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THE GCR-SIMPLE SOLVER AND THE SIMPLE-TYPE PRECONDITIONING FOR INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

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Abstract. *The discretization of incompressible Navier-Stokes equation leads to a large linear system with a nonsymmetric and indefinite coefficient matrix. Many methods are known to overcome these difficulties: Uzawa method, SIMPLE-type methods, penalty method, pressure correction method, etc. In this paper, Krylov accelerated versions of the SIMPLE(R) methods: GCR-SIMPLE(R) are investigated, where SIMPLE(R) are used as preconditioners. Preconditioning plays a key role in Krylov methods. We analyze the spectral structure of these SIMPLE-type preconditioners. Some formulations are established to characterize the eigenvalue distributions of these preconditioning methods. Spectral comparisons with Elman's preconditioner and ILU preconditioners are discussed. This gives insight into the convergence of the iterative methods. Some numerical test results are presented.*

1 INTRODUCTION

The steady state incompressible Navier-Stokes equations

$$\begin{cases} -\nu\Delta u + u\cdot \operatorname{grad} u + \operatorname{grad} p & = f, \\ -\operatorname{div} u & = 0, \end{cases}$$

combined with some boundary conditions, are widely used to simulate the incompressible flow of a fluid. Discretization and linearization of the equations leads to the following large sparse linear algebraic system

$$\begin{pmatrix} Q & G \\ G^T & O \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \tag{1}$$

where $Q \in \mathbb{R}^{n\times n}, G \in \mathbb{R}^{n\times m}, m \leqslant n, \det(Q) \neq 0, \operatorname{rank}(G) = m$; $u \in \mathbb{R}^n$ and $p \in \mathbb{R}^m$ are the velocity vector and the pressure vector respectively. For problems with three space dimensions, iterative solvers are required. Preconditioning often determines the numerical performance of the Krylov subspace solvers [2].

In [11, 12], Vuik proposed GCR-SIMPLE(R) algorithm for solving the large linear system (1). The algorithm can be considered as a combination of the Krylov subspace method GCR [3] with the SIMPLE(R) algorithm[7]. In this combined algorithm, the SIMPLE(R) iteration is collaborated as a preconditioner with the GCR method. Numerical tests indicate that the SIMPLE(R) preconditioning is effective and competitive for practical use.

In this paper, we focus on the eigenvalue analysis of the SIMPLE preconditioned matrix $\widetilde{A}$. Two related formulations are derived to describe the spectrum of $\widetilde{A}$. The spectrum has some connection with that of the Schur complement of the matrix A. The relationship between the two different formulations has been investigated by using the theory of matrix singular value decomposition. A diagonal scaling technique proposed by Vuik[11] is studied. Some useful eigenvalue bounds have been got in symmetric situation. Numerical tests are used to illustrate the theoretical bounds.

In the remaining parts of this paper, the linear system (1) is abbreviated as $Ax = b$, where $A \in \mathbb{R}^{(n+m)\times(n+m)}, b \in \mathbb{R}^{n+m}$. Notations have the same meaning with references [12, 11]. $\sigma(A)$ represents the set of all eigenvalues of matrix A, for example. Besides, we assume that the matrix Q, its diagonal matrix $D := \operatorname{diag}(Q)$, are all nonsingular in this paper.

2 SIMPLE TYPE METHODS AND GCR-SIMPLE(R) ALGORITHMS

In this section, we review briefly the SIMPLE and SIMPLER methods and its combination with one Krylov methods:GCR-SIMPLE(R) algorithm.

2.1 SIMPLE type methods

If we denote $R := -G^T D^{-1} G$, where $D := \operatorname{diag}(Q)$ is the diagonal of the matrix Q, then the SIMPLE method proposed by Patanker [7] is given by the following algorithms:

Algorithm 2.1. *SIMPLE algorithm*
1. *Choose an initial guess* p^*.
2. *Solve* $Qu^* = b_1 - Gp^*$.
3. *Solve* $R\delta p = b_2 - G^T u^*$.
4. *Compute* $u = u^* - D^{-1}G\delta p$, *and* $p := p^* + \delta p$.
5. *If not converged take* $p^* = p$ *and go to 2.*

The SIMPLE method can be seen as a distributive method. Instead of solving the system $Ax = b$ the system $ABy = b, x = By$ will be solved.Choosing B and M as:

$$B = \begin{pmatrix} I & -D^{-1}G \\ O & I \end{pmatrix}, \quad M = \begin{pmatrix} Q & O \\ G^T & R \end{pmatrix}, \tag{2}$$

and using the splitting $AB = M - N$, the following iteration is equivalent to the SIMPLE method [13]:

$$x^{k+1} = x^k + BM^{-1}(b - Ax^k), \ k = 0, 1, 2, \cdots, niter.$$

When the velocity vector u is known, p is a solution of the system:

$$Rp = b_2 - G^T D^{-1}((D - Q)u + b_1).$$

The idea is used in the SIMPLER method:

