Deflated Krylov Acceleration of the Schwarz Domain Decomposition Method

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Delft University of Technology

GAMM Workshop Numerical Linear Algebra

University of Bielefeld, Germany, September 13-14, 2002

Contents

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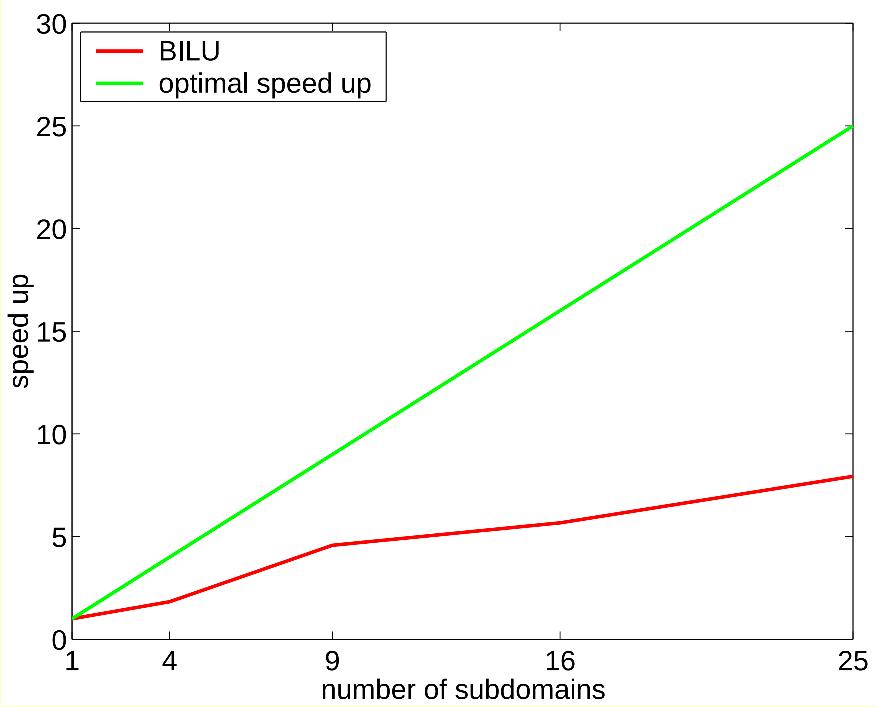
1. Introduction

