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Optimal control for the Oseen equation with a distributed control function

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Contents of the talk

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Problem formulation

Consider the velocity tracking problem for the stationary Oseen equation:
Find the velocity $\mathbf{u} \in H_0^1(\Omega)^d$,
the pressure $p \in L_0^2(\Omega)$, where $L_0^2(\Omega) = \{q \in L^2(\Omega)\}, \int_\Omega q\,dx = 1\}$, and
the control function $\mathbf{f}$, that minimize the cost function

$$\mathcal{J}(\mathbf{u}, \mathbf{f}) = \frac{1}{2}\|\mathbf{u} - \mathbf{u}_d\|^2 + \frac{1}{2}\alpha\|\mathbf{f}\|^2,$$

subject to state equation for an incompressible fluid velocity $\mathbf{u}$, such that

$$\begin{cases} -\Delta\mathbf{u} + (\mathbf{b}\cdot\nabla)\mathbf{u} + \nabla p & = & \mathbf{f} & \text{in } \Omega \\ \nabla\cdot\mathbf{u} & = & 0 & \text{in } \Omega \end{cases}$$

and boundary conditions $\mathbf{u} = \mathbf{0}$ on $\partial\Omega_1$, $\mathbf{u}\cdot\mathbf{n}$ on $\partial\Omega_2 = \partial\Omega\backslash\partial\Omega_1$, where $\mathbf{n}$ denotes the outward normal vector to the boundary $\partial\Omega$.

Problem formulation, cont.

Here:
– $\mathbf{u}_d$ is the desired solution,
– $\alpha > 0$ is a regularization parameter, used to penalize too large values of the control function.
– $\mathbf{b}$ is a given, smooth vector. For simplicity we assume that $\mathbf{b} = \mathbf{0}$ on $\partial\Omega_1$ and $\mathbf{b} \cdot \mathbf{n} = 0$ on $\partial\Omega_2$.

In a Navier-Stokes problem, solved by a Picard iteration using the frozen coefficient framework, $\mathbf{b}$ equals the previous iterative approximation of $\mathbf{u}$, in which case normally $\nabla \cdot \mathbf{b} = 0$ in Ω. For simplicity, we assume that this holds here also.

The variational form of the state, i.e., constraint equation reads as follows:

$$\begin{cases} (\nabla \mathbf{u}, \nabla \widetilde{\mathbf{u}}) + (\mathbf{b} \cdot \nabla \mathbf{u}, \widetilde{\mathbf{u}}) - (\nabla \widetilde{\mathbf{u}}, p) & = & (\mathbf{f}, \widetilde{\mathbf{u}}) & \forall \widetilde{\mathbf{u}} \in H_0^1(\Omega)^d \\ (\nabla \cdot \mathbf{u}, \widetilde{p}) & = & 0 & \forall \widetilde{p} \in L_0^2(\Omega) \end{cases}$$

The Lagrangian functional, corresponding to the optimization problem is given by

$$\mathcal{L}(\mathbf{u}, p, \mathbf{v}, q, \mathbf{f}) = \mathcal{J}(\mathbf{u}, \mathbf{f}) + (\nabla \mathbf{u}, \nabla \mathbf{v}) + (\mathbf{b} \cdot \nabla \mathbf{u}, \mathbf{v}) - (\nabla \cdot \mathbf{v}, p) - (\nabla \cdot \mathbf{u}, q) - (\mathbf{f}, \mathbf{v}),$$

where $\mathbf{v}$ is the Lagrange multiplier function for the state equation and q for its divergence constraint. Applying the divergence theorem, the divergence condition $\nabla \cdot \mathbf{b} = 0$ and the boundary conditions, we can write

$$\int_\Omega \mathbf{b} \cdot \nabla \widetilde{\mathbf{u}} \cdot \mathbf{v} d\Omega = - \int_\Omega (\mathbf{b} \cdot \underline{\nabla} \mathbf{v}) \cdot \widetilde{\mathbf{u}} d\Omega \, .$$