Algorithm 2.2. *SIMPLER algorithm*
1. *Solve* $Rp^* = b_2 - G^T D^{-1}((D - Q)u^k + b_1)$.
2. *Solve* $Qu^* = b_1 - Gp^*$.
3. *Solve* $R\delta p = b_2 - G^T u^*$.
4. *Compute* $u^{k+1} = u^* - D^{-1}G\delta p$, *and* $p := p^* + \delta p$.
5. *If not converged take* $p^* = p$ *and go to 2.*

Similar to SIMPLE method, SIMPLER method can also be expressed as:

$$x^{k+1} = x^k + P(b - Ax^k), \ k = 0, 1, 2, \cdots, niter.$$

Where,

$$P = BM^{-1} - BM^{-1}AM_L^{-1}B_L + M_L^{-1}B_L, \tag{3}$$

$$B_L = \begin{pmatrix} I & O \\ -G^T D^{-1} & I \end{pmatrix}, \quad M_L = \begin{pmatrix} Q & G \\ O & R \end{pmatrix},$$

2.2 GCR-SIMPLE(R) algorithms

Many Krylov subspace methods are known to solve non-symmetric linear systems. We choose the GCR method [3] because the method is robust,minimizes the residual and allows a variable preconditioner [12]. The preconditioned GCR method is described as following:

Algorithm 2.3. *Preconditioned GCR Method*

$r^0 = b - Ax^0$
for $k = 0, 1, \cdots, ngcr$
$s^{k+1} = P_k^{-1} r^k$
$v^{k+1} = As^{k+1}$
for $i = 0, 1, \cdots, k$
$v^{k+1} = v^{k+1} - (v^{k+1}, v^i)v^i$
$s^{k+1} = s^{k+1} - (v^{k+1}, v^i)s^i$
end for
$v^{k+1} = v^{k+1}/\left\|v^{k+1}\right\|_2$
$s^{k+1} = s^{k+1}/\left\|v^{k+1}\right\|_2$
$x^{k+1} = x^k + (r^k, v^{k+1})s^{k+1}$
$r^{k+1} = r^k - (r^k, v^{k+1})v^{k+1}$
end for

Here, P_k^{-1} is a preconditioner. We call it GCR-SIMPLE algorithm when P_k^{-1} is chosen as BM^{-1} defined by (2), and GCR-SIMPLER algorithm when P_k^{-1} is chosen as P defined by (3).

GCR-SIMPLE algorithm and GCR-SIMPLER algorithm are efficient and practical. For the numerical tests relating with industrial modelling we refer to [11], [12] for reference. Next sections will be devoted to explore the theoretical aspects by using spectral analysis.

3 TWO FORMULATIONS OF THE SPECTRUM OF THE SIMPLE PRECONDITIONED MATRIX

Consider the right preconditioning to the linear system (1)

$$AP^{-1}y = b, \quad x = P^{-1}y. \tag{4}$$

When the SIMPLE algorithm is used as preconditioning, it is equivalent to choose the preconditioner P^{-1} as[12, 13]

$$P^{-1} = BM^{-1} \quad, P = MB^{-1}, \tag{5}$$

where,

$$B = \begin{pmatrix} I & -D^{-1}G \\ O & I \end{pmatrix}, \quad M = \begin{pmatrix} Q & O \\ G^T & R \end{pmatrix}, \quad D = \mathrm{diag}(Q), \; R = -G^T D^{-1} G.$$

We call this preconditioning a SIMPLE preconditioning, and the preconditioner P^{-1} as SIMPLE preconditioner. For SIMPLE preconditioning, we have the following result:

Proposition 3.1. *If the right preconditioner P^{-1} is taken to be the matrix defined by* (5)*, then the preconditioned matrix is*

$$\widetilde{A} := AP^{-1} = \begin{pmatrix} I-(I-QD^{-1})GR^{-1}G^TQ^{-1} & (I-QD^{-1})GR^{-1} \\ O & I \end{pmatrix}. \tag{6}$$

And, therefore, the spectrum of the SIMPLE *preconditioned matrix $\widetilde{A}$ is*

$$\sigma(\widetilde{A}) = \{1\} \cup \sigma(I-(I-QD^{-1})GR^{-1}G^TQ^{-1}). \tag{7}$$