Block ILU



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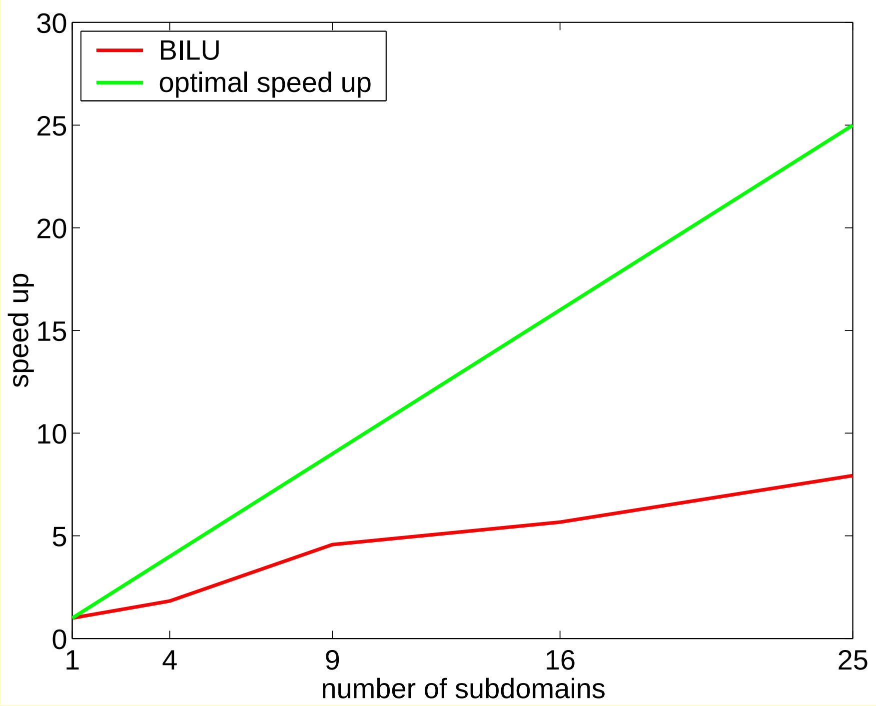
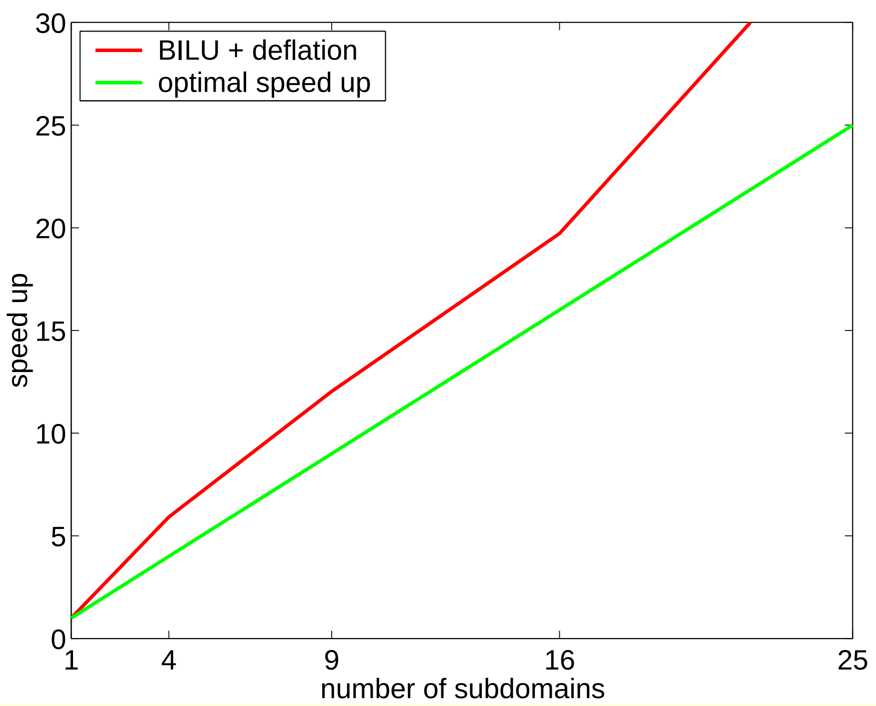
1. Introduction

Block ILU



Block ILU with deflation



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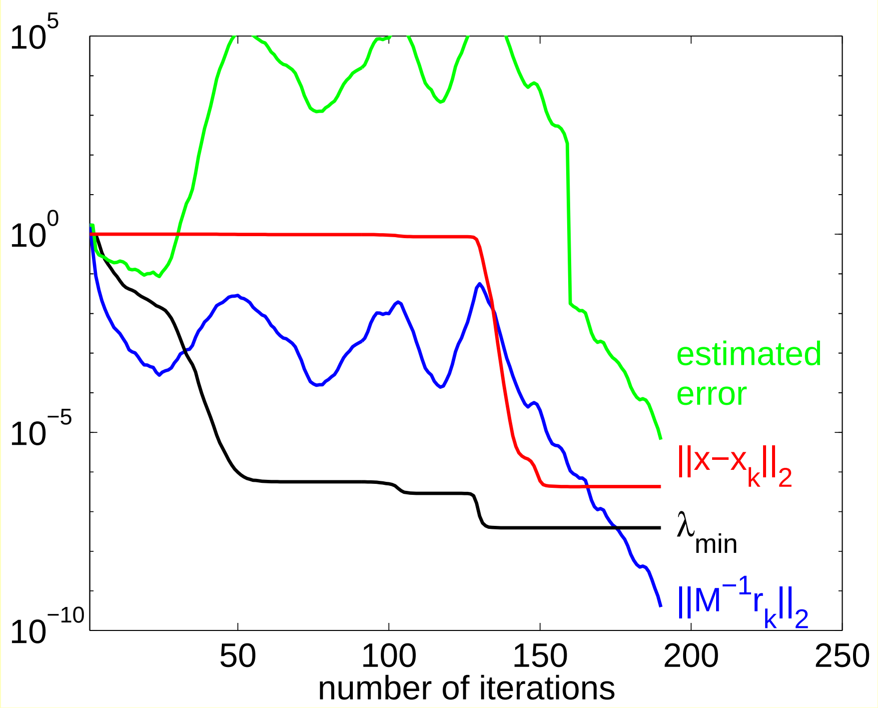
Results oil flow problem