The five first order necessary conditions for an optimal solution:

$$
\begin{aligned}
(\mathbf{u}, \widetilde{\mathbf{u}}) + (\nabla \mathbf{v}, \nabla \widetilde{\mathbf{u}}) - (\mathbf{b} \cdot \nabla \mathbf{v}, \widetilde{\mathbf{u}}) - (\nabla \cdot \widetilde{\mathbf{u}}, q) &= (\mathbf{u}_d, \widetilde{\mathbf{u}}) && \forall \widetilde{\mathbf{u}} \in H_0^1(\Omega)^d \\
(\nabla \cdot \mathbf{v}, \widetilde{p}) &= 0 && \forall \widetilde{p} \in L_0^2(\Omega) \\
(\nabla \mathbf{u}, \nabla \widetilde{\mathbf{v}}) + (\mathbf{b} \cdot \nabla \mathbf{u}, \widetilde{\mathbf{v}}) - (\nabla \cdot \widetilde{\mathbf{v}}, p) - (\mathbf{f}, \widetilde{\mathbf{v}}) &= 0 && \forall \widetilde{\mathbf{v}} \in H_0^1(\Omega)^d \\
(\nabla \cdot \mathbf{u}, \widetilde{q}) &= 0 && \forall \widetilde{q} \in L_0^2(\Omega) \\
\alpha(\mathbf{f}, \widetilde{\mathbf{f}}) - (\widetilde{\mathbf{f}}, \mathbf{v}) &= 0 && \forall \widetilde{\mathbf{f}} \in L^2(\Omega)
\end{aligned}
$$

Here $\mathbf{u}, p, \mathbf{f}$ are the solutions of the optimal control problem with $\mathbf{v}, q$ as Lagrange multipliers for the state equation and $\widetilde{\mathbf{u}}, \widetilde{\mathbf{v}}, \widetilde{p}, \widetilde{q}, \widetilde{\mathbf{f}}$ denote corresponding test functions.

The control function $\mathbf{f}$ can be eliminated, $\mathbf{f} = \alpha^{-1}\mathbf{v}$, resulting in the reduced system:

$$\begin{aligned}
(\mathbf{u}, \widetilde{\mathbf{u}}) + (\nabla\mathbf{v}, \nabla\widetilde{\mathbf{u}}) - (\mathbf{b}\cdot\nabla\mathbf{v}, \widetilde{\mathbf{u}}) - (\nabla\cdot\widetilde{\mathbf{u}}, q) &= (\mathbf{u}_d, \widetilde{\mathbf{u}}) \quad \forall\widetilde{\mathbf{u}} \in H_0^1(\Omega)^d \\
(\nabla\mathbf{u}, \nabla\widetilde{\mathbf{v}}) + (\mathbf{b}\cdot\nabla\mathbf{u}, \widetilde{\mathbf{v}}) - (\nabla\cdot\widetilde{\mathbf{v}}, p) - \alpha^{-1}(\mathbf{v}, \widetilde{\mathbf{v}}) &= 0 \quad \forall\widetilde{\mathbf{v}} \in H_0^1(\Omega)^d \\
(\nabla\cdot\mathbf{v}, \widetilde{p}) &= 0 \quad \forall\widetilde{p} \in L_0^2(\Omega) \\
(\nabla\cdot\mathbf{u}, \widetilde{q}) &= 0 \quad \forall\widetilde{q} \in L_0^2(\Omega)
\end{aligned}$$

To discretize: use an LBB-stable pair of finite element spaces for the pair $(\mathbf{u}, \mathbf{v})$ and (p, q).
Taylor-Hood pair with $\{Q1, Q1, Q2, Q2\}$, namely, piece-wise bi-quadratic basis functions for $\mathbf{u}, \mathbf{v}$ and piece-wise bi-linear basis functions for p, q.

2D: $\mathscr{A}_0$ is an 8×8 block matrix

The linear system to be solved is then the following:

$$\mathscr{A}_0 \begin{bmatrix} \mathbf{u} \\ -\mathbf{v} \\ p \\ q \end{bmatrix} = \begin{bmatrix} \mathcal{M} & -\mathcal{K}^T & 0 & \mathcal{D}^T \\ \mathcal{K} & \alpha^{-1}\mathcal{M} & \mathcal{D}^T & 0 \\ 0 & \mathcal{D} & 0 & 0 \\ \mathcal{D} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ -\mathbf{v} \\ p \\ q \end{bmatrix} = \begin{bmatrix} M\mathbf{u}_d \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

where we have changed the sign of $\mathbf{v}$.

Notational conventions: the matrices $\mathcal{M}, \mathcal{K}, \mathcal{D}$ are two-by-two block matrices (which reflects the fact that we solve two-dimensional problems), namely,

$$\mathcal{M} = \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}, \ \mathcal{K} = \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix}, \ \mathcal{D} = \begin{bmatrix} B_1 & B_2 \end{bmatrix},$$

M – the velocity mass matrix, $K = L + C$ – the velocity stiffness matrix,
L – discrete diffusion operator, C – discrete convection operator,
B_i^T and $B_i, i = 1, 2$ – discrete gradient and divergence operators.

