Uniformly effective numerical methods for hyperbolic systems

P. Wesseling*, D.R. van der Heul*† and C. Vuik*

Abstract

We discuss numerical schemes for hyperbolic systems occurring in fluid dynamics that are uniformly accurate and efficient with respect to changes in the equation of state or of the Mach number. A unified method for compressible and incompressible flows and arbitrary equation of the state is presented. The method is an extenssion of an incompressible scheme to the compressible case using a staggered grid. An application is given for cavitating flow around a hydrofoil, in which the Mach number ranges between 10^{-3} and 25, and the equation of state is nonconvex. For Riemann problems for a perfect gas similar accuracy is obtained as for well-established colocated schemes.

Key words: Computational fluid dynamics, nonconvex hyperbolic systems, finite volume methods, cavitation..

AMS subject classifications: 65M, 76M..

1 Introduction

Standard numerical methods for gasdynamics break down as the Mach number $\mathrm{M} \downarrow 0$. Furthermore, in many cases they need to be redesigned if the equation of state is different from the perfect gas law, expecially if it is nonconvex. We call a numerical method for hyperbolic systems uniformly effective, if it is uniformly applicable to compressible ($\mathrm{M} > 0$) and incompressible ($\mathrm{M} = 0$) flow, and if it can handle arbitrary equations of state. In this paper we present some developments concerning such methods.

2 A simple hyperbolic system

The simplest nonlinear hyperbolic system is

$$V_t - u_y = 0 \ , \qquad u_t + p(V)_y = 0 \ .$$

This is called the p-system. It describes one-dimensional flow of an inviscid barotropic medium in the Lagrange coordinate y (a derivation may be found in [17]), with V the specific volume, p the pressure and u the velocity. The linearized version, rewritten as

$$d_t + u_x = 0 \ , \qquad u_t + g d_x = 0 \ ,$$

gives the linearized shallow-water (SW) equations, with d the depth, g the acceleration of gravity, x a Cartesian coordinate, and unit average depth.

Starting with [10], the SW equations are usually discretized on a staggered grid, with depth nodes at $x_j = (j - 1/2)h$ and velocity nodes at $x_{j+1/2} = jh$, with h the mesh size. The popular schemes of [3], [7], [13] and [14] for the full SW equations are strongly related to the following efficient explicit scheme:

$$\begin{aligned} d_j^{n+1} - d_j^n + \lambda u^n|_{j-1/2}^{j+1/2} &= 0 \ , \qquad \lambda = \tau/h \ , \\ u_{j/+1/2}^{n+1} - u_{j+1/2}^n + \lambda g d^{n+1/2}|_j^{j+1} &= 0 \ , \qquad d^{n+1/2} = (d^n + d^{n+1})/2 \ , \end{aligned}$$

*J.M. Burgers Center and Delft University of Technology, Faculty of Information Technology and Systems, Mekelweg 4, 2628 CD Delft, The Netherlands, e-mail: P.Wesseling@its.tudelft.nl

†Supported by the Netherlands Organization for Scientific Research (NWO)

with τ a time step. Such a scheme is seldom used in gasdynamics, whewre colocated schemes predominate, giving us two flavors (staggered and colocated) of numerical schemes for the hyperbolic systems of computational fluid dynamics. Because the above scheme has low computing cost and favorable dispersion and dissipation properties, we want to try it out on the p-system in the Eulerian coordinate x, given by

$$\rho_t + m_x = 0 \ , \qquad m_t + (um + p(\rho))_x = 0 \ , \qquad \rho = 1/V \ , \qquad m = \rho u \ .$$

These are the barotropic Euler equations.

In order to obtain a unified method for compressible and incompressible flow, we deviate from the above scheme by taking momentum implicit in the first equation. The resulting scheme is given by

(1) $$\rho_j^{n+1} - \rho_j^n + \lambda m^{n+1}|_{j-1/2}^{j+1/2} = 0 \ ,$$

(2) $$m_{j+1/2}^{n+1} - m_{j+1/2}^n + \lambda(u^n m^n + p^{n+1/2})|_j^{j+1} = 0 \ , \quad p^{n+1/2} = (p^n + p^{n+1})/2 \ .$$