ICCG



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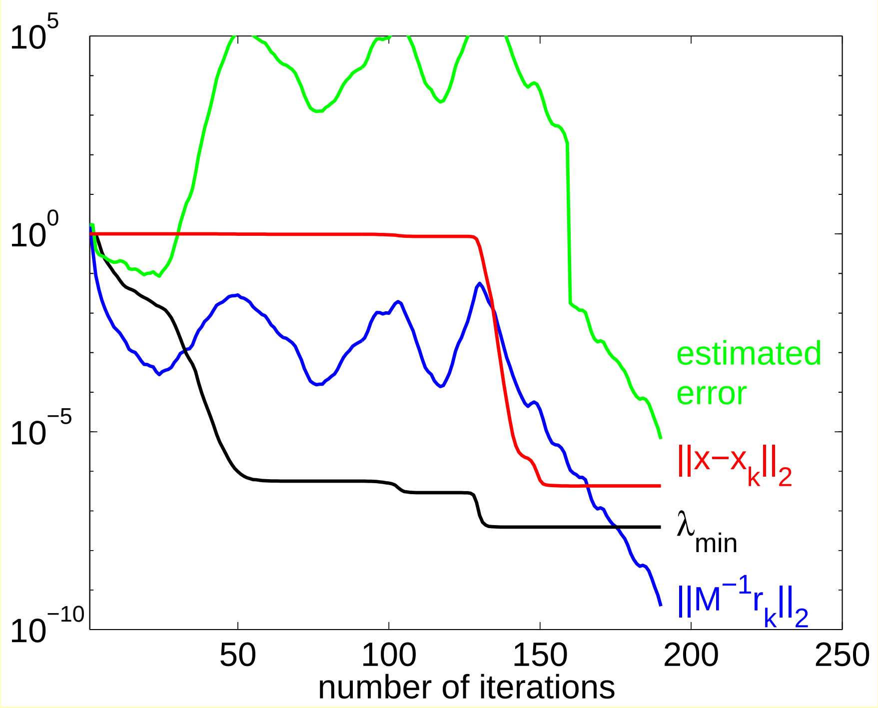
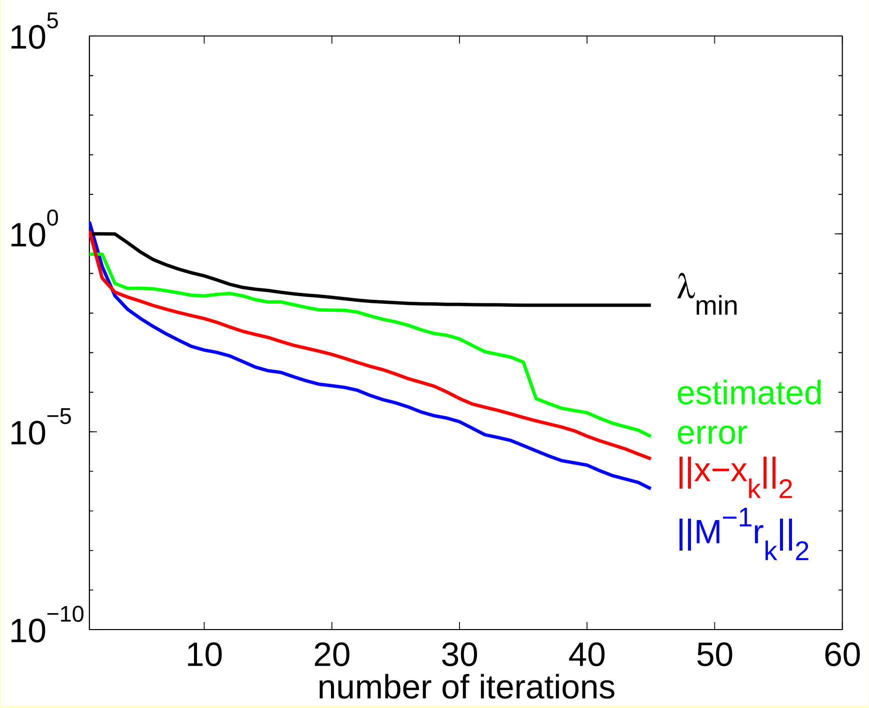


