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Numerical stability for modelling of dynamic two-phase interaction

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SUMMARY

Dynamic two-phase interaction of soil can be modelled by a displacement-based, two-phase formulation. The finite element method together with a semi-implicit Euler–Cromer time-stepping scheme renders a discrete equation that can be solved by recursion. By experience, it is found that the CFL stability condition for undrained wave propagation is not sufficient for the considered two-phase formulation to be numerically stable at low values of permeability. Because the stability analysis of the two-phase formulation is onerous, an analysis is performed on a simplified two-phase formulation that is derived by assuming an incompressible pore fluid. The deformation of saturated porous media is now captured in a single, second-order partial differential equation, where the energy dissipation associated with the flow of the fluid relative to the soil skeleton is represented by a damping term. The paper focuses on the different options to discretize the damping term and its effect on the stability criterion. Based on the eigenvalue analyses of a single element, it is observed that in addition to the CFL stability condition, the influence of the permeability must be included. This paper introduces a permeability-dependent stability criterion. The findings are illustrated and validated with an example for the dynamic response of a sand deposit. Copyright © 2015 John Wiley & Sons, Ltd.

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KEY WORDS: dynamic two-phase interaction; finite element method; Euler–Cromer scheme; numerical stability; stability criterion

1. INTRODUCTION

The numerical simulation of dynamic problems involving saturated porous media often requires taking into account the interaction between the solid and fluid. One might think of the settlement of saturated soil subject to loading via a foundation. Terzaghi [1] was the first to describe one-dimensional consolidation under a constant load where the soil was assumed to consist of a solid skeleton filled with pore fluid. Biot [2] extended this theory to three-dimensional consolidation under a time-dependent load. Research has provided enhancements to these models over the years. Currently, many different equations are known to describe consolidation, each with their own assumptions regarding (an)isotropy of the material, (in)compressibility of the pore fluid and (non-)linearity of the stress–strain relation of the solid phase; compare with [3].

This paper considers the displacement-based, two-phase formulation of Zienkiewicz *et al.* [3]. Adopting a finite element discretization and a semi-implicit Euler–Cromer scheme, the solution algorithm is conditionally stable. For this class of problem, the Courant–Friedrichs–Lewy (CFL) stability condition [4] has been used in the past, with the critical time step depending on the shortest

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time that it takes for a compression wave to travel through an element. The CFL condition is a necessary stability condition, but for two-phase interaction problems, it is not sufficient.

Various methods exist to estimate the critical time step for single-step time-stepping schemes, including the perturbation method, von Neumann's method and the matrix method; see, for example, Wood [5] and Hoffman [6]. Because these methods are quite onerous to apply to systems of differential equations, the displacement-based, two-phase formulation is simplified by assuming an incompressible pore fluid for the specific boundary-valued problem of zero flux and rigid base along the bottom boundary. This allows the deformation of saturated porous media to be captured by a second-order partial differential equation, where the energy dissipation associated with the flow of the fluid relative to the soil skeleton is represented by a damping term.

The analysis present in this paper was intended to identify a stability criterion that is suitable for use in the material point method that relies on the Cromer–Euler time-stepping scheme; see, for example, Jassim *et al.* [7]. Originally, the CFL condition had been adopted to estimate an approximate critical time step, but it became clear that it was not sufficient. The analysis presented here is considerably simpler than that presented in [8]. The simplifications help flush out dependencies. Other relatively recent contributions to the subject are presented in [9] and [10]. In these, the emphasis is on the time-stepping scheme.

The equations of the displacement-based, two-phase formulation are given in Section 2 for the specific case of a one-dimensional material response together with the assumption of linear-elastic, stress–strain behaviour. Section 3 analyses the critical time steps for consolidation and undrained wave propagation separately. In Section 4, the field equations are simplified with help of the assumption of incompressible pore fluid. The finite element discretization and Euler–Cromer scheme are also introduced. Section 5 performs the stability analysis for both a lumped and a reduced integration damping matrix. The findings are illustrated and validated by an example of a sand deposit in Section 6, followed by conclusions in Section 7.