Proof. It is easy to verify that

$$M^{-1} = \begin{pmatrix} Q^{-1} & O \\ -R^{-1}G^TQ^{-1} & R^{-1} \end{pmatrix}, \tag{8}$$

and

$$\begin{aligned} \widetilde{A} &= AP^{-1} = ABM^{-1} \\ &= \begin{pmatrix} Q & G \\ G^T & O \end{pmatrix} \begin{pmatrix} I & -D^{-1}G \\ O & I \end{pmatrix} \begin{pmatrix} Q^{-1} & O \\ -R^{-1}G^TQ^{-1} & R^{-1} \end{pmatrix} \\ &= \begin{pmatrix} I-(I-QD^{-1})GR^{-1}G^TQ^{-1} & (I-QD^{-1})GR^{-1} \\ O & I \end{pmatrix}. \end{aligned}$$

So, the fact about the spectrum of $\widetilde{A}$, described by (7), follows. □

Now, we study the spectrum defined by (7) in more detail. By multiplying with matrices Q^{-1} and Q from the left- and right-hand side of the matrix $I-(I-QD^{-1})GR^{-1}G^TQ^{-1}$ respectively, we get

$$\begin{aligned} \sigma(I-(I-QD^{-1})GR^{-1}G^TQ^{-1}) &= \sigma(I-(Q^{-1}-D^{-1})GR^{-1}G^T) \\ &= \sigma(I-D^{-1}(D-Q)Q^{-1}GR^{-1}G^T) \ , \\ &= \sigma(I-J\ Q^{-1}GR^{-1}G^T) \end{aligned}$$

in which, the matrix $J := D^{-1}(D-Q)$ is the *Jacobi* iteration matrix for the matrix Q. This observation leads to the following proposition:

Proposition 3.2. *For the* SIMPLE *preconditioned matrix $\widetilde{A}$,*

1. 1 *is an eigenvalue with multiplicity at least of m, and*

2. *the remaining eigenvalues are $1-\mu_i,\ i = 1, 2, \cdots, n$, where μ_i is the $i-$ th eigenvalue of the generalized eigenvalue problem*

$$ZEx = \mu x, \tag{9}$$

where,

$$E = GR^{-1}G^T \in \mathbb{R}^{n\times n}, \quad Z = JQ^{-1} \in \mathbb{R}^{n\times n}.$$

Next, to investigate the spectrum of $\widetilde{A}$ more accurately, we derive another formulation of it. Consider the eigenvalue problem

$$\widetilde{A}x = \lambda x,$$

i.e.,

$$AP^{-1}x = \lambda x. \tag{10}$$

We know that AP^{-1} has the same spectrum as $P^{-1}A$ except for some possible zero eigenvalues [1]. When matrices A and P are both nonsingular, it holds that $\sigma(AP^{-1}) = \sigma(P^{-1}A)$. So, the eigenvalue problem (10) is equivalent to the generalized eigenvalue problem

$$Ax = \lambda Px. \tag{11}$$

Here,

$$A = \begin{pmatrix} Q & G \\ G^T & O \end{pmatrix},$$

and

$$P = MB^{-1} = \begin{pmatrix} Q & O \\ G^T & R \end{pmatrix} \begin{pmatrix} I & D^{-1}G \\ O & I \end{pmatrix} = \begin{pmatrix} Q & QD^{-1}G \\ G^T & O \end{pmatrix}.$$

The generalized eigenvalue problem (11) can be written as

$$\begin{pmatrix} Q & G \\ G^T & O \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \lambda \begin{pmatrix} Q & QD^{-1}G \\ G^T & O \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix}, \tag{12}$$

that is

$$\begin{cases} Qu + Gp & = \lambda(Qu + QD^{-1}Gp), \\ G^T u & = \lambda G^T u. \end{cases}$$

Multiply by Q^{-1} from the left to the first equation, and re-arrange the two equations as

$$\begin{cases} (1-\lambda)u & = (\lambda D^{-1} - Q^{-1})Gp, \\ G^T(1-\lambda)u & = 0. \end{cases} \tag{13}$$

From (13), we see that 1 is an eigenvalue of (12). From the right-hand side of the first equation of (13)($\lambda = 1$),we can see that the eigenvectors corresponding to eigenvalue 1 are:

$$v_i = \begin{pmatrix} u_i \\ 0 \end{pmatrix} \in \mathbb{R}^{(n+m)}, u_i \in \mathbb{R}^n,\ i = 1, 2, \cdots, n,$$

where, $\{u_i\}_{i=1}^n$ is a basis of $\mathbb{R}^n$.