Deflated ICCG



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Literature review

- Robust preconditioners
  (M)ICCG vd Vorst, Meijering, Gustafsson
  ILUT Saad, MRILU Ploeg, Wubs
  Navier-Stokes Elman, Silvester, Wathen, Golub
  RIF Benzi, Tuma

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  Block variants see above
  ILU Bastian, Horton, Vuik, Nooyen, Wesseling
  SPAI Grote, Huckle, Benzi, Tuma, Chow, Saad

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- Parallel preconditioners
  Block variants see above
  ILU Bastian, Horton, Vuik, Nooyen, Wesseling
  SPAI Grote, Huckle, Benzi, Tuma, Chow, Saad
- Acceleration of parallel preconditioners
  CGC Notay, vd Velde, Benzi, Frommer, Nabben, Szyld, Chan, Mathew, Dryja, Widlund, Padiy, Axelsson, Polman
  Deflation Nicolaides, Mansfield, Frank, Vuik
  Morgan, Chapman, Saad, Burrage, Ehrel, Pohl
  FETI Farhat, Roux, Mandel, Klawonn, Widlund

2. Deflated Krylov methods

A is SPD, Conjugate Gradients

$P = I - AZE^{-1}Z^T$ with $E = Z^TAZ$

and $Z = [z_1 ... z_m]$, where $z_1, ..., z_m$ are independent deflation vectors.

Properties

1. $P^TZ = 0$ and $PAZ = 0$
2. $P^2 = P$
3. $AP^T = PA$

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Deflated ICCG

$$x = (I - P^T)x + P^T x,$$

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$$x = (I - P^T)x + P^T x,$$

$$(I - P^T)x = ZE^{-1}Z^TAx = ZE^{-1}Z^Tb,$$

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Deflated ICCG

$$x = (I - P^T)x + P^Tx,$$

$$(I - P^T)x = ZE^{-1}Z^TAx = ZE^{-1}Z^Tb, \qquad AP^Tx = PAx = Pb.$$

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Deflated ICCG

$$x = (I - P^T)x + P^T x,$$

$$(I - P^T)x = ZE^{-1}Z^T Ax = ZE^{-1}Z^T b, \qquad AP^T x = PAx = Pb.$$

DICCG

$k = 0,\ \hat{r}_0 = Pr_0,\ p_1 = z_1 = L^{-T}L^{-1}\hat{r}_0;$

while $\|\hat{r}_k\|_2 > \varepsilon$ **do**

$k = k + 1;$

$\alpha_k = \frac{(\hat{r}_{k-1}, z_{k-1})}{(p_k, PAp_k)};$

$x_k = x_{k-1} + \alpha_k p_k;$

$\hat{r}_k = \hat{r}_{k-1} - \alpha_k PAp_k;$

$z_k = L^{-T}L^{-1}\hat{r}_k;$

$\beta_k = \frac{(\hat{r}_k, z_k)}{(\hat{r}_{k-1}, z_{k-1})}; \qquad p_{k+1} = z_k + \beta_k p_k;$

end while

Deflation for non-symmetric matrices

$$P = I - AZE^{-1}Y^T \text{ with } E = Y^TAZ$$
$$Q = I - ZE^{-1}Y^TA$$

and $Z = [z_1 ... z_m]$, $Y = [y_1 ... y_m]$ where $z_1, ..., z_m$ and $y_1, ..., y_m$ are independent sets of deflation vectors.