2. FIELD EQUATIONS

A saturated porous medium is considered, whose one-dimensional deformation history is described in terms of displacements u and w of the solid and fluid, respectively. The total vertical stress σ is decomposed into the effective stress component σ' that is associated with the soil skeleton and pore pressure p according to the well-known Terzaghi equation $\sigma = \sigma' + p$. We deviate here from the usual soil mechanics convention by assuming that suction pressure is positive. In terms of mixture theory, the bulk density ρ depends on porosity n and the densities of the fluid ρ_w and solid grains ρ_s according to $\rho = (1-n)\rho_s + n\rho_w$.

The governing equations of the dynamic problem are derived from the conservation of (linear) momentum, conservation of mass and the constitutive law. Here, linear elasticity is assumed. For one dimension, the equations can be written as

Momentum equation of mixture: $$(\sigma' + p)_{,z} - (1-n)\rho_s \ddot{u} - n\rho_w \ddot{w} - \gamma = 0 \tag{1}$$

Momentum equation of fluid: $$p_{,z} - \rho_w \ddot{w} - \frac{n\gamma_w}{k}(\dot{w} - \dot{u}) - \gamma_w = 0 \tag{2}$$

Mass balance of mixture: $$\frac{n}{K_w}\dot{p} = (1-n)\dot{u}_{,z} + n\dot{w}_{,z} \tag{3}$$

Constitutive law for solid: $$\dot{\sigma}' = E_c \dot{u}_{,z} \tag{4}$$

The equations are considered in a Lagrangian description [11], such that the comma in the subscript denotes differentiation with respect to the material coordinate z and the superposed dot denotes the time derivative of the considered quantity. As is often the case, the convective acceleration of the fluid

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relative to the soil skeleton is ignored. The parameters γ_w and γ represent the unit weight of the fluid and the mixture, respectively. Furthermore, we have the Darcy permeability k [m/s], the bulk modulus of the fluid K_w and the confined modulus of the soil skeleton E_c. The solid particles themselves are incompressible.

3. CHARACTERISTIC TIMES FOR TWO-PHASE ANALYSIS

When considering the dynamics of the mixture and consolidation separately, there are two timescales of interest, namely that of excess pore pressure dissipation related to consolidation and that of compression wave propagation within the solid–fluid mixture. In mathematics, consolidation is seen as parabolic behaviour, while wave propagation is hyperbolic behaviour. As far as the dynamic behaviour is concerned, it is possible to further refine the analysis by considering the wave propagation in the fluid and solid separately rather than for the mixture. This will not be considered here as we are more interested in the transition from hyperbolic to parabolic dominated behaviour, which represents partially drained conditions.

Before deriving a time step criterion for the complex interaction of the two phases, it is prudent to consider the critical time steps for each phenomenon separately. The differential equations describing one-dimensional consolidation and confined, undrained wave propagation are, respectively,

$$c_v p_{,zz} = \dot{p} \quad \text{and} \quad v_c^2 u_{,zz} = \ddot{u} \tag{5}$$

in which $c_v = kE_c/\gamma_w$ is the coefficient of consolidation and $v_c = \sqrt{E_u/\rho}$ is the speed of a compression wave in an undrained medium, with E_u being the undrained, confined modulus. An estimate of the undrained modulus can be obtained by $E_u \approx E_c + K_w/n$; see, for example, Verruijt [12].

The associated critical time steps according to the CFL conditions are

$$\Delta t_{crit}^c = \frac{h^2}{2c_v} \quad \text{and} \quad \Delta t_{crit}^u = \frac{h}{v_c} \tag{6}$$

in which h represents the minimum element size. For the extreme cases in which consolidation influences or inertial effects are negligible, the criterion is fairly straightforward. The first criterion of Eq. (6) shows that for low values of k, a larger critical time step can be found. However, a sensitivity analysis with respect to permeability indicates that for low values of k, the stability of the coupled equations depends on permeability in a way that is opposite to what is expected. The question arises, how does permeability influence the critical time step for a dynamic behaviour between the extreme cases?

4. STATEMENT OF THE PROBLEM

Regardless of the stability analysis approach, the process of obtaining a time step criterion can be quite onerous, particularly when dealing with multiple balance equations such as Eqs (1) to (4). To simplify the problem, the full set of equations is reduced to a single differential equation that captures the salient features of the two-phase interaction. This equation must capture the reciprocal solid–fluid interaction, in addition to the simultaneous consolidation and dynamic processes. In the following, the problem of a one-dimensional column with an impermeable, rigid base and a load applied on top is considered. The height of the column is denoted by H.