For $\lambda \neq 1$, it follows from the second equation in (13) that $G^T u = 0$. Multiplying the first equation in (13) with G^T shows that

$$\begin{aligned} 0 &= -G^T Q^{-1} Gp + \lambda G^T D^{-1} Gp, \\ -G^T Q^{-1} Gp &= -\lambda G^T D^{-1} Gp. \end{aligned}$$

It follows that

$$Sp = \lambda Rp,$$

in which, $S = -G^T Q^{-1} G \in \mathbb{R}^{m\times m}$ is the *Schur* complement of the matrix A, and $R = -G^T D^{-1} G \in \mathbb{R}^{m\times m}$.

To conclude the above analysis, the following proposition is derived.

Proposition 3.3. *For the* SIMPLE *preconditioned matrix* $\widetilde{A}$,

1. 1 *is an eigenvalue with multiplicity of* n, *and*

2. *the remaining eigenvalues are defined by the generalized eigenvalue problem*

$$Sp = \lambda Rp. \tag{14}$$

In the following section, we investigate the generalized eigenvalue problems in more detail.

4 THE RELATION BETWEEN BOTH SPECTRAL FORMULATIONS

In section 3, two different generalized eigenvalue problems (9) and (14) have been derived to describe the spectrum of $\widetilde{A}$. In this section, we shall show that the two generalized eigenvalue problems are closely related.

Firstly, we investigate the generalized eigenvalue problem (14). Re-write matrix R as

$$R = -G^T D^{-1} G = -(D^{-\frac{1}{2}}G)^T(D^{-\frac{1}{2}}G).$$

Making the singular value decomposition of the matrix $D^{-\frac{1}{2}}G \in \mathbb{R}^{n\times m}$, we have

$$D^{-\frac{1}{2}}G = U\Sigma V^T, \tag{15}$$

in which,$U \in \mathbb{R}^{n\times n}, V \in \mathbb{R}^{m\times m}$ are unitary matrices,i.e., $U^TU = I \in \mathbb{R}^{n\times n},\quad V^TV = I \in \mathbb{R}^{m\times m}$, and

$$\Sigma = \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & O & \\ & O & \ddots & \\ & & & \sigma_m \end{pmatrix} \in \mathbb{R}^{n\times m},$$

$\sigma_i, i = 1, 2, \cdots, m$, are the singular values of the matrix $D^{-\frac{1}{2}}G$, which are all positive numbers since $\mathrm{rank}(D^{-\frac{1}{2}}G) = m$. So,

$$\begin{aligned} G &= D^{\frac{1}{2}}U\Sigma V^T, \\ R &= -(U\Sigma V^T)^T(U\Sigma V^T) = -V\Sigma^T\Sigma V^T, \\ S &= -G^TQ^{-1}G \\ &= -(D^{\frac{1}{2}}U\Sigma V^T)^TQ^{-1}(D^{\frac{1}{2}}U\Sigma V^T) \\ &= -V\Sigma^TU^TD^{\frac{1}{2}}Q^{-1}D^{\frac{1}{2}}U\Sigma V^T. \end{aligned}$$

It follows that

$$R^{-1}S = V(\Sigma^T\Sigma)^{-1}\Sigma^T U^T D^{\frac{1}{2}} Q^{-1} D^{\frac{1}{2}} U\Sigma V^T. \tag{16}$$

To study the generalized eigenvalue problem (9), by using the same singular value decomposition for matrix $D^{-\frac{1}{2}}G$, we have

$$\begin{aligned} E &= GR^{-1}G^T \\ &= (D^{\frac{1}{2}}U\Sigma V^T)(-V(\Sigma^T\Sigma)^{-1}V^T)(D^{\frac{1}{2}}U\Sigma V^T)^T \\ &= -D^{\frac{1}{2}}U\Sigma(\Sigma^T\Sigma)^{-1}\Sigma^T U^T D^{\frac{1}{2}}, \end{aligned}$$

and

$$Z = JQ^{-1} = D^{-1}(D-Q)Q^{-1} = (Q^{-1} - D^{-1}).$$

Finally, we get

$$ZE = -(Q^{-1} - D^{-1})D^{\frac{1}{2}}U\Sigma(\Sigma^T\Sigma)^{-1}\Sigma^T U^T D^{\frac{1}{2}}. \tag{17}$$

Multiplying by $U^T D^{\frac{1}{2}}$ and $D^{-\frac{1}{2}}U$ to (17) from the left-side and right-side respectively, a spectrum equivalent matrix is produced as

$$U^T D^{\frac{1}{2}} ZED^{-\frac{1}{2}}U = -U^T D^{\frac{1}{2}} Q^{-1} D^{\frac{1}{2}} U\Sigma(\Sigma^T\Sigma)^{-1}\Sigma^T + \Sigma(\Sigma^T\Sigma)^{-1}\Sigma^T.$$