By taking $\rho \equiv 1$, the incompressible case is recovered. The incompressible one-dimensional case is trivial, but the multidimensional case is not. With $\rho \equiv 1$, equation (1) becomes a kinematic constraint (solenoidality condition on the velocity) which has to be satisfied at the new time level. The freedom to satisfy this constraint is provided by the unknown pressure values p_j^{n+1} in (2), which can be regarded as Lagrange multipliers. With $\rho \equiv 1$, the system (1), (2) reduces to the classical incompressible staggered scheme of [4], so that (1) and (2) make sense both in the compressible and incompressible case. In (2) we use the first order upwind scheme: $(um)_j^n = (um)_{j-1/2}^n$. In one dimension, solving the implicit system (1), (2) takes little computing time. In more dimensions this can be done efficiently with the pressure-correction method, introduced in [4]. First, a prediction is made for the momentum using the old pressure:

$$m_{j+1/2}^* - m_{j+1/2}^n + \lambda(u^n m^n + p^n)|_j^{j+1} = 0 \ .$$

Next, a correction $\delta m = m^{n+1} - m^*$ is postulated of the following form:

$$\delta m_{j+1/2} = -\lambda \delta p|_j^{j+1} \ , \quad \delta p = (p^{n+1} - p^n)/2 \ .$$

We substitute $m^{n+1} = m^* + \delta m$ in (1), and obtain

$$\rho_j^{n+1} - \rho_j^n - \lambda^2 \delta p|_j^{j+1} + \lambda^2 \delta p|_{j-1}^j = -\lambda m^*|_{j-1/2}^{j+1/2} \ .$$

This is linearized as follows:

$$\rho^{n+1} - \rho^n \cong 2(\frac{d\rho}{dp})^n \delta p \ .$$

It remains to see whether the resulting scheme converges in the compressible case to a genuine weak solution that satisfies the entropy condition. To this end we compare with an exact solution of a Riemann problem, given by the drawn line in Fig. 1. As can be seen from its graph, the equation of state is nonconvex. The symbols in this graph are to be disregarded here; they have to do with the construction of the exact solution and the application of Oleinik's entropy condition, discussed in [17], [16]. The quality of the numerical solution in Fig. 1 is found to be equivalent to that of Osher's scheme in [17], and suggests convergence to the correct weak solution. The staggered scheme above is significantly simpler than colocated schemes using approximate Riemann solvers (such as Osher's scheme) or flux splitting (such as the AUSM scheme), because only simple central and upwind differences are involved, and therefore the above scheme requires less computing time. It is also versatile, because it allows the equation of state to be arbitrary.

3 Application to hydrodynamic flow with cavitation

The homogeneous equilibrium model (HEM) is a mathematical model for hydrodynamic flow with cavitation, in which the physics is simplified by assuming a one-phase fluid and constant temperature. An artificial equation of state $p = p(\rho)$ is adopted, corresponding to water for p larger than a certain value p_2, and to vapor below a certain value p_1, with a smooth artificial transition for $p_1 < p < p_2$. This leads to the kind of equation of state shown in Fig. 1. The HEM approach to cavitation is followed in [2], [5], [9], [12], [16]. In the wet part of the flow, the Mach number is very low, e.g. $\mathrm{M} \cong 10^{-3}$. In the transition regime $p_1 < p < p_2$ the speed of sound $c = \sqrt{dp/d\rho}$ becomes