– For the Stokes problem $C = 0$.

– For the Oseen problem, the convection vector field is divergence-free, thus, the matrix C is skew-symmetric (or nearly skew-symmetric in finite arithmetic), thus, $C^T = -C$. Then we have $K^T = L^T + C^T = L - C$.

– Due to the use of an inf-sup (LBB) stable pairs of finite element spaces, the divergence matrix D has full rank.

To get a form, better suitable for the solution algorithms, we scale the equations and unknowns as follows. Let $\widehat{\mathbf{v}} = \mathbf{v}/\sqrt{\alpha}$, $\widehat{q} = q/\sqrt{\alpha}$. We also multiply the second and the fourth equations with $\sqrt{\alpha}$:

$$\mathscr{A} \begin{bmatrix} \mathbf{u} \\ -\widehat{\mathbf{v}} \\ p \\ \widehat{q} \end{bmatrix} = \begin{bmatrix} \mathcal{M} & -\sqrt{\alpha}\mathcal{K}^T & 0 & \sqrt{\alpha}\mathcal{D}^T \\ \sqrt{\alpha}\mathcal{K} & \mathcal{M} & \sqrt{\alpha}\mathcal{D}^T & 0 \\ 0 & \sqrt{\alpha}\mathcal{D} & 0 & 0 \\ \sqrt{\alpha}\mathcal{D} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ -\widehat{\mathbf{v}} \\ p \\ \widehat{q} \end{bmatrix}$$

We denote the matrix as

$$\mathscr{A} = \begin{bmatrix} \widetilde{\mathcal{A}} & \widetilde{\mathcal{D}}^T \\ \widetilde{\mathcal{D}} & 0 \end{bmatrix},$$

$$\widetilde{\mathcal{A}} = \begin{bmatrix} \mathcal{M} & -\sqrt{\alpha}\mathcal{K}^T \\ \sqrt{\alpha}\mathcal{K} & \mathcal{M} \end{bmatrix}, \; \widetilde{\mathcal{D}} = \sqrt{\alpha} \begin{bmatrix} 0 & \mathcal{D}^T \\ \mathcal{D} & 0 \end{bmatrix}.$$

General viewpoint - analysis of saddle point matrices

Consider a constraint saddle point system

$$\mathcal{A}\mathbf{x} = \mathbf{b}, \quad \text{where } \mathcal{A} = \begin{bmatrix} A & B^T \\ B & -C \end{bmatrix},$$

– A symmetric and positive definite (spd),
– C is symmetric and positive semidefinite (spsd),
– B has full rank.

We analyse the following preconditioners $\mathcal{B}_1$ and $\mathcal{B}_2$ to $\mathcal{A}$.

$$\mathcal{B}_1 = \begin{bmatrix} I & 0 \\ BP_A^{-1} & I \end{bmatrix} \begin{bmatrix} P_A & B^T \\ 0 & -P_S \end{bmatrix} = \begin{bmatrix} P_A & B^T \\ B & BP_A^{-1}B^T - P_S \end{bmatrix}$$

and

$$\mathcal{B}_2 = \begin{bmatrix} P_A & B^T \\ 0 & -P_S \end{bmatrix},$$

P_A and P_S – spd preconditioners to A and to the Schur complement matrix, $S = C + BA^{-1}B^T$, respectively.

Consider the generalized eigenvalue problem

$$\lambda \mathcal{B}_i \mathbf{x} = \mathcal{A}\mathbf{x},\ \mathbf{x} \neq 0, i = 1, 2.$$

Since, by assumptions, both A and S are nonsingular, it follows that $\lambda \neq 0$.

Spectral properties of $\mathcal{B}_i^{-1}\mathcal{A}$

To find the spectral properties of $\lambda\mathcal{B}_i\mathbf{x} = \mathcal{A}\mathbf{x}$ we find it convenient to use a congruence transformation,

$$\begin{bmatrix} A^{-1/2} & 0 \\ 0 & S^{-1/2} \end{bmatrix} \begin{bmatrix} A & B^T \\ B & -C \end{bmatrix} \begin{bmatrix} A^{-1/2} & 0 \\ 0 & S^{-1/2} \end{bmatrix}.$$

Then the transformed matrix $\mathcal{A}$ takes the form

$$\begin{bmatrix} I & \widetilde{B}^T \\ \widetilde{B} & -\widetilde{C} \end{bmatrix},$$

where $\widetilde{B} = S^{-1/2}BA^{-1/2}$ and $\widetilde{C} = S^{-1/2}CS^{-1/2} = I - \widetilde{B}\widetilde{B}^T$. Note that $\| \widetilde{B} \| \leq 1$.