Properties

1. $PAZ = Y^TP = 0$ and $Y^TAQ = QZ = 0$
2. $P^2 = P$ and $Q^2 = Q$
3. $PA = AQ$

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Deflated Krylov methods

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Deflated Krylov methods

$$x = (I - Q)x + Qx,$$

$$(I - Q)x = ZE^{-1}Y^T Ax = ZE^{-1}Y^T b, \qquad AQx = PAx = Pb.$$

Deflated Krylov methods

$$x = (I - Q)x + Qx,$$

$$(I - Q)x = ZE^{-1}Y^TAx = ZE^{-1}Y^Tb, \qquad AQx = PAx = Pb.$$

Preconditioning

$$K^{-1}PA\tilde{x} = K^{-1}Pb, \quad Qx = Q\tilde{x}$$

$$PAK^{-1}\tilde{y} = Pb, \quad Qx = QK^{-1}\tilde{y}$$

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Deflated Krylov methods

$$x = (I - Q)x + Qx,$$

$$(I - Q)x = ZE^{-1}Y^TAx = ZE^{-1}Y^Tb, \qquad AQx = PAx = Pb.$$

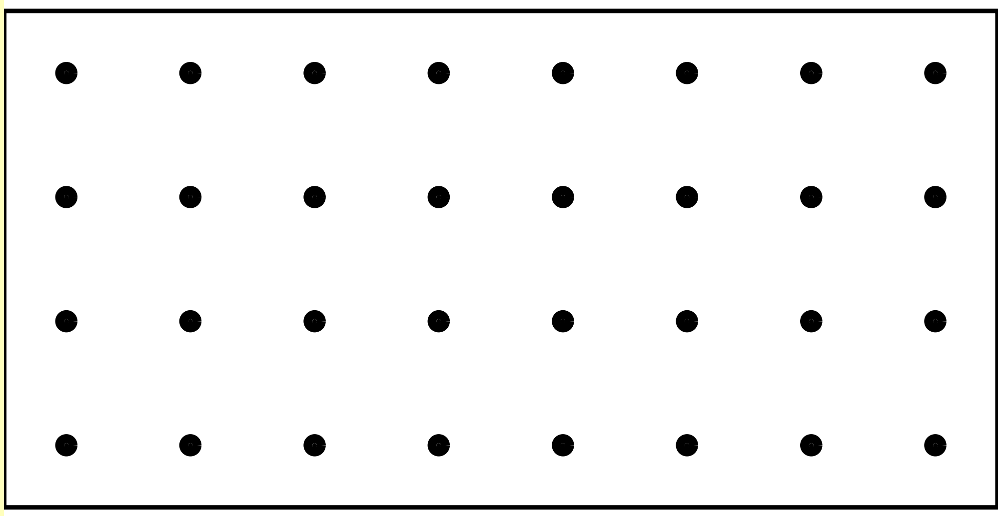
Preconditioning

