mathrm{T}}Q^{-1}D = O\\ \tilde{\mathscr{A}}_{22} &= -D_R^{-1}G^{\mathrm{T}}D^{-1}G\mathscr{R}^{-1} = I\end{aligned}$$

Finally, it follows that

$$\tilde{\mathscr{A}} = \begin{pmatrix} I - D^{-1}(I - QD^{-1})GR^{-1}G^{\mathrm{T}}Q^{-1}D & D^{-1}(I - QD^{-1})GR^{-1}D_R \\ O & I \end{pmatrix} \tag{29}$$

Comparing the matrix $\tilde{\mathscr{A}}$ in (29) with the matrix $\tilde{A}$ defined by (4), we find that the spectra of both matrices are exactly the same. However in practice we see a difference in convergence, which again shows that the eigenvalues are only a limited tool to predict the convergence of non-symmetric Krylov solvers.

5. NUMERICAL EXAMPLES

Two numerical test results are reported here to illustrate the discussions above.

Example 6
In this example, the coefficient matrix is taken from a discretized Navier–Stokes equations on a 16×16 grid [3] (length $= 2$, $v = 1$). The dimensions are $n = 544$, $m = 256$, and $n + m = 800$. $A \in \mathbb{R}^{800\times 800}$ is a non-symmetric matrix.

The eigenvalues of the preconditioned matrix $\tilde{A}$ has been computed by both Propositions 3 and 4. The computed results are the same, which coincide with the theoretical analysis. Spectra of A and $\tilde{A}$ are plotted in Figure 1, and some extreme eigenvalues are listed in Table I.

From this example, we can see that the eigenvalues of the SIMPLE preconditioned matrix $\tilde{A}$ are clustered in a smaller region in the right-half plane. GCR applied to the original system requires 410 iterations, whereas GCR–SIMPLE needs only 48 iterations.

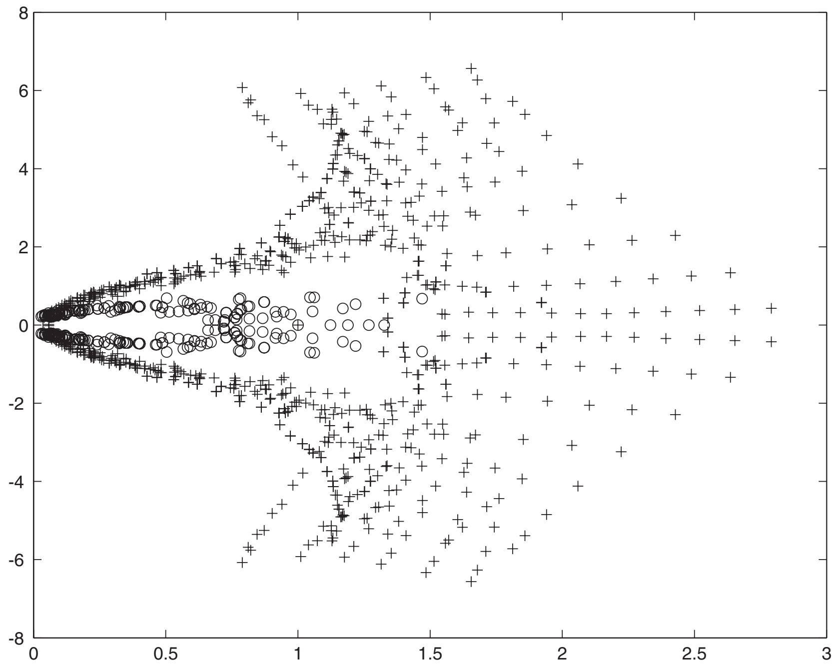

Figure 1. Spectrum of A and $\tilde{A}$. The '+' represents for the eigenvalues of A, while 'o' for that of the preconditioned $\tilde{A}$.

Table I. The extreme eigenvalues of A and $\tilde{A}$.