We denote this equation by

$$U^T D^{\frac{1}{2}} ZED^{-\frac{1}{2}}U = -MN + N, \tag{18}$$

in which,

$$M = U^T D^{\frac{1}{2}} Q^{-1} D^{\frac{1}{2}} U \in \mathbb{R}^{n\times n},$$

and

$$N = \Sigma(\Sigma^T\Sigma)^{-1}\Sigma^T = \begin{pmatrix} I_m & O \\ O & O \end{pmatrix} \in \mathbb{R}^{n\times n}.$$

Partitioning matrix M according to the structure of N, (18) can be written in a sub-matrix form

$$\begin{aligned} U^T D^{\frac{1}{2}} ZED^{-\frac{1}{2}}U &= -MN + N \\ &= -\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}\begin{pmatrix} I_m & O \\ O & O \end{pmatrix} + \begin{pmatrix} I_m & O \\ O & O \end{pmatrix} \\ &= \begin{pmatrix} I_m - M_{11} & O \\ -M_{21} & O \end{pmatrix}. \end{aligned} \tag{19}$$

Its characteristic polynomial is

$$\det(\mu I - U^T D^{\frac{1}{2}} ZED^{-\frac{1}{2}}U) = \mu^{n-m}\det((\mu - 1)I_m + M_{11}).$$

So, we get to know that 0 is an eigenvalue of ZE with multiplicity of $n-m$, and the remaining eigenvalues are $\mu_i = 1 - \eta_i, i = 1, 2, \cdots, m$, where η_i is the $i-$th nonzero eigenvalue of the sub-matrix M_{11}. From (19),η_i is also an eigenvalue of MN at the same time, since that

$$\det(\eta I - MN) = \eta^{n-m} \det(\eta I_m - M_{11}).$$

By proposition 2.2, we have

$$\sigma(\widetilde{A}) = \{1\} \cup \{1 - \mu_i\} = \{1\} \cup \{\eta_i\}, \tag{20}$$

in which, the eigenvalue 1 has the multiplicity of $m + (n - m) = n$, and $\eta_i \in \sigma(MN), \eta_i \neq 0, i = 1, 2, \cdots, m$.

On the other hand, if we denote

$$T_1 := U^T D^{\frac{1}{2}} Q^{-1} D^{\frac{1}{2}} U \Sigma \in \mathbb{R}^{n \times m},$$

and

$$T_2 := (\Sigma^T \Sigma)^{-1} \Sigma^T \in \mathbb{R}^{m \times n},$$

then $MN = T_1 T_2$. We know that $T_1 T_2 \in \mathbb{R}^{n \times n}$ and $T_2 T_1 \in \mathbb{R}^{m \times m}$ have the same spectrum except for the possible zero eigenvalue [1, pp.69]. The spectrum of $T_2 T_1$ is

$$\begin{aligned} \sigma(T_2 T_1) &= \sigma((\Sigma^T \Sigma)^{-1} \Sigma^T U^T D^{\frac{1}{2}} Q^{-1} D^{\frac{1}{2}} U \Sigma) \\ &= \sigma(V (\Sigma^T \Sigma)^{-1} \Sigma^T U^T D^{\frac{1}{2}} Q^{-1} D^{\frac{1}{2}} U \Sigma V^T) \\ &= \sigma(R^{-1} S). \end{aligned}$$

The last equation is based on the fact of equation (16). This relation motivates the following proposition.

Proposition 4.1. *For the two generalized eigenvalue problem (9) and (14), suppose that $\mu_i \in \sigma(ZE), i = 1, 2, \cdots, n$, and $\lambda_i \in \sigma(R^{-1}S), i = 1, 2, \cdots, m$, the relationship between the two problems is that $\mu = 0$ is an eigenvalue of (9) with multiplicity of $n - m$, which can be denoted as $\mu_{m+1} = \mu_{m+2} = \cdots = \mu_n = 0$, and that $\lambda_i = 1 - \mu_i$, $i = 1, 2, \cdots, m$, holds for the remaining m eigenvalues.*

5 THE DIAGONAL SCALING

Vuik [11, 12] proposed a diagonal scaling strategy for practical implementation of the SIMPLE preconditioning. Scale the coefficient matrix A by (left) multiplying the diagonal matrix

$$\widehat{D} := \begin{pmatrix} D^{-1} & O \\ O & D_R^{-1} \end{pmatrix}, \tag{21}$$

where,

$$D = \mathrm{diag}(Q), D_R = \mathrm{diag}(R), R = -G^T D^{-1} G.$$

After the scaling, the coefficient matrix becomes to be

$$\mathcal{A} := \widehat{D} A = \begin{pmatrix} D^{-1} Q & D^{-1} G \\ D_R^{-1} G^T & O \end{pmatrix}. \tag{22}$$