When assuming incompressible pore fluid, the mass balance for a porous medium given by Eq. (3) is replaced by

$$(1-n)\dot{u}_{,z} + n\dot{w}_{,z} = 0 \tag{7}$$

This is a strong constraint when considering that it is often desirable to include some fluid compressibility to tie the pressure field to the volumetric strain that helps mitigate non-physical pressure variations. When it is assumed that the influence of variations in density and porosity in space and time are negligible, Eq. (7) can be replaced by the constraint

$$(1-n)\dot{u} + n\dot{w} = C(t) \tag{8}$$

As a consequence of the impermeable, rigid base, the right-hand side of Eq. (8) is equal to 0. Elimination of the variables w, p and σ from Eqs (1), (2), (4) and (8) renders

$$\widetilde{\rho}\ddot{u} + \frac{\gamma_w}{k}\dot{u} - \left(E_c u_{,z}\right)_{,z} = 0 \tag{9}$$

with $\widetilde{\rho} = \rho + \left(\frac{1}{n} - 2\right)\rho_w$. It should be noted that gravity is left out. Equation (9) has the form of the standard equation for damped wave propagation, in which consolidation, a hydraulic lag phenomenon, acts as a damping term. The equation expresses the effective dynamic equilibrium of the soil skeleton, properly taking into account the interaction of the soil skeleton and the pore water.

The weak form of Eq. (9) is given by

$$\int_H \left(\Delta u \tilde{\rho}\ddot{u} + \Delta u \frac{\gamma_w}{k}\dot{u} + \Delta u_{,z} E_c u_{,z}\right) dz = \Delta u \sigma'|_{z=H} \tag{10}$$

where δu is the weighting function that is consistent with the Galerkin method [13]. After introducing the finite element approximation with linear interpolation, Eq. (10) becomes in discretized form

$$\mathbf{M}\ddot{\mathbf{a}} + \mathbf{C}\dot{\mathbf{a}} + \mathbf{K}\mathbf{a} = \mathbf{F} \tag{11}$$

such that **M**, **C** and **K** are the mass, damping and stiffness matrices, respectively, and **F** is the load vector. Vector **a** contains the solid displacement degrees of freedom. For one element, Eq. (11) becomes

$$\frac{\widetilde{\rho}h}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{a}_1 \\ \ddot{a}_2 \end{Bmatrix} + \mathbf{C}\begin{Bmatrix} \dot{a}_1 \\ \dot{a}_2 \end{Bmatrix} + \frac{E_c}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} \tag{12}$$

where a lumped mass matrix [13] is introduced to avoid the time-consuming calculation of the inverse mass matrix. Equation (12) is assembled to a larger system of equations when considering more than one element.

In this paper, we consider two forms of the damping matrix. Besides a lumped damping matrix, a reduced integration damping matrix proposed by Mieremet [14] is also considered:

- lumped form $$\mathbf{C} = \frac{\gamma_w h}{2k}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{13a}$$

- reduced integration form $$\mathbf{C} = \frac{\gamma_w h}{4k}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{13b}$$

The lumped damping matrix is obtained using Newton–Côtes integration, while Eq. (13b) is obtained with one Gauss integration point [15].

A modified Euler–Cromer scheme [16] is applied to Eq. (12). It first determines the velocity using the forward difference approximation and then updates the displacement with the backward difference approximation. Because the regular Euler–Cromer scheme does not include the damping term, the implicitness of this term is varied with help of the standard θ method to investigate its influence. The implicitness parameter θ varies between 0 and 1; the extreme values represent explicit and implicit time integration, respectively.

5. STABILITY ANALYSIS

The matrix method is considered for the stability analyses in this paper. For a detailed description, the reader is referred to Hirsch [17]. According to the matrix method, the stability of a time-stepping scheme depends on the largest eigenvalue associated with the recursion equation, which in turn depends on the time algorithm that is adopted. Irons [18] shows that the largest eigenvalue of a finite element is larger than that of the system. Therefore, only one element is considered, that is, Eq. (12). Because the load can been neglected as only the bounding on the free vibration prediction is of significance to the stability analysis, we work with the following equation:

$$\frac{\widetilde{\rho}h}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{Bmatrix}\ddot{a}_1\\ \ddot{a}_2\end{Bmatrix}+\mathbf{C}\begin{Bmatrix}\dot{a}_1\\ \dot{a}_2\end{Bmatrix}+\frac{E_c}{h}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}a_1\\ a_2\end{Bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix} \tag{14}$$

As mentioned before, we consider two cases, one with a lumped damping matrix and the other with a reduced integration damping matrix. It should be noted that with the latter, only explicit time integration ($\theta=0$) is considered to avoid inverse matrix calculations.