The transformed matrix can be factorized as

$$\left[\begin{array}{cc} I & \widetilde{B}^T \\ \widetilde{B} & \widetilde{B}\widetilde{B}^T - I \end{array}\right] = \left[\begin{array}{cc} I & 0 \\ \widetilde{B} & I \end{array}\right] \left[\begin{array}{cc} I & \widetilde{B}^T \\ 0 & -I \end{array}\right]$$

For the preconditioner $\mathcal{B}_1 = \begin{bmatrix} I & 0 \\ BP_A^{-1} & I \end{bmatrix} \begin{bmatrix} P_A & B^T \\ 0 & -P_S \end{bmatrix}$, the transformed generalized eigenvalue problem takes the form,

$$\frac{1}{\lambda} \left[\begin{array}{cc} I & \widetilde{B}^T \\ \widetilde{B} & -\widetilde{C} \end{array}\right] \left[\begin{array}{c} \widetilde{\mathbf{x}} \\ \widetilde{\mathbf{y}} \end{array}\right] = \left[\begin{array}{cc} \widetilde{P}_A & \widetilde{B}^T \\ \widetilde{B} & \widetilde{B}\widetilde{P}_A^{-1}\widetilde{B}^T - \widetilde{P}_S \end{array}\right] \left[\begin{array}{c} \widetilde{\mathbf{x}} \\ \widetilde{\mathbf{y}} \end{array}\right].$$

Proposition: Assume that

$$P_A \mathbf{x} = \alpha A \mathbf{x}, \mathbf{x} \neq \mathbf{0},$$
$$P_S \mathbf{y} = \beta S \mathbf{y}, \mathbf{y} \neq \mathbf{0},$$

where $0 < \alpha_0 \leq \alpha \leq \alpha_1,\ 0 < \beta_0 \leq \beta \leq \beta_1$, and $\frac{1}{2} \leq \alpha_0 \leq 1,\ 1 \leq \alpha_1 \leq \beta_1$, and that

$$\widetilde{P}_S \geq \widetilde{B}(I - \widetilde{P}_A)\widetilde{P}_A^{-1}(I - \widetilde{P}_A) + \alpha_0 I\,,$$

where $\widetilde{P}_S = S^{-1/2} P_S S^{-1/2}$. This holds if

$$\beta_0 \geq \frac{(1-\alpha_0)^2}{\alpha_0} + \alpha_0 \quad \text{where } 1 \geq \beta_0 \geq \frac{2}{1+\sqrt{2}} \text{ for } \frac{1}{2} \leq \alpha_0 \leq 1\,.$$

Then for the preconditioner $\mathcal{B}_1$ to $\mathcal{A}$ it holds.

(i) $\alpha_0 - 1 \leq \mu_0 = Re(\mu) \leq \beta_1 - 1$

(ii) $\mid \mu_1 \mid = Im(\mu) \leq \max\{1-\alpha_0, \alpha_1 - 1\} = \mid \mu \mid_{\max}$.

If $\lambda = \lambda_0 + i\lambda_1$, λ_0, λ_1 real, then

$$\min\left\{\frac{1}{\alpha_0 + \frac{|\mu|^2_{\max}}{\alpha_0}}, \frac{1}{\beta_1 + \frac{|\mu|^2_{\max}}{\beta_1}}\right\} \leq \lambda_0 \leq \frac{1}{\alpha_0}$$

$$\mid \lambda_1 \mid \leq \frac{1}{\alpha_0^2} \max\left\{1-\alpha_0, \alpha_1 - 1\right\} \leq 4 \max\left\{1-\alpha_0, \alpha_1 - 1\right\} .$$

Remark:

- It is seen that the imaginary part of λ depends only on the accuracy of the preconditioner P_A to A.
- The same holds for the upper bound of the real parts of the eigenvalues.

This is important since it shows that one can control the rate of convergence of a generalized conjugate gradient method essentially by solving the pivot block matrix more accurately and, since the lower eigenvalue bound depends on β_1, scaling P_S properly if P_S is a sufficiently accurate preconditioner to S.