At this moment,

$$\mathcal{D} = \mathrm{diag}(D^{-1}Q) = I \in \mathbb{R}^{n\times n}, \mathcal{R} = -(D_R^{-1}G^T)\mathcal{D}^{-1}(D^{-1}G) = D_R^{-1}R \in \mathbb{R}^{m\times m},$$

and

$$\mathcal{B} = \begin{pmatrix} I & -D^{-1}G \\ O & I \end{pmatrix}, \mathcal{M} = \begin{pmatrix} D^{-1}Q & O \\ D_R^{-1}G^T & \mathcal{R} \end{pmatrix}, \mathcal{M}^{-1} = \begin{pmatrix} Q^{-1}D & O \\ -\mathcal{R}^{-1}D_R^{-1}G^TQ^{-1}D & \mathcal{R}^{-1} \end{pmatrix}.$$

The SIMPLE preconditioned matrix now is

$$\begin{aligned}
\widetilde{\mathcal{A}} &= \mathcal{A}\mathcal{B}\mathcal{M}^{-1} \\
&= \begin{pmatrix} D^{-1}Q & D^{-1}G \\ D_R^{-1}G^T & O \end{pmatrix} \begin{pmatrix} I & -D^{-1}G \\ O & I \end{pmatrix} \begin{pmatrix} Q^{-1}D & O \\ -\mathcal{R}^{-1}D_R^{-1}G^TQ^{-1}D & \mathcal{R}^{-1} \end{pmatrix} \\
&= \begin{pmatrix} \widetilde{\mathcal{A}}_{11} & \widetilde{\mathcal{A}}_{12} \\ \widetilde{\mathcal{A}}_{21} & \widetilde{\mathcal{A}}_{22} \end{pmatrix},
\end{aligned}$$

in which, by doing some elementary matrix calculation, these sub-matrices are:

$$\begin{aligned}
\widetilde{\mathcal{A}}_{11} &= I + D^{-1}\left[QD^{-1}G\mathcal{R}^{-1}D_R^{-1}G^TQ^{-1} - G\mathcal{R}^{-1}D_R^{-1}G^TQ^{-1}\right]D \\
&= I - D^{-1}Q(Q^{-1} - D^{-1})GR^{-1}G^TQ^{-1}D, \\
\widetilde{\mathcal{A}}_{12} &= -D^{-1}QD^{-1}G\mathcal{R}^{-1} + D^{-1}G\mathcal{R}^{-1} = D^{-1}(I - QD^{-1})G\mathcal{R}^{-1}, \\
\widetilde{\mathcal{A}}_{21} &= D_R^{-1}G^TQ^{-1}D + D_R^{-1}G^TD^{-1}G\mathcal{R}^{-1}D_R^{-1}G^TQ^{-1}D = O, \\
\widetilde{\mathcal{A}}_{22} &= -D_R^{-1}G^TD^{-1}G\mathcal{R}^{-1} = I.
\end{aligned}$$

Finally, it follows that

$$\widetilde{\mathcal{A}} = \begin{pmatrix} I - D^{-1}Q(Q^{-1} - D^{-1})GR^{-1}G^TQ^{-1}D & D^{-1}(I - QD^{-1})GR^{-1}D_R \\ O & I \end{pmatrix}. \tag{23}$$

Comparing the matrix $\widetilde{\mathcal{A}}$ in (23) with the matrix $\widetilde{A}$ defined by (6), we find that the spectra of both matrices are exactly the same. So, theoretically speaking, there is no influence to the spectrum of the SIMPLE preconditioned matrix by the diagonal scaling (21).

6 EIGENVALUE BOUNDS FOR SYMMETRIC CASE

In this section, we assume that Q is symmetric positive definite, which corresponds to the cases when term $u \,\mathrm{grad}\, u$ is deleted from Navier-Stokes equations in incompressible flow. In this case, the coefficient matrix A is symmetric and indefinite.

Consider the generalized eigenvalue problem (14)

$$Sp = \lambda Rp. \tag{24}$$

It is obvious that the problem $-Sp = -\lambda Rp$ is completely equivalent to the problem $Sp = \lambda Rp$. Since both $-S$ and $-R$ are s.p.d. matrices, we call (24) as a s.p.d. generalized eigenvalue problem by neglecting the negative signs in both sides. For the s.p.d. generalized eigenvalue problem, the extreme eigenvalues ($\lambda_{\max}$ and $\lambda_{\min}$) are the extreme values of [1, pp.379]:

$$\frac{p^T Sp}{p^T Rp} = \frac{p^T G^T Q^{-1} Gp}{p^T G^T D^{-1} Gp}, \quad p \neq 0, p \in \mathbb{R}^m, \tag{25}$$

which is the ratio of the Rayleigh quotients of S and R. So,

$$\lambda_{\max} = \max_{p\neq 0} \frac{p^T G^T Q^{-1} Gp}{p^T G^T D^{-1} Gp} = \max_{p\neq 0} \frac{(Gp)^T Q^{-1} (Gp)}{(Gp)^T D^{-1} (Gp)}. \tag{26}$$