Matrix	$\max \Re(\lambda_i)$	$\min \Re(\lambda_i)$	$\max \Im(\lambda_i)$	$\max \lvert\lambda_i\rvert$	$\min \lvert\lambda_i\rvert$
A	2.79074	0.03559	6.56341	6.76892	0.06018
$\tilde{A}$	1.46960	0.03000	0.70700	1.61894	0.21395

Example 7

The matrix A is obtained from a discretized Stokes equation on a 16×16 grid by removing the Dirichlet boundary conditions. The resulted coefficient matrix $A \in \mathbb{R}^{800\times 800}$ is symmetric, and $Q \in \mathbb{R}^{544\times 544}$ is a s.p.d. matrix.

The extreme eigenvalues of A and $\tilde{A}$ are listed in Table II.

The results of this example agree with the theoretical eigenvalue bounds in Section 3, which are:

$$\frac{\lambda_{\min}(D)}{\lambda_{\max}(Q)} = \frac{0.96}{2.547} = 0.377 \leqslant \lambda(\tilde{A}) \leqslant 103.9 = \frac{1.6}{0.0154} = \frac{\lambda_{\max}(D)}{\lambda_{\min}(Q)}$$

Note that the eigenvalues of $\tilde{A}$ are all positive. GCR applied to the original system requires 178 iterations, whereas GCR–SIMPLE needs only 19 iterations. For more numerical experiments with GCR–SIMPLE(R) we refer to References [2, 3].

Table II. The extreme eigenvalues of A and $\tilde{A}$ for Example 5.2.

Matrix	$\lambda_{\min}$	$\min\|\lambda_i\|$	$\lambda_{\max}$	$\kappa(\cdot)$
A	−23.4555	0.0501	25.3762	1729.5
$\tilde{A}$	0.5049	0.5049	46.7880	344.1
Q	0.0154	0.0154	2.5477	232.9
D	0.9600	0.9600	1.6000	1.6

6. CONCLUDING REMARKS

We have derived two formulations to describe the spectrum of the SIMPLE preconditioned matrix $\tilde{A}$. These theoretical results are helpful to achieve new insights for this preconditioner. The methodology in this paper is instructive for the eigenvalue analysis for this type of preconditioning (for example, the SIMPLER preconditioning). The eigenvalue bounds in the symmetric case are useful for evaluating the efficiency of the SIMPLE preconditioned iterative solvers for the Stokes equations.

The results for general non-symmetric matrix in this paper mainly have some theoretical meaning. More accurate and more practical estimations about the spectrum of $\tilde{A}$ need to be done. The main issues towards this aim are the investigations to the specific generalized eigenvalue problems (7) and (13). Pseudo-spectra analysis [32, 33] might be needed to analyze these non-symmetric problems.

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REFERENCES

1. Bruaset AM. *A Survey of Preconditioned Iterative Methods*. Pitman research notes in mathematics series 328. Longman Scientific and Technical: Harlow, 1995.
2. Vuik C, Saghir A, Boerstoel GP. The Krylov accelerated SIMPLE(R) method for flow problems in industrial furnaces. *International Journal for Numerical Methods in Fluids* 2000; **33**:1027–1040.
3. Vuik C, Saghir A. The Krylov accelerated SIMPLE(R) method for incompressible flow. *Report 02-01*, Delft University of Technology, Department of Applied Mathematical Analysis, Delft, 2002. http://ta.twi.tudelft.nl/nw/users/vuik/papers/Vui02S.pdf.
4. Eisenstat SC, Elman HC, Schultz MH. Variational iterative methods for nonsymmetric systems of linear equations. *SIAM Journal on Numerical Analysis* 1983; **20**:345–357.
5. Patankar SV. *Numerical Heat Transfer and Fluid Flow*. McGraw-Hill: New York, 1980.
6. Temam R. *Navier–Stokes equations. Theory and Numerical Analysis* (3rd edn). North-Holland: Amsterdam, 1984.
7. Vincent C, Boyer R. A preconditioned conjugate gradient Uzawa-type method for the solution of the Stokes problem by mixed Q1-P0 stabilized finite elements. *International Journal for Numerical Methods in Fluids* 1992; **14**:289–298.
8. Elman HC. Multigrid and Krylov subspace methods for the discrete Navier–Stokes equations. *International Journal for Numerical Methods in Fluids* 1996; **22**:755–770.
9. Elman HC, Golub GH. Inexact and preconditioned Uzawa algorithms for saddle-point problems. *SIAM Journal on Numerical Analysis* 1994; **31**:1645–1661.
10. Bramble JH, Pasciak JE, Vassilev AT. Analysis of the inexact Uzawa algorithm for saddle point problems. *SIAM Journal on Numerical Analysis* 1997; **34**:1072–1092.