$\mathcal{B}_2^{-1}\mathcal{A}$

Consider now the block triangular preconditioner $\mathcal{B}_2 = \begin{bmatrix} P_A & B^T \\ 0 & -P_S \end{bmatrix}$.

Using the same congruence transformations as before, the corresponding generalized eigenvalue problem takes the form

$$\frac{1}{\lambda}\begin{bmatrix} I & \widetilde{B}^T \\ \widetilde{B} & -\widetilde{C} \end{bmatrix}\begin{bmatrix} \widetilde{\mathbf{x}} \\ \widetilde{\mathbf{y}} \end{bmatrix} = \begin{bmatrix} \widetilde{P}_A & 0 \\ \widetilde{B} & -\widetilde{P}_S \end{bmatrix}\begin{bmatrix} \widetilde{\mathbf{x}} \\ \widetilde{\mathbf{y}} \end{bmatrix}.$$

It follows that: $\left(\frac{1}{\lambda} - 1\right) \begin{bmatrix} \widehat{\mathbf{x}} \\ \widehat{\mathbf{y}} \end{bmatrix} =$

$$\begin{bmatrix} \widetilde{P}_A - I & (\widetilde{P}_A - I)\widetilde{B}^T \\ \widetilde{B}(I - \widetilde{P}_A) & \widetilde{P}_S - I - \widetilde{B}(\widetilde{P}_A - I)\widetilde{B}^T \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{x}} \\ \widehat{\mathbf{y}} \end{bmatrix} + \begin{bmatrix} 0 & \widetilde{B}^T \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{x}} \\ \widehat{\mathbf{y}} \end{bmatrix}.$$

In this case due to the term $\begin{bmatrix} 0 & \widetilde{B}^T \\ 0 & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & \widetilde{B}^T \\ \widetilde{B} & 0 \end{bmatrix} + \begin{bmatrix} 0 & \widetilde{B}^T \\ -\widetilde{B} & 0 \end{bmatrix}$,

there is a strong influence on both the real and imaginary parts of the eigenvalues from the off-diagonal matrix block and cannot be controlled by solving the inner system with A sufficiently accurately.

Only if also the Schur complement system is solved to full precision we get a nilpotent preconditioned matrix, resulting in only two iterations.

Schur complement matrix approximation; outer level

Basic Lemma:
Assume that A and W are nonsingular of order $n \times n$ and $m \times m$, respectively, and B of order $n \times m$ has full rank. Then

$$\left(B(A+\gamma B^T W^{-1} B)^{-1} B^T\right)^{-1} = \gamma W^{-1} + (BA^{-1}B^T)^{-1}.$$

Proof. Well-known, but follows easily from

$$A_1(I + A_1 A_2)^{-1} A_2 = I + (A_1 A_2)^{-1},$$

which follows by multiplication with $I + A_1 A_2$. Take then inverses,

$$\left(A_1(I + A_1 A_2)^{-1} A_2\right)^{-1} = I + (A_1 A_2)^{-1}$$

and let $A_1 = L_2^{-1} B^T U_1^{-1}$, $A_2 = L_1^{-1} B^T U_2^{-1}$, where $A = L_1 U_1$ and $W = L_2 U_2$. ■

Application of the lemma:

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} A_\gamma & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ 0 \end{bmatrix}$$

where $A_\gamma = A + \gamma B^T W^{-1} B$, the Augmented Lagrangian matrix. Then

$$\begin{bmatrix} A_\gamma & 0 \\ B & -S_\gamma \end{bmatrix} \begin{bmatrix} I_1 & A_\gamma^{-1} B^T \\ 0 & I_2 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ 0 \end{bmatrix}$$

where $S_\gamma = B A_\gamma^{-1} B^T$ and $S_\gamma^{-1} = \gamma W^{-1} + (B A^{-1} B^T)^{-1}$.

For the two-by-two block matrix application, for small α:

$$\begin{aligned} -S_{\mathscr{A}}^{-1} &= \gamma \widetilde{W}^{-1} + \left(\widetilde{\mathcal{D}}\widetilde{\mathcal{A}}^{-1}\widetilde{\mathcal{D}}^T\right)^{-1} \\ &= \gamma \begin{bmatrix} W^{-1} & 0 \\ 0 & W^{-1} \end{bmatrix} + \left(\alpha \begin{bmatrix} 0 & \mathcal{D} \\ \mathcal{D} & 0 \end{bmatrix} \begin{bmatrix} \mathcal{M} & -\sqrt{\alpha}\mathcal{K}^T \\ \sqrt{\alpha}\mathcal{K} & \mathcal{M} \end{bmatrix}^{-1} \begin{bmatrix} 0 & \mathcal{D}^T \\ \mathcal{D}^T & 0 \end{bmatrix}\right)^{-1} \\ &\approx \gamma \begin{bmatrix} W^{-1} & 0 \\ 0 & W^{-1} \end{bmatrix} + \left(\alpha \begin{bmatrix} \mathcal{D}\mathcal{M}^{-1}\mathcal{D}^T & 0 \\ 0 & \mathcal{D}\mathcal{M}^{-1}\mathcal{D}^T \end{bmatrix}\right)^{-1} \\ &\approx \gamma \begin{bmatrix} W^{-1} & 0 \\ 0 & W^{-1} \end{bmatrix} + \frac{1}{\alpha} \begin{bmatrix} K_p^{-1} & 0 \\ 0 & K_p^{-1} \end{bmatrix}, \end{aligned}$$

where we have simply approximated

$$\begin{bmatrix} \mathcal{M} & -\sqrt{\alpha}\mathcal{K}^T \\ \sqrt{\alpha}\mathcal{K} & \mathcal{M} \end{bmatrix}^{-1} \approx \begin{bmatrix} \mathcal{M}^{-1} & 0 \\ 0 & \mathcal{M}^{-1} \end{bmatrix}.$$

Inner iterations: the pivot block - a two-by-two block matrix, its preconditioner

Let A and B be square matrices. Consider matrices in the form

$$\mathcal{A} = \begin{bmatrix} A & -aB^T \\ -bB & A \end{bmatrix},$$

where a, b are real numbers such that $ab > 0$.
Assume that A and $B + B^T$ are positive semidefinite and

$$\ker(A) \cap \ker(B) = \{\emptyset\} \ .$$

It follows readily that under these assumptions, $\mathcal{A}$ is nonsingular.

We let

$$\mathcal{B} = \begin{bmatrix} A & aB^T \\ -bB & A + \sqrt{ab}(B + B^T) \end{bmatrix},$$

be a preconditioner to $\mathcal{A}$. Clearly $\mathcal{B}$ is also nonsingular.

The exact inverse of $\mathcal{B}$ has the form

$$\mathcal{B}^{-1} = \begin{bmatrix} H_1^{-1} + H_2^{-1} - H_2^{-1} A H^{-1} & \sqrt{\frac{a}{b}}(I - H_2^{-1}A)H_1^{-1} \\ -\sqrt{\frac{b}{a}} H_2^{-1}(I - AH_1^{-1}) & H_2^{-1} A H_1^{-1} \end{bmatrix},$$

$H_i = A + \sqrt{ab}B_i\ ,\ i = 1, 2$ and $B_1 = B, B_2 = B^T$. It follows readily that, besides some matrix vector multiplications and vector additions, an action of $\mathcal{B}^{-1}$ involves just a solution of a system with H_1 and with H_2, namely, the computation of $\begin{bmatrix} x \\ y \end{bmatrix} = \mathcal{B}^{-1} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ can take place in the following order:

(i) Solve $H_1 \mathbf{g} = \mathbf{f}_1 + \sqrt{\frac{a}{b}} \mathbf{f}_2$.

(ii) Compute $A\mathbf{g}$ and $\mathbf{f}_1 - A\mathbf{g}$.

(iii) Solve $H_2 \mathbf{h} = \mathbf{f}_1 - A\mathbf{g}$.

(iv) Compute $\mathbf{x} = \mathbf{g} + \mathbf{h}$ and $\mathbf{y} = -\sqrt{\frac{b}{a}}\mathbf{h}$.

Proposition: Let A and $B + B^T$ be symmetric and positive semi-definite and assume that $\ker(A) \cap \ker(B) = \{\emptyset\}$. Then the following holds true.

(i) The eigenvalues λ of $\mathcal{B}^{-1}\mathcal{A}$ satisfy $\frac{1}{2} \leq \frac{1}{1+q} \leq \lambda \leq 1$, where

$$q = \sup_{\widetilde{\mathbf{x}},\mathbf{y}} \frac{2(\widetilde{\mathbf{x}}^*(B + B^T)\mathbf{y})}{\widetilde{\mathbf{x}}^*(B + B^T)\widetilde{x} + \mathbf{y}^*(B + B^T)\mathbf{y}} \leq 1,$$

where $\widetilde{\mathbf{x}} = \sqrt{\frac{b}{a}}\mathbf{x}$ and $\mathbf{x}, \mathbf{y}$ are eigenvectors of the generalized eigenvalue problem $\lambda\mathcal{B}\begin{bmatrix}\mathbf{x}\\ \mathbf{y}\end{bmatrix} = \mathcal{A}\begin{bmatrix}\mathbf{x}\\ \mathbf{y}\end{bmatrix}$. Here $\lambda = 1$ if and only if $\mathbf{y} \in \mathcal{N}(B + B^T)$.