$$K^{-1}PA\tilde{x} = K^{-1}Pb, \quad Qx = Q\tilde{x}$$

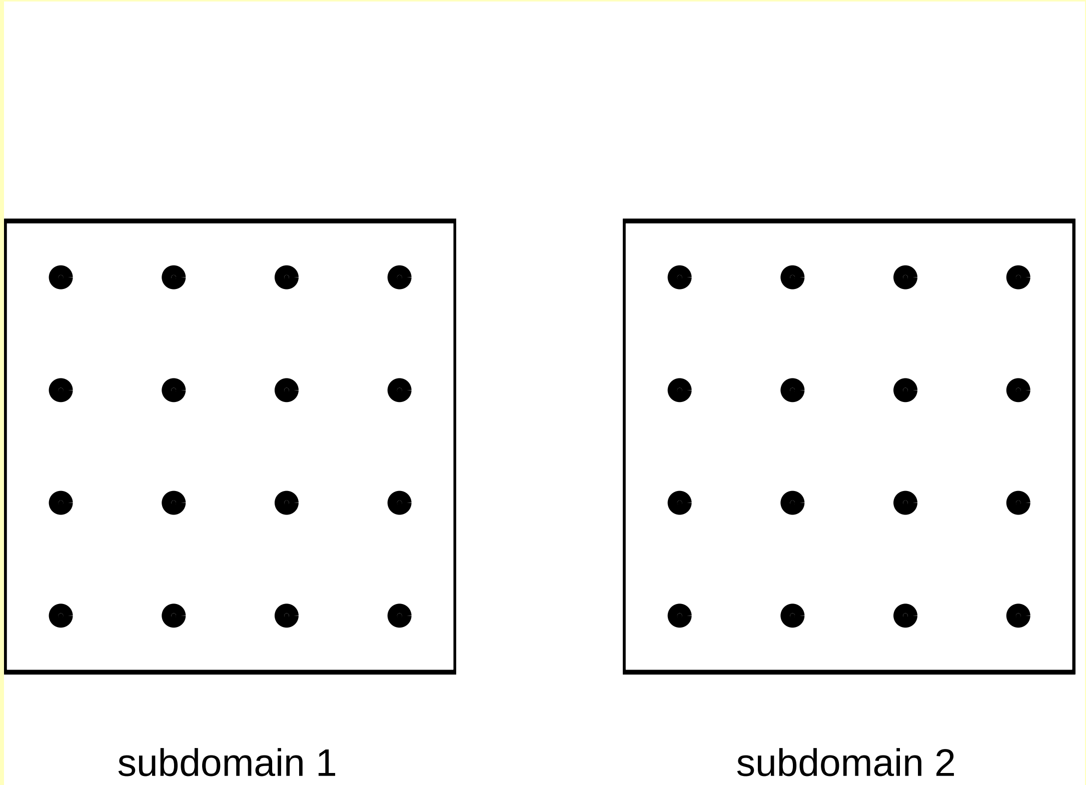
$$PAK^{-1}\tilde{y} = Pb, \quad Qx = QK^{-1}\tilde{y}$$

Systems can be solved by: GMRES, GCR, Bi-CGSTAB,...

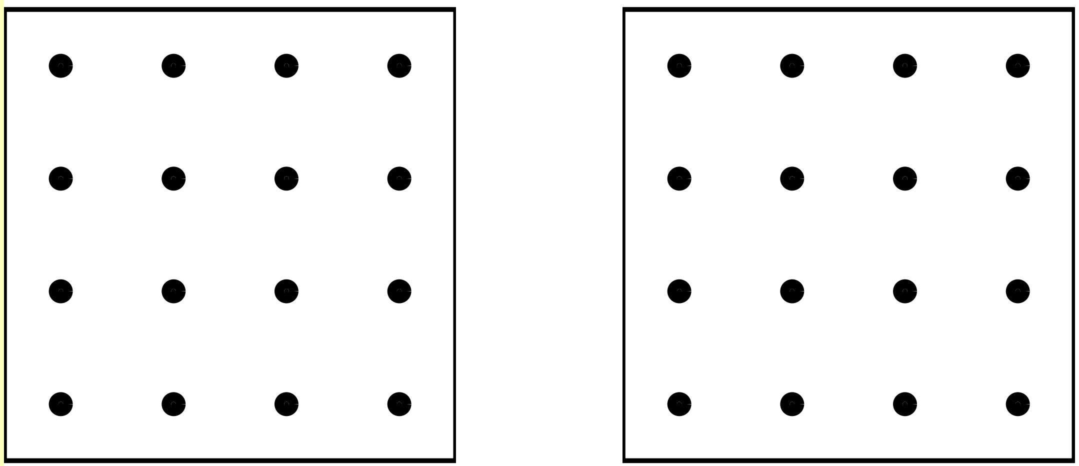
Decomposisition of a cell centered domain



Decomposisition of a cell centered domain



Decomposisition of a cell centered domain



$$\bar{\Omega} = \bigcup_{i=1}^{m} \bar{\Omega}_i$$

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Choice of the deflation vectors

m is number of subdomains $z_1, ..., z_m$ deflation vectors

- $z_i = 1$ on $\bar{\Omega}_i$
- $z_i = 0$ on $\Omega \setminus \bar{\Omega}_i$

Remarks

- The matrix E is sparse
- $K_{eff}(PA)$ decreases for increasing m
- Work to invert E increases for increasing m
- Optimal value of m?