5.1. Consideration of a lumped damping matrix

Setting $\mathbf{a}=\tilde{\mathbf{a}}e^{-\lambda t}$ in Eq. (14), where $\tilde{\mathbf{a}}$ is an eigenvector and λ an eigenvalue, renders a quadratic eigenvalue problem. The problem is characterized by four eigenvalues corresponding to two eigenvectors, from which the stability criteria are derived as follows:

(a) Eigenvector $\widetilde{\mathbf{a}}_1=\langle -1 \quad 1\rangle^T$ corresponds to the deformation mode of the soil skeleton that yields the equation

$$\ddot{c}_1+2v\omega\dot{c}_1+\omega^2c_1=0 \tag{15}$$

in which c_1 represents the participation factor for $\tilde{\mathbf{a}}_1$, $2v\omega=\gamma_w/\widetilde{\rho}k$, $\omega^2=4E_c/\widetilde{\rho}h^2$ with h being the length of the element and v representing a 'damping ratio'.

The modified Euler–Cromer scheme is applied to solve Eq. (15). First, the velocity $\dot{c}_1$ is determined using the forward difference approximation taking into account the standard θ method for the damping term. Thereafter, the displacement c_1 is updated with the backward difference approximation. This sequence of steps results in a recursion equation:

$$[1+2\theta v\omega\Delta t]c_1^{n+1}-\left[2-2(1-2\theta)v\omega\Delta t-(\omega\Delta t)^2\right]c_1^n+[1-2(1-\theta)v\omega\Delta t]c_1^{n-1}=0 \tag{16}$$

The pattern $c_1^{n+1}=r_1c_1^n$ yields the following characteristic equation:

$$[1+2\theta v\omega\Delta t]r_1^2-\left[2-2(1-2\theta)v\omega\Delta t-(\omega\Delta t)^2\right]r_1+[1-2(1-\theta)v\omega\Delta t]=0 \tag{17}$$

Numerical stability requires the roots $|r_1|\leq 1$ for the solution to be bounded, that is,

$$\left|\frac{1-(1-2\theta)v\omega\Delta t-(\omega\Delta t)^2\pm\sqrt{\left(1-(1-2\theta)v\omega\Delta t-(\omega\Delta t)^2\right)^2-(1+2\theta v\omega\Delta t)(1-2(1-\theta)v\omega\Delta t)}}{(1+2\theta v\omega\Delta t)}\right|\leq 1 \tag{18}$$

(b) The second eigenvector $\widetilde{\mathbf{a}}_2=\langle 1 \quad 1\rangle^T$ corresponds to the flow of fluid, delivering

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$$\ddot{c}_2 + 2v\omega\dot{c}_2 = 0 \tag{19}$$

where c_2 represents the participation factor for the second mode. For the case of no consolidation, the second eigenvector represents a rigid-body mode, which is neglected for stability analysis. With consolidation, this mode corresponds to the movement of water relative to the soil skeleton.

The recursion equation for the Euler–Cromer scheme for Eq. (19) may be written as

$$[1 + 2\theta v\omega\Delta t]c_2^{n+1} - [2 - 2(1 - 2\theta)v\omega\Delta t]c_2^n + [1 - 2(1 - \theta)v\omega\Delta t]c_2^{n-1} = 0 \tag{20}$$

Following the pattern $c_2^{n+1} = r_2 c_2^n$, we have

$$[1 + 2\theta v\omega\Delta t]{r_2}^2 - [2 - 2(1 - 2\theta)v\omega\Delta t]r_2 + [1 - 2(1 - \theta)v\omega\Delta t] = 0 \tag{21}$$

and numerical stability is obtained when $|r_2| \leq 1$; that is,

$$\left| \frac{1 - (1 - 2\theta)v\omega\Delta t \pm \sqrt{(1 - (1 - 2\theta)v\omega\Delta t)^2 - (1 + 2\theta v\omega\Delta t)(1 - 2(1 - \theta)v\omega\Delta t)}}{(1 + 2\theta v\omega\Delta t)} \right| \leq 1 \tag{22}$$

The implicitness factor θ is not yet assigned in Eqs (18) and (22). We give the simplified versions of the stability criteria for the most commonly used values in Table I.