Since that the matrix G has column full rank, i.e. $\mathrm{rank}(G) = m$, $Gp = 0$ if and only if $p = 0$. Denoting $y = Gp$, it follows that

$$\lambda_{\max} \leqslant \max_{y\neq 0} \frac{y^T Q^{-1} y}{y^T D^{-1} y}. \tag{27}$$

Let μ_1, μ_n be the largest and the smallest eigenvalues of the matrix Q, and d_1, d_n be the largest and the smallest diagonal elements of Q respectively, then

$$\lambda_{\max} \leqslant \frac{d_1}{\mu_n}. \tag{28}$$

It is easy to show that

$$\lambda_{\min} \geqslant \frac{d_n}{\mu_1} \tag{29}$$

by a similar argument.

So, combining (28), (29) and proposition 2.3, we get the following bounds for the eigenvalues of the preconditioned matrix $\widetilde{A}$:

$$\min\left\{1, \frac{d_n}{\mu_1}\right\} \leqslant \lambda \leqslant \max\left\{1, \frac{d_1}{\mu_n}\right\}, \quad \forall \lambda \in \sigma(\widetilde{A}). \tag{30}$$

If the both sides of (30) are taken to be $\frac{d_n}{\mu_1}$ and $\frac{d_1}{\mu_n}$ respectively, then

$$\kappa(\widetilde{A}) = \frac{\lambda_{\max}}{\lambda_{\min}} \leqslant \frac{d_1}{d_n} \cdot \frac{\mu_1}{\mu_n} = \frac{d_1}{d_n}\kappa(Q), \tag{31}$$

where, $\kappa(\cdot)$ represents for the (spectral) condition number.

7 NUMERICAL TESTS

Several numerical test results are reported here to illustrate the discussions above.

Example 7.1. *In this example, the coefficient matrix is taken from a discretised Navier-Stokes equations on a* 16×16 *grid [12](length=* 2*,* $\nu = 1$*).The dimensions are* $n = 544, m = 256,$ *and* $n + m = 800$*.* $A \in \mathbb{R}^{800\times 800}$ *is a nonsymmetric matrix.*

The eigenvalues of the preconditioned matrix $\widetilde{A}$ were computed by both proposition 3.2 and proposition 3.3. The computing results were the same, which coincided with the theoretical analysis. Spectra of A and $\widetilde{A}$ are plotted in Fig. 7.1, and some extreme eigenvalues are listed in table 7.1.

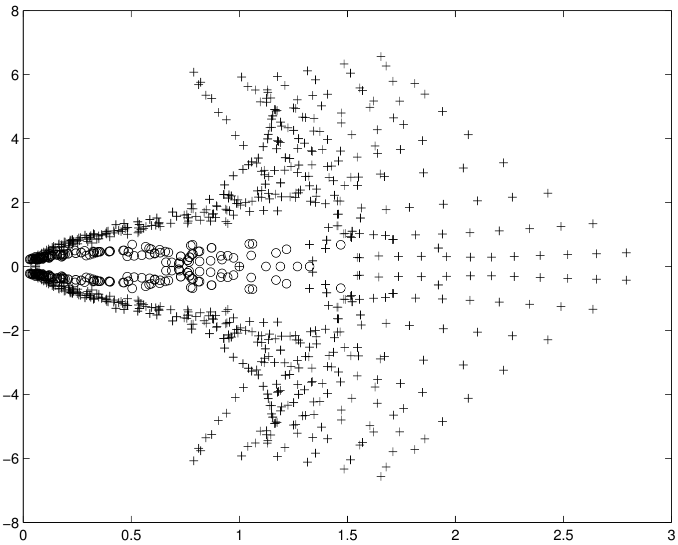

Fig.7.1: Spectrum of A and $\widetilde{A}$.
The '+' represents for the eigenvalues of A, while 'o' for that of the preconditioned $\widetilde{A}$.

matrix	$\max \Re(\lambda_i)$	$\min \Re(\lambda_i)$	$\max \Im(\lambda_i)$	$\max \lvert\lambda_i\rvert$	$\min \lvert\lambda_i\rvert$
A	2.79074	0.03559	6.56341	6.76892	0.06018
$\widetilde{A}$	1.46960	0.03000	0.70700	1.61894	0.21395

Table 7.1: The extreme eigenvalues of A and $\widetilde{A}$.