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3. Comparison of Deflation with Coarse Grid Correction

Definition: $P_D = I - AZE^{-1}Z^T$.

$x = (I - P_D^T)x + P_D^T x,$
where $(I - P_D^T)x = ZE^{-1}Z^T b$ and $AP_D^T x = P_D A x = P_D b$

DICCG

$k = 0,\ \hat{r}_0 = P_D r_0,\ p_1 = z_1 = L^{-T}L^{-1}\hat{r}_0;$

while $\|\hat{r}_k\|_2 > \varepsilon$ **do**

$k = k + 1;$

$\alpha_k = \frac{(\hat{r}_{k-1}, z_{k-1})}{(p_k, P_D A p_k)};$

$x_k = x_{k-1} + \alpha_k p_k;$

$\hat{r}_k = \hat{r}_{k-1} - \alpha_k P_D A p_k;$

$z_k = L^{-T}L^{-1}\hat{r}_k;$

$\beta_k = \frac{(\hat{r}_k, z_k)}{(\hat{r}_{k-1}, z_{k-1})};$ $\quad p_{k+1} = z_k + \beta_k p_k;$

end while

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Coarse Grid Correction of ICCG

Definition

- $Z \in \mathbb{R}^{n \times m}$ with independent columns.
- $E = Z^T A Z \in \mathbb{R}^{m \times m}$, E is SPD.
- $P_C = L^{-T} L^{-1} + \gamma Z E^{-1} Z^T$.

CICCG

$k = 0,\ r_0 = b - Ax_0,\ p_1 = z_1 = L^{-T} L^{-1} r_0;$

while $\|r_k\|_2 > \varepsilon$ **do**

$$k = k + 1;$$

$$\alpha_k = \frac{(r_{k-1}, z_{k-1})}{(p_k, Ap_k)};$$

$$x_k = x_{k-1} + \alpha_k p_k;$$

$$r_k = r_{k-1} - \alpha_k A p_k;$$

$$z_k = P_C r_k = L^{-T} L^{-1} r_k + \gamma Z E^{-1} Z^T r_k;$$

$$\beta_k = \frac{(r_k, z_k)}{(r_{k-1}, z_{k-1})}; \qquad p_{k+1} = z_k + \beta_k p_k;$$

end while

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Properties of Deflation and CGC

$$P_D = I - AZE^{-1}Z^T \qquad\qquad P_C = I + ZE^{-1}Z^T$$

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Properties of Deflation and CGC

$P_D = I - AZE^{-1}Z^T$ $\qquad$ $P_C = I + ZE^{-1}Z^T$

Properties of P_D

- $P_D A$ is symmetric and positive semidefinite
- P_D is a projection, $P_D AZ = 0$
- since $P_D A$ is singular, a good termination criterion is important

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Properties of Deflation and CGC

$P_D = I - AZE^{-1}Z^T$ $P_C = I + ZE^{-1}Z^T$

Properties of P_D

- $P_D A$ is symmetric and positive semidefinite
- P_D is a projection, $P_D AZ = 0$
- since $P_D A$ is singular, a good termination criterion is important

Properties of P_C

- P_C is symmetric positive definite
- $A^{\frac{1}{2}}(P_C - I)A^{\frac{1}{2}}$ is a projection

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Properties of Deflation and CGC

Definition

Eigenpair $\{\lambda_i, v_i\}$, so $Av_i = \lambda_i v_i$ with $0 < \lambda_1 \leq \ldots \leq \lambda_n$.

Take $Z = [v_1 \ldots v_m]$.

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Properties of Deflation and CGC

Definition

Eigenpair $\{\lambda_i, v_i\}$, so $Av_i = \lambda_i v_i$ with $0 < \lambda_1 \leq \ldots \leq \lambda_n$.
Take $Z = [v_1 \ldots v_m]$.

Theorem

- the spectrum of $P_D A$ is $\{0, \ldots, 0, \lambda_{m+1}, \ldots, \lambda_n\}$

- the spectrum of $P_C A$ is $\{1 + \lambda_1, \ldots, 1 + \lambda_m, \lambda_{m+1}, \ldots, \lambda_n\}$

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Properties of Deflation and CGC

Definition

Eigenpair $\{\lambda_i, v_i\}$, so $Av_i = \lambda_i v_i$ with $0 < \lambda_1 \leq \ldots \leq \lambda_n$.
Take $Z = [v_1 \ldots v_m]$.