From Table I, we may conclude that for $\theta = 0$, the critical time step is permeability-dependent according to

$$\Delta t_{crit} = \frac{-2v\omega + \sqrt{(2v\omega)^2 + 4\omega^2}}{\omega^2} \tag{23}$$

This relation between the critical time step and permeability, through $2v\omega = \gamma_w / \widetilde{\rho}k$ and $\omega^2 = 4E_c/\widetilde{\rho}h^2$, is consistent with the sensitivity analysis on the full two-phase formulation and will therefore be considered in the example of the sand deposit that follows in Section 6.

For the case of $\theta = 1/2$, no dependency on permeability is found. Strictly speaking, this stability criterion is valid for the simplified two-phase formulation but does not properly capture the numerical stability for the more complex full two-phase formulation that accommodates some fluid compressibility.

With an implicit damping term, that is, $\theta = 1$, the stability criterion again depends on the permeability. Because a low permeability provides a larger critical time step, which is opposite to the findings from the sensitivity analysis of the full two-phase formulation, this stability criterion will not be taken into account in the example of the sand deposit.

5.2. *Consideration of a reduced integration damping matrix*

Given the stability analysis presented previously, we only state the obtained stability criteria for the explicit reduced integration damping matrix, that is, $\theta = 0$.

Table I. Stability criteria for different values of the implicitness factor.

Implicitness factor θ	Stability criterion for eigenvector $\widetilde{\mathbf{a}}_1 = \langle -1 \quad 1 \rangle^T$	Stability criterion for eigenvector $\widetilde{\mathbf{a}}_2 = \langle 1 \quad 1 \rangle^T$
0	$0 \leq (\omega\Delta t)^2 \leq 4(1 - v\omega\Delta t)$	$0 \leq 2v\omega\Delta t \leq 2$
$\frac{1}{2}$	$0 \leq (\omega\Delta t)^2 \leq 4$	$0 \leq 2v\omega\Delta t$
1	$0 \leq (\omega\Delta t)^2 \leq 4(1 + v\omega\Delta t)$	$0 \leq 2v\omega\Delta t$

(a) The eigenvector $\widetilde{\mathbf{a}}_1 = \langle -1 \quad 1 \rangle^T$ provides the stability criterion

$$0 \leq (\omega \Delta t)^2 \leq 4 \tag{24}$$

(b) The eigenvector $\widetilde{\mathbf{a}}_2 = \langle 1 \quad 1 \rangle^T$ renders

$$0 \leq 2 v \omega \Delta t \leq 2 \tag{25}$$

It should be noted that Eq. (24) is mesh-dependent through $\omega^2 = 4E/\widetilde{\rho}h^2$, while Eq. (25) is permeability-dependent through $2v\omega = \gamma_w/\widetilde{\rho}k$. The critical time step is determined as the minimum of the limitations obtained from eigenvectors $\tilde{\mathbf{a}}_1$ and $\tilde{\mathbf{a}}_2$:

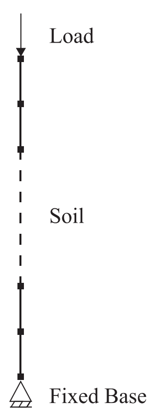
$$\Delta t_{crit} = \min\left(\frac{2}{\omega}, \frac{2}{2v\omega}\right) \tag{26}$$

6. EXAMPLE: DYNAMIC RESPONSE OF A SAND DEPOSIT

The previous analyses focused on the dynamic characteristics of a single element in isolation. In this section, we consider a 10 m thick saturated sand deposit supported by rigid, impermeable bedrock onto which a new layer of sand is placed. The sand deposit, which is shallow compared with its horizontal dimensions, can be modelled as a one-dimensional problem subject to an instantaneous surface load of 100 kPa. The corresponding mesh is shown in Figure 1 together with the material properties that are kept fixed.

Simulations were performed with the finite element equivalent of Eq. (9) with the reduced integration damping matrix to which the Euler–Cromer scheme was applied. Both 20 and 200 elements were considered to show the effect of the element size on the critical time step. The permeability was varied to see its effect on the critical time step, adopting values of $3.0 \cdot 10^3$ and $0.3 \cdot 10^3$ m/s. It should be noted that a low value for the confined modulus was selected to demonstrate the appropriateness of the stability criteria.