(ii) If A is spd, then $\max\left\{\frac{1}{1+q}, \frac{1}{1+\sqrt{ab}\,\sigma_0}\right\} \leq \lambda \leq 1\,,$ where $\sigma_0 = \sigma(A^{-1/2}(B + B^T)A^{-1/2})$ and $\sigma(.)$ denotes the spectral radius.

The pivot block matrix in our application

Consider next the solution of systems with $\widetilde{\mathcal{A}}_\gamma$. To this end, we use the block-structure of the matrices involved:

$$\begin{aligned}
\widetilde{\mathcal{A}}_\gamma &= \begin{bmatrix} \mathcal{M} & -\sqrt{\alpha}\mathcal{K}^T \\ \sqrt{\alpha}\mathcal{K} & \mathcal{M} \end{bmatrix} + \gamma\alpha \begin{bmatrix} 0 & \mathcal{D}^T \\ \mathcal{D}^T & 0 \end{bmatrix} \begin{bmatrix} W^{-1} & 0 \\ 0 & W^{-1} \end{bmatrix} \begin{bmatrix} 0 & \mathcal{D} \\ \mathcal{D} & 0 \end{bmatrix} \\
&= \begin{bmatrix} \mathcal{M} & -\sqrt{\alpha}\mathcal{K}^T \\ \sqrt{\alpha}\mathcal{K} & \mathcal{M} \end{bmatrix} + \widetilde{\gamma} \begin{bmatrix} \mathcal{D}^T W^{-1}\mathcal{D} & 0 \\ 0 & \mathcal{D}^T W^{-1}\mathcal{D} \end{bmatrix} \\
&= \begin{bmatrix} \mathcal{M} + \widetilde{\gamma}\mathcal{D}^T W^{-1}\mathcal{D} & -\sqrt{\alpha}\mathcal{K}^T \\ \sqrt{\alpha}\mathcal{K} & \mathcal{M} + \widetilde{\gamma}\mathcal{D}^T W^{-1}\mathcal{D} \end{bmatrix}.
\end{aligned}$$

Here $\widetilde{\gamma} = \gamma\alpha$.
Note: α is a small parameter, while γ is large, so, $\widetilde{\gamma}$ can take more moderate values.

The matrix $\widetilde{\mathcal{A}}_\gamma$ is of the form $\begin{bmatrix} A & -B \\ B & A \end{bmatrix}$ and we can approximate it very efficiently as already described.
Note: the block $\mathcal{M} + \widetilde{\gamma}\mathcal{D}^T W^{-1}\mathcal{D}$ is spd.

In the suggested computational procedure we replace $\widetilde{\mathcal{A}}_\gamma$ by its corresponding approximation, referred to as $\mathcal{P}_\gamma$, namely,

$$\mathcal{P}_\gamma = \begin{bmatrix} \mathcal{M} + \widetilde{\gamma}\mathcal{D}^T W^{-1}\mathcal{D} & -\sqrt{\alpha}\mathcal{K}^T \\ \sqrt{\alpha}\mathcal{K} & \mathcal{M} + \widetilde{\gamma}\mathcal{D}^T W^{-1}\mathcal{D} + \sqrt{\alpha}(\mathcal{K} + \mathcal{K}^T) \end{bmatrix}.$$

The solution of systems with the matrix $\mathcal{P}_\gamma$ boils down to solutions with the matrices

$$\mathcal{H}_\gamma^{(1)} = \mathcal{M} + \widetilde{\gamma}\mathcal{D}^T W^{-1}\mathcal{D} + \sqrt{\alpha}\mathcal{K} = (\mathcal{M} + \sqrt{\alpha}\mathcal{K}) + \widetilde{\gamma}\mathcal{D}^T W^{-1}\mathcal{D}$$
$$\mathcal{H}_\gamma^{(2)} = \mathcal{M} + \widetilde{\gamma}\mathcal{D}^T W^{-1}\mathcal{D} + \sqrt{\alpha}\mathcal{K}^T = (\mathcal{M} + \sqrt{\alpha}\mathcal{K}^T) + \widetilde{\gamma}\mathcal{D}^T W^{-1}\mathcal{D}$$

Both matrices $\mathcal{H}_\gamma^{(1)}$ and $\mathcal{H}_\gamma^{(2)}$ have very similar structure.