Theorem

- the spectrum of $P_D A$ is $\{0, \ldots, 0, \lambda_{m+1}, \ldots, \lambda_n\}$

- the spectrum of $P_C A$ is $\{1 + \lambda_1, \ldots, 1 + \lambda_m, \lambda_{m+1}, \ldots, \lambda_n\}$

Corollary

DICCG converges faster than CICCG.

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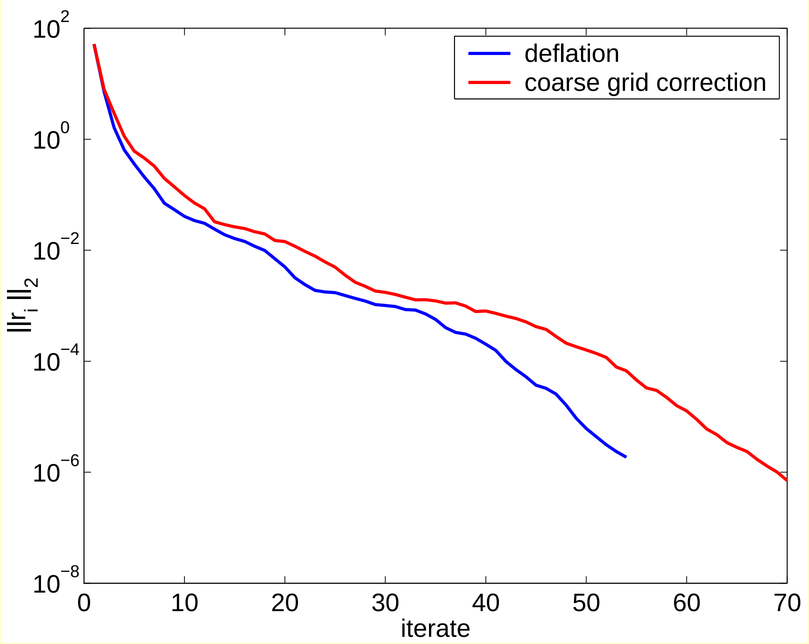
Experiments with Deflation and CGC

Residual with Block IC



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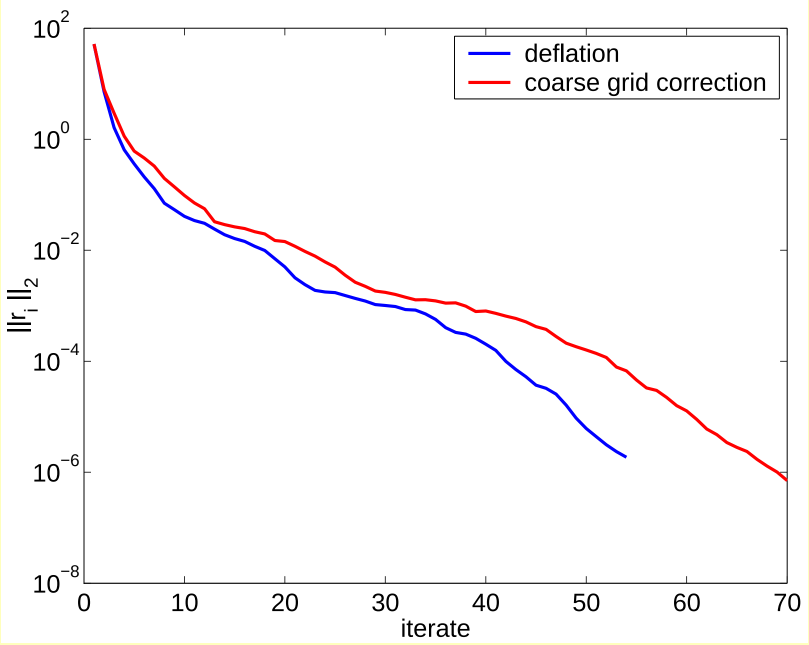