Table II summarizes the critical time steps calculated according to Eq. (26). We see that for a coarse mesh with 20 elements, the permeability-dependent criterion determines the critical time step. A refinement to 200 elements renders a switch to the mesh-dependent criterion at a higher permeability, while the permeability-dependent criterion stays the dominating criterion for a lower permeability.



Parameter	Value
Density of sand (ρ_s, kg/m^3)	2600
Density of water (ρ_w, kg/m^3)	1000
Porosity (n)	0.45
Confined modulus (E, kPa)	10000
Gravitational Acceleration (g, m/s^2)	10

Figure 1. One-dimensional finite element mesh for the saturated sand deposit with fixed material properties.

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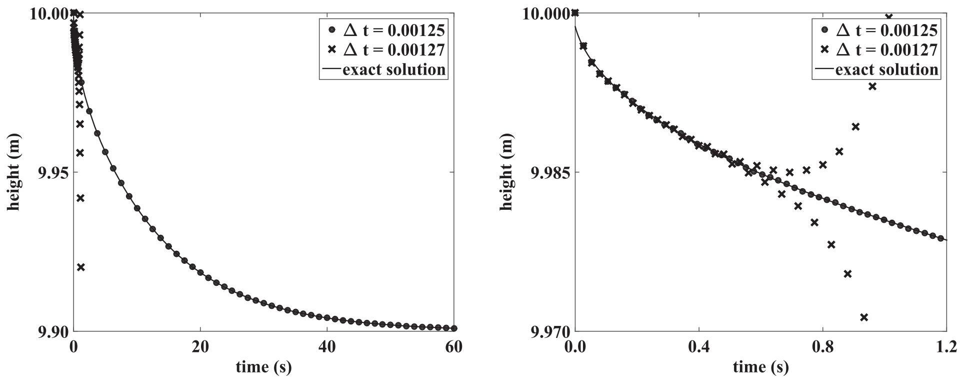
Table II. Critical time steps for the saturated sand deposit considering an explicit reduced integration damping matrix, with Δt_1 referring to the mesh-dependent criterion and Δt_2 to the permeability-dependent criterion.

Element size h (m)	Permeability k (m/s)	Time step $\Delta t_1 = h\sqrt{\widetilde{\rho}/E}$ (s)	Time step $\Delta t_2 = 2\widetilde{\rho}k/\gamma_w$ (s)	Critical time step $\Delta t_{crit} = \min(\Delta t_1, \Delta t_2)$ (s)
0.50	$3.0 \cdot 10^3$	$7.250 \cdot 10^3$	$1.261 \cdot 10^3$	$1.261 \cdot 10^3$
0.50	$0.3 \cdot 10^3$	$7.250 \cdot 10^3$	$0.126 \cdot 10^3$	$0.126 \cdot 10^3$
0.05	$3.0 \cdot 10^3$	$0.725 \cdot 10^3$	$1.261 \cdot 10^3$	$0.725 \cdot 10^3$
0.05	$0.3 \cdot 10^3$	$0.725 \cdot 10^3$	$0.126 \cdot 10^3$	$0.126 \cdot 10^3$

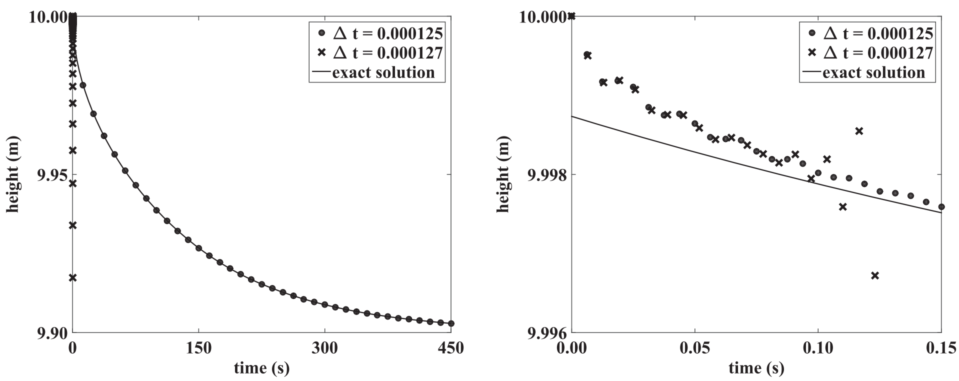
Figures 2 and 3 compare the stable and unstable numerical solutions for the finite element model with 20 elements and different values of permeability to the analytical solution of the considered benchmark. It was obtained from Eq. (9) as presented in [14]. The analytical solution shows an error around $t = 0$ as a result of cutting off an infinite sum.