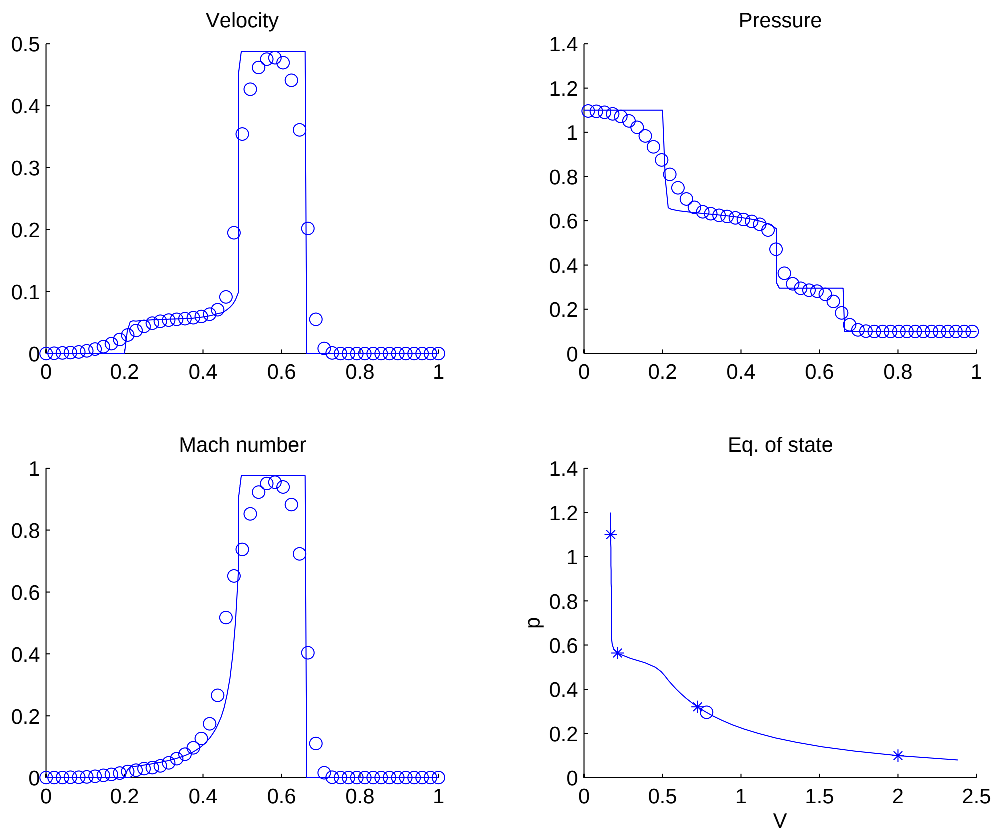

Figure 1: Solution of Riemann problem with staggered scheme; $h = 1/48, \lambda = 0.4, t = 0.2$.

very low, so that we have M as high as 25 in the application shown below. Therefore, to use the HEM approach as it stands, a numerical implementation is required that is truly uniformly effective, allowing both very low and more than hypersonic (ultrasonic?) Mach numbers, and an arbitrary equation of state. In some works, useful results have been obtained with numerical schemes not satisfying these demands, by modification of the HEM model; for example, by making the water artificially compressible. Such compromises are not needed with the present numerical method.

To apply HEM to cavitating flow around bodies, the scheme (1), (2) needs to be generalized to multi space dimensions and boundary-fitted coordinates. This is done in [16], [18], [19]. Fig. 2 shows a result concerning unsteady sheet cavitation on the EN ([6]) hydrofoil. Darker shading corresponds to lower density. Cavitation bubbles are captured as regions of low density. The flow pattern is quite similar to what is found in experiment.

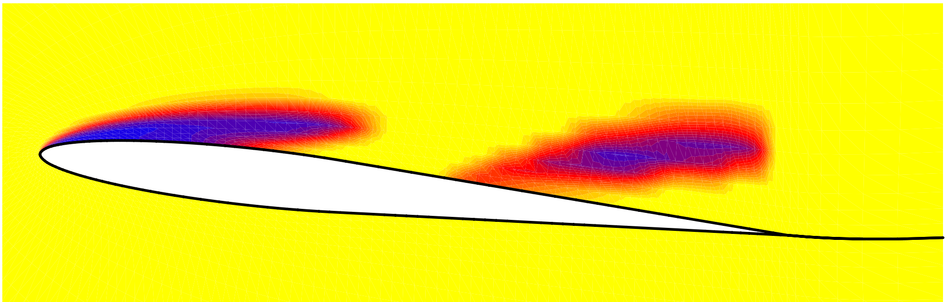

Figure 2: Density distribution in cavitating flow around EN hydrofoil.

4 Unified methods for compressible and incompressible flows

We will now say a bit more about unification of numerical methods for compressible and incompressible flows. The case of flow of a (non-barotropic) perfect gas is considered. The governing equations are the Euler equations of gasdynamics. It suffices to consider the one-dimensional case:

$$\rho_t + m_x = 0\ , \quad m_t + (um + p)_x = 0\ ,$$
$$(\rho E)_t + (mH)_x = 0\ , \quad E = e + \frac{1}{2}u^2\ , \quad H = \gamma e + \frac{1}{2}u^2\ , \quad \gamma = 7/5\ .$$

The equation of state of a perfect gas is $p = (\gamma - 1)\rho e$. Colocated compressible methods may be extended to low Mach numbers by preconditioning; see [1] and references quoted there. Here we generalize the (inviscid version of) the staggered incompressible scheme of [4] to the compressible case; see [1] for references to publications following this approach. Because $\rho =$ constant in the incompressible case, p must be introduced as primary variable. To avoid rounding errors for very low Mach numbers, it is advisable to work with the fluctuation of the pressure rather than the pressure itself. Therefore the dimensionless pressure $\tilde{p}$ is chosen as follows:

$$\tilde{p} = \frac{p - p_0}{\rho_r u_r^2}\ ,$$

where p_0 is the ambient pressure and ρ_r and u_r are units of density and velocity. The ambient pressure is given at a subsonic outflow or supersonic inflow boundary. By manipulation of thermodynamic relations and the governing equations the equation for ρE can be replaced by an equation for p. The resulting dimensionless equations are (deleting tildes):

$$\rho_t + m_x = 0\ , \quad m_t + (um + p)_x = 0\ ,$$
$$M_r^2[p_t + (up)_x + (\gamma - 1)pu_x] + u_x = 0\ , \quad \mathrm{M}_r^2 = \frac{\rho_r u_r^2}{\gamma p_0}\ .$$