Example 7.2. *The matrix* A *is obtained from a discretised Stokes equation on a* 16×16 *grid by removing the Dirichlet boundary conditions. The resulted coefficient matrix* $A \in \mathbb{R}^{800\times 800}$ *is symmetric, and* $Q \in \mathbb{R}^{544\times 544}$ *is a s.p.d. matrix.*

The extreme eigenvalues of A and $\widetilde{A}$ are listed in Table 7.2.

matrix	$\lambda_{\min}$	$\min \lvert\lambda_i\rvert$	$\lambda_{\max}$	$\kappa(\cdot)$
A	-23.4555	0.0501	25.3762	1.7295×10^3
$\widetilde{A}$	0.5049	0.5049	46.7880	344.1452
Q	0.0154	0.0154	2.5477	232.9809
diag(Q)	0.9600	0.9600	1.6000	1.6667

Table 7.2: The extreme eigenvalues of A and $\widetilde{A}$ for example 7.2.

From example 7.1, we can see that the eigenvalues of the SIMPLE preconditioned matrix $\widetilde{A}$ are clustered in a smaller region in the right -half plane. The results of example 7.2 agree with the theoretical eigenvalue bounds in section 6, which are:

$$\frac{0.96}{2.547} = 0.377, \text{ and } \quad \frac{1.6}{0.0154} = 103.9.$$

We have also achieved some theoretical results on the spectral analysis of the SIMPLER preconditioning. We refer to [6] for reference. Next test is about the SIMPLER preconditioning.

Example 7.3. *In this example, the coefficient matrix is taken from a discretised Navier-Stokes equations on a* 24×24 *grid [12](lengthy* $= 2$, $\nu = 1$*).The dimensions are* $n = 1200, m = 576,$ *and* $n + m = 1776$. $A \in \mathbb{R}^{1776 \times 1776}$ *is a nonsymmetric matrix.*

The eigenvalues of the SIMPLER preconditioned matrix $\widetilde{A}$ are plotted in Fig. 7.2 and Fig. 7.3, and some comparisons for the numerical performance of GCR (without any preconditioning), GCR-SIMPLE, and GCR-SIMPLER are listed in Table 7.3.

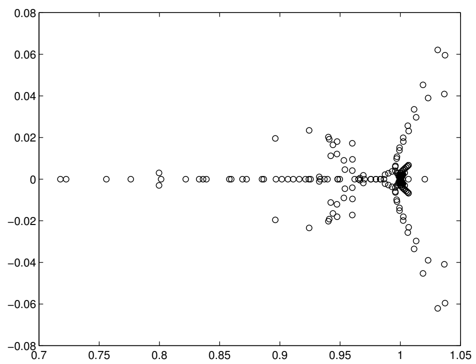

Fig.7.2: Spectrum of the SIMPLER preconditioned matrix $\widetilde{A}$. 'o' for the eigenvalues of $\widetilde{A}$.

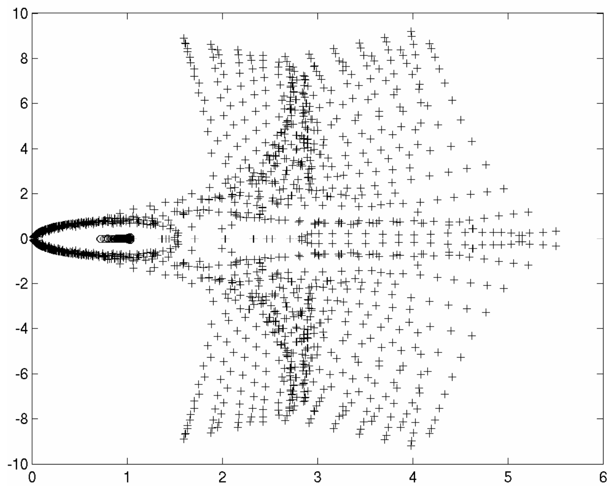

Fig. 7.3: Spectrum of A and $\widetilde{A}$.
The '+' represents for the eigenvalues of A, while 'o' for that of $\widetilde{A}$.

	GCR	GCR-SIMPLE	GCR-SIMPLER
iteration	907	64	10
CPU (s)	189.45	23.08	18.83

Table 7.3: Comparison of Example 7.3.

8 CONCLUSIONS

- GCR-SIMPLE(R) is a good combination of Krylov subspace method with SIMPLE type preconditioning. It is an efficient and robust method to simulate incompressible flows.
- The GCR acceleration can easily be added in an existing CFD code.
- SIMPLE(R) preconditioners are effective in terms of its spectrum clustering.
- The eigenvalue analysis gives a good explanation of the convergence acceleration for the Stokes problem.
- The eigenvalue analysis here could be helpful to analyze the other preconditioners for solving linear system (1).

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