It should be noted that the first figure for each case (a) corresponds to the full simulation period with the second (b) showing the details at the beginning. Because the instabilities that develop are of an oscillatory nature, only the amplitudes are shown to avoid overcrowding. Because the time step sizes of the stable and unstable numerical solutions are within a 1% range of the critical time step, the permeability-dependent criterion is hereby validated.



(a) (b)

Figure 2. Validation of the critical time step $\Delta t_{crit} = 1.261 \cdot 10^3$ s for the saturated sand deposit considering an element size $h = 0.50$ m and permeability $k = 3.0 \cdot 10^3$ m/s: (a) over 60 s and (b) details over 1.2 s.



(a) (b)

Figure 3. Validation of the critical time step $\Delta t_{crit} = 0.126 \cdot 10^3$ s for the saturated sand deposit considering an element size $h = 0.50$ m and permeability $k = 0.3 \cdot 10^3$ m/s: (a) over 450 s and (b) details over 0.15 s.

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Figures 4 and 5 give similar numerical results for the case with 200 elements. Because the critical time step in the problem with a permeability of $3.0 \cdot 10^3$ m/s is determined by the mesh-dependent criterion, Figure 4 validates this criterion. Figure 5, which shows the results for both lower permeability and smaller element size, is added for completeness.

The same validation was performed with an explicit lumped damping matrix, with the critical time steps belonging to different element sizes and permeability being presented in Table III. Once again, the critical time steps are validated. The figures for the lumped damping case are however left out because the numerical results are comparable to those shown in Figures 2–5.

We next demonstrate how the permeability influences the numerical stability of the Euler–Cromer scheme for the full two-phase formulation. Figure 6 shows the variation in the critical time step

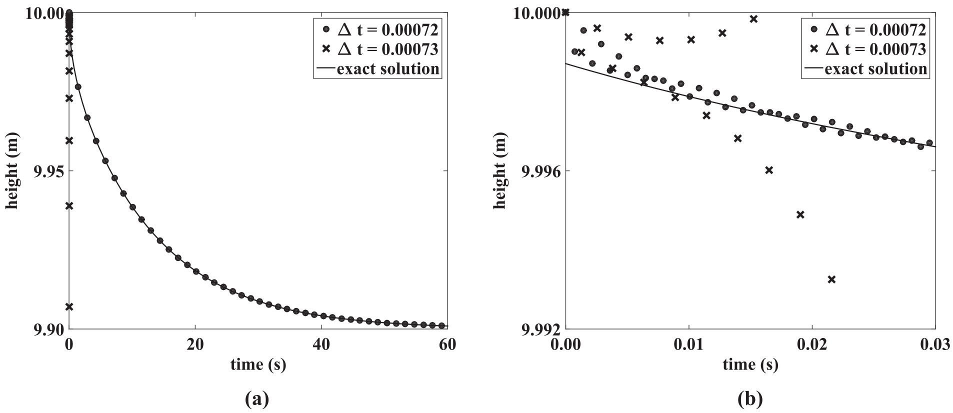

Figure 4. Validation of the critical time step $\Delta t_{crit} = 0.725 \cdot 10^3$ s for the saturated sand deposit considering an element size $h = 0.05$ m and permeability $k = 3.0 \cdot 10^3$ m/s: (a) over 60 s and (b) details over 0.15 s.