Due to lack of time the details will not be discussed now.

Numerical illustrations: Stokes problem, **V2:**

Problem size	Iterations				
	$It(\mathcal{A})$	$It^{av}(Q)$	$It^{av}(\mathcal{M}+\sqrt{\alpha}\mathcal{K})$	$It^{av}(\widetilde{S}_{Q,1})$	$It^{av}(K_p)$
$\alpha_{opt} = 10^{-4}, \gamma = 10^4, \tilde{\gamma} = 1$					
19078	9	14	6	6	6
75014	9	19	6	6	6
297478	8	26	7	6	7
$\alpha_{opt} = 10^{-6}, \gamma = 10^6, \tilde{\gamma} = 1$					
19078	8	11	5	5	6
75014	8	14	6	6	6
297478	8	19	6	6	7
$\alpha_{opt} = 10^{-8}, \gamma = 10^8, \tilde{\gamma} = 1$					
19078	6	10	4	4	6
75014	7	12	4	5	6
297478	8	14	5	5	6

Full-block factorized outer preconditioner

Numerical illustrations: Oseen problem, **V2**

Problem size	Iterations				
	$It(\mathcal{A})$	$It^{av}(Q)$	$It^{av}(\mathcal{M}+\sqrt{\alpha}\mathcal{K})$	$It^{av}(\widetilde{S}_{Q,1})$	$It^{av}(K_p)$
$\alpha_{opt}=10^{-4}, \gamma=10^4, \tilde{\gamma}=1$					
19078	11	14	6	6	6
75014	11	19	6	6	7
297478	10	26	7	6	7
$\alpha_{opt}=10^{-6}, \gamma=10^6, \tilde{\gamma}=1$					
19078	10	11	5	5	6
75014	12	15	6	6	7
297478	13	20	6	6	7
$\alpha_{opt}=10^{-8}, \gamma=10^8, \tilde{\gamma}=1$					
19078	10	10	4	5	6
75014	14	12	4	5	6
297478	16	14	5	6	7

Block-triangular outer preconditioner

Numerical illustrations: Oseen problem, **V3**

Problem size	Iterations					
	It $(\mathcal{A})$	It^{av} (Q_M)	It^{av} $(\tilde{S}_{Q_M})$	It^{av} $(\mathcal{M}+\sqrt{\alpha}\mathcal{K})$	It^{av} $(S_{\tilde{Q}_M+Q_K})$	It^{av} (K_p)
$\alpha_{opt}=10^{-4}, \gamma=10^4, \tilde{\gamma}=1$						
19078	23	14	5	5	7	6
75014	35	15	5	5	7	6
297478	116	17	5	5	8	7
$\alpha_{opt}=10^{-6}, \gamma=10^6, \tilde{\gamma}=1$						
19078	12	13	5	5	7	6
75014	14	14	5	5	7	6
297478	19	16	5	5	7	7
$\alpha_{opt}=10^{-8}, \gamma=10^8, \tilde{\gamma}=1$						
19078	7	12	5	4	7	6
75014	8	13	5	4	7	6
297478	13	18	5	5	7	6

Block-triangular outer preconditioner

Conclusions:

Although optimal control problems for PDE's involve several levels of iterations, it has been shown that the inner systems can be solved efficiently using different types of preconditioners on the different levels.

Some improvements are still needed for the preconditioner on the lowest level to solve the two matrices H_i, $i = 1, 2$ appearing in the action of the inverse of the preconditioner from the two-by-two pivot block.

Some references:

- J.-L. Lions, *Optimal Control of Systems Governed by Partial Differential Equations*, Springer–Verlag, Berlin, 1971.
- W. Zulehner, Nonstandard norms and robust estimates for saddle–point problems, *SIAM J. Matrix Anal. Appl.*, 32(2011), 536–560.
- M.D. Gunzburger and S. Manservisi, Analysis and approximation of the velocity tracking problem for Navier–Stokes flows with distributed control, *SIAM J. Numer. Anal.* 37(2000), 1481–1512.
- O. Axelsson, Preconditioners for some matrices of two-by-two block form, with applications, I.

Thank you for your attention!