We see that if the reference Mach number $\mathrm{M}_r \downarrow 0$ the solenoidality condition of incompressible flow is recovered. The pressure equation is not in conservation form, so it remains to be seen if the Rankine-Hugoniot conditions are satisfied.

To enhance accuracy, higher order flux-limited schemes (MUSCL) for spatial discretization and Runge-Kutta time stepping can be used. These are familiar techniques in computational gasdynamics using colocated schemes; see for example [8]. These techniques are also used in the following scheme:

$$\rho_j^{(m+1)} - \rho_j^n + \alpha_{m+1}\lambda(u^{(m)}\rho^{(m)})|_{j-1/2}^{j+1/2} = 0\ ,$$
$$m_{j+1/2}^{(m+1)} - m_{j+1/2}^n + \alpha_{m+1}\lambda(u^{(m)}m^{(m)} + p^n)|_j^{j+1} = 0\ ,$$
$$\rho_j^{n+1} = \rho_j^{(4)}\ , \quad m_{j+1/2}^{n+1} = m_{j+1/2}^{(4)} - \frac{1}{2}\lambda\delta p|_j^{j+1}\ , \quad \delta p = p^{n+1} - p^n\ ,$$
$$\mathrm{M}_r^2\{\delta p_j + \lambda(u^{n+1}p^n)|_{j-1/2}^{j+1/2} + \lambda(\gamma - 1)p_j^n u^{n+1}|_{j-1/2}^{j+1/2}\} + \lambda u^{n+1}|_{j-1/2}^{j+1/2} = 0\ .$$

Here the superscript $m \in \{1, 2, 3, 4\}$ is a Runge-Kutta stage counter. The Runge-Kutta method of [11] is used. For second order spatial accuracy, $\rho_{j+1/2}^{(m)}$, $(u^{(m)}m^{(m)})_j$ and $p_{j+1/2}^n$ are approximated with a flux-limited scheme, using the van Albada ([15]) limiter. Substitution of $u_{j+1/2}^{n+1} = (m/\rho)_{j+1/2}^{n+1}$ gives for δp a linear system, that is reminiscent of a discretized convection-diffusion equation. Pressure correction is not included in the Runge-Kutta stages, to save computing time. The numerical solution of a Riemann problem is compared with the exact solution in Fig. 3. The maximum Mach number is 2. In this and similar problems the accuracy is found to be comparable to that of well-established second order (MUSCL) colocated schemes, such as the Osher and AUSM schemes. Apparently, the staggered scheme satisfies the Rankine-Hugoniot and entropy conditions. Extension to two-dimensional examples is given in [1]. Stationary contact discontinuities are found to be preserved. Obviously, as $\mathrm{M}_r \downarrow 0$ the scheme reduces to the classical incompressible scheme of [4]. As a consequence, the performance for M_r arbitrary small is good.

We conclude that staggered schemes with pressure correction provide a viable approach to uniformly effective numerical methods for hyperbolic systems.

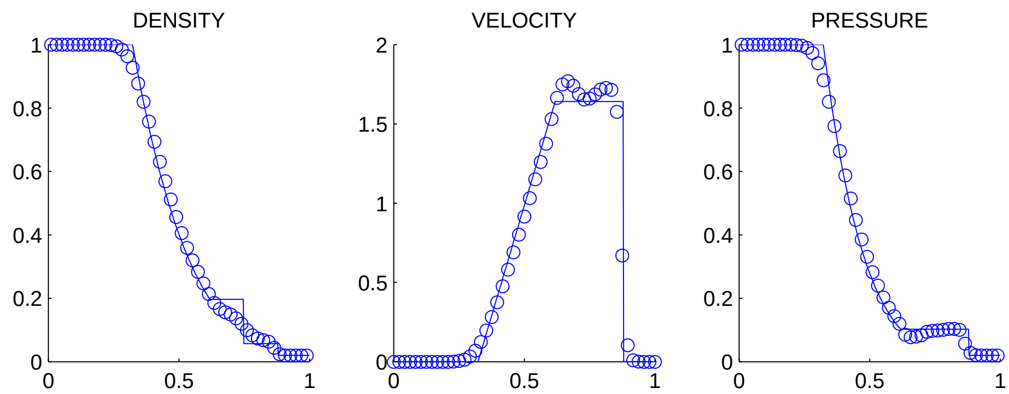

Figure 3: Solution of a Riemann problem; $\lambda = 0.3, h = 1/48, t = 0.15$.