11. Mingrong Cui. A sufficient condition for the convergence of the inexact Uzawa algorithm for saddle point problems. *Journal of Computational and Applied Mathematics* 2002; **139**:189–196.
12. Ferziger JH, Peric M. *Computational Methods for Fluid Dynamics* (2nd edn). Springer: Berlin, 1999.
13. Bercovier M. Perturbation of a mixed variational problem, applications to mixed finite element methods. *RAIRO. Analyse Numérique* 1978; **12**:211–236.
14. van Kan J. A second-order accurate pressure-correction scheme for viscous incompressible flow. *SIAM Journal on Statistical and Scientific Computing* 1986; **7**:870–891.
15. Barten IE. Comparison of SIMPLE- and PISO-type algorithms for transient flows. *International Journal for Numerical Methods in Fluids* 1998; **26**:459–483.
16. Bramble JH, Pasciak JE. A preconditioning technique for indefinite systems resulting from mixed approximations of elliptic problems. *Mathematics of Computation* 1988; **50**:1–17.
17. Golub GH, Wathen AW. An iteration for indefinite systems and its application to the Navier–Stokes equations. *SIAM Journal on Scientific Computing* 1998; **19**:530–539.
18. Kay D, Loghin D, Wathen AJ. A preconditioner for the steady-state Navier–Stokes equations. *SIAM Journal on Scientific Computing* 2002; **24**:237–256.
19. Axelsson O, Neytcheva M. Preconditioning methods for linear systems arising in constrained optimization problems. *Numerical Linear Algebra with Applications* 2003; **10**:3–31.
20. Shaw GJ, Sivaloganathan S. On the smoothing of the SIMPLE pressure correction algorithm. *International Journal for Numerical Methods in Fluids* 1988; **8**:441–462.
21. Wittum G. On the convergence of multi-grid methods with transforming smoothers. *Numerische Mathematik* 1990; **57**:15–38.
22. Wesseling P. *An Introduction to Multigrid Methods*. Wiley: Chichester, 1992. available at www.mgnet.org/mgnet-books-wesseling.html.
23. Braess D, Sarazin R. An efficient smoother for the Stokes problem. *Applied Numerical Mathematics* 1996; **23**:3–19.
24. Wesseling P. *Principles of Computational Fluid Dynamics*. Springer Series in Computational Mathematics, vol. 29. Springer: Heidelberg, 2000.
25. Pernice M, Tocci MD. A multigrid-preconditioned Newton–Krylov method for the incompressible Navier–Stokes equations. *SIAM Journal on Scientific Computing* 2001; **23**:398–418.
26. Quarteroni A, Valli A. *Numerical Approximation of Partial Differential Equations*. Springer series in Computational Mathematics, vol. 23. Springer: Berlin, 1994.
27. Saad Y, Schultz MH. GMRES: A generalized minimal residual algorithm for solving nonsymmetric linear systems. *SIAM Journal on Statistical and Scientific Computing* 1986; **7**:856–869.
28. van der Vorst HA, Vuik C. The superlinear convergence behaviour of GMRES. *Journal of Computational and Applied Mathematics* 1993; **48**:327–341.
29. Greenbaum A, Ptak V, Strakos Z. Any nonincreasing convergence curve is possible for GMRES. *SIAM Journal on Matrix Analysis and Applications* 1996; **17**:465–469.
30. Axelsson O. *Iterative Solution Methods*. Cambridge University Press: Cambridge, 1994.
31. Golub GH, van Loan CF. *Matrix Computations* (3rd edn). The Johns Hopkins University Press: Baltimore, 1996.
32. Trefethen LN. Pseudospectra of linear operators. *SIAM Review* 1997; **39**:383–406.
33. Trefethen LN. Spectra and pseudospectra. In *The Graduate Student's Guide to Numerical Analysis '98*, Ainsworth M, Levesley J, Marletta M (eds). Springer: Berlin, 1997; 217–250.