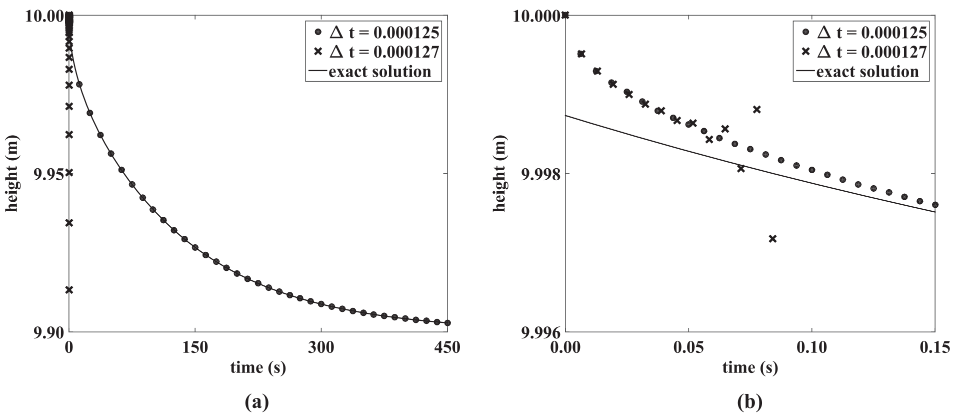

Figure 5. Validation of the critical time step $\Delta t_{crit} = 0.126 \cdot 10^3$ s for the saturated sand deposit considering an element size $h = 0.05$ m and permeability $k = 0.3 \cdot 10^3$ m/s; (a) over 450 s and (b) details over 0.15 s.

Table III. Critical time steps for the saturated sand deposit considering an explicit lumped damping matrix.

Element size h (m)	Permeability k (m/s)	Critical time step $\Delta t_{crit} = \frac{-\gamma_w/\tilde{\rho}k + \sqrt{(\gamma_w/\tilde{\rho}k)^2 + 16E/\tilde{\rho}h^2}}{4E/\tilde{\rho}h^2}$ (s)
0.50	$3.0 \cdot 10^3$	$1.226 \cdot 10^3$
0.50	$0.3 \cdot 10^3$	$0.126 \cdot 10^3$
0.05	$3.0 \cdot 10^3$	$0.546 \cdot 10^3$
0.05	$0.3 \cdot 10^3$	$0.123 \cdot 10^3$

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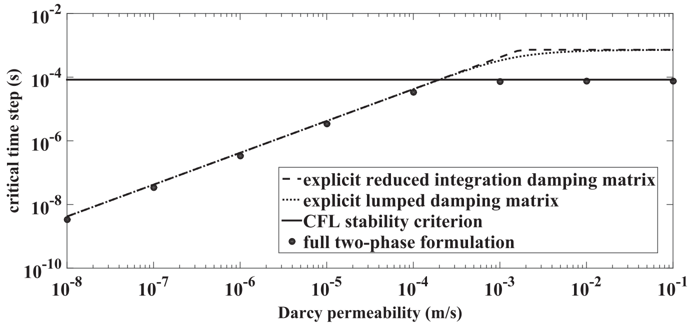

Figure 6. Comparison between different time step criteria for the simplified and full two-phase formulations.

given the full two-phase formulation assuming various values of permeability and adopting 200 elements. Also shown are the CFL stability condition and the criterion corresponding to the simplified two-phase formulation with the lumped and the reduced integration damping matrix. We clearly observe that the permeability-dependent criterion controls numerical stability for low values of permeability, while the CFL stability condition for undrained wave propagation controls numerical stability for high values of permeability. In mathematical notation, it is written as

$$\Delta t_{crit} = \min\left(\frac{h}{\sqrt{E_u/\rho}}, \frac{2\widetilde{\rho}k}{\gamma_w}\right) \tag{27}$$

It should be noted that with the simplified two-phase formulation, higher time steps are possible than with the full two-phase formulation, particularly at higher permeability.

7. CONCLUDING REMARKS

In this paper, the displacement-based, two-phase formulation of Zienkiewicz [3] was simplified with the assumption of an incompressible pore fluid and one-dimensional confined deformation. Stability analyses were performed on this simplified two-phase formulation with lumped and reduced integration damping matrices. The results for dynamic simulations of a saturated sand deposit showed that the numerical stability of the full two-phase formulation could be best captured by taking the minimum of the permeability-dependent criterion and the CFL stability condition. The analysis presented here is useful for two-dimensional and three-dimensional problems when properly taking into account the characteristic length of the elements.

A challenge, which depends on computational resources and what the analyst is interested in, is to use a model that captures the information of interest. We showed that for the displacement-based, two-phase formulation, the critical time step size decreases with permeability, possibly leading to an unacceptably small time step size when considering low permeability. In problems characterized by high-frequency behaviour, it may be best to neglect consolidation and consider undrained dynamic behaviour to reduce the computational cost. Similarly, from both a numerical and physical point of view, the inertial term can be dropped if low frequency response is of interest. In all other cases, the displacement-based, two-phase formulation can be used with the obtained stability criterion.

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