References

[1] H. Bijl and P. Wesseling. A unified method for computing incompressible and compressible flows in boundary-fitted coordinates. *J. Comp. Phys.*, 141:153–173, 1998.

[2] Y. Dellanoy and J.L. Kueny. Two phase flow approach in unsteady cavitation modelling. In O. Furuya, editor, *Cavitation and Multiphase Flow Forum, Toronto, June 4-7, 1990*, FED-98, pages 153–158, New York, 1990. ASME.

[3] W. Hansen. Theorie zur Errechnung des Wasserstandes und der Strömungen in Randmeeren nebst Anwendungen. *Tellus*, 8:289–300, 1956.

[4] F.H. Harlow and J.E. Welch. Numerical calculation of time-dependent viscous incompressible flow of fluid with a free surface. *The Physics of Fluids*, 8:2182–2189, 1965.

[5] H.W.M. Hoeijmakers, M.E. Janssens, and W. Kwan. Numerical simulation of sheet cavitation. In J.M. Michel and H. Kato, editors, *Third International Symposium on Cavitation, April 7–10,1998, Grenoble*, volume 2, pages 257–262, 1998.

[6] A. Kubota, H.Kato, H. Yamaguchi, and M. Maeda. Unsteady structure measurement of cloud cavitation on a foil section using conditional sampling technique. *J. Fluids Engrn*, 111:204–210, 1989.

[7] J.J. Leendertse. *Aspects of a computational model for long period water-wave propagation*. PhD thesis, Delft University of Technology, 1967. Also appeared as Rand Memorandum RM-5294-PR, Rand Corporation, Santa Monica, California, 1967.

[8] R.J. LeVeque. *Numerical methods for conservation laws*. Birkhäuser, Basel, 1992.

[9] C.L. Merkle, J.Z. Feng, and P.E.O. Buelow. Computational modeling of the dynamics of sheet cavitation. In J.M. Michel and H. Kato, editors, *Third International Symposium on Cavitation, April 7–10,1998, Grenoble*, volume 2, pages 307–311, 1998.

[10] L.F. Richardson. *Weather prediction by numerical process*. Cambridge Univ. Press, London, 1922. Reprinted, Dover, New York, 1965.

[11] B.P. Sommeijer, P.J. van der Houwen, and J. Kok. Time integration of three-dimensional numerical transport models. *Appl. Numer. Math.*, 16:201–225, 1994.

[12] C.C.S. Song and J. He. Numerical simulation of cavitating flows by single-phase flow approach. In J.M. Michel and H. Kato, editors, *Third International Symposium on Cavitation, April 7–10,1998, Grenoble*, volume 2, pages 295–300, 1998.

[13] G.S. Stelling. *On the construction of computational methods for shallow water flow problems.* PhD thesis, Delft University of Technology, 1983. Also appeared as Rijkswaterstaat Communications 35, 1984. Rijkswaterstaat, The Hague.

[14] G.S. Stelling, A.K. Wiersma, and J.B.T.M. Willemse. Practical aspects of accurate tidal computations. *J. Hydr. Eng.*, 112:802–817, 1986.

[15] G.D. van Albada, B. van Leer, and W.W. Roberts. A comparative study of computational methods in cosmic gas dynamics. *Astron. Astrophys.*, 108:76–84, 1982.

[16] D.R. van der Heul, C. Vuik, and P. Wesseling. A staggered scheme for hyperbolic conservation laws applied to unsteady sheet cavitation. *Computing and Visualization in Science*, 2:63–68, 1999.

[17] P. Wesseling. Non-convex hyperbolic systems. In H. Deconinck, editor, *30th Computational Fluid Dynamics Course*, pages IV.1 – IV.20, Brussels, 1999. von Karman Institute. Lecture Series 1999-03.

[18] P. Wesseling, A. Segal, and C.G.M. Kassels. Computing flows on general three-dimensional nonsmooth staggered grids. *J. Comp. Phys.*, 149:333–362, 1999.

[19] P. Wesseling, A. Segal, C.G.M. Kassels, and H. Bijl. Computing flows on general two-dimensional nonsmooth staggered grids. *J. Eng. Math.*, 34:21–44, 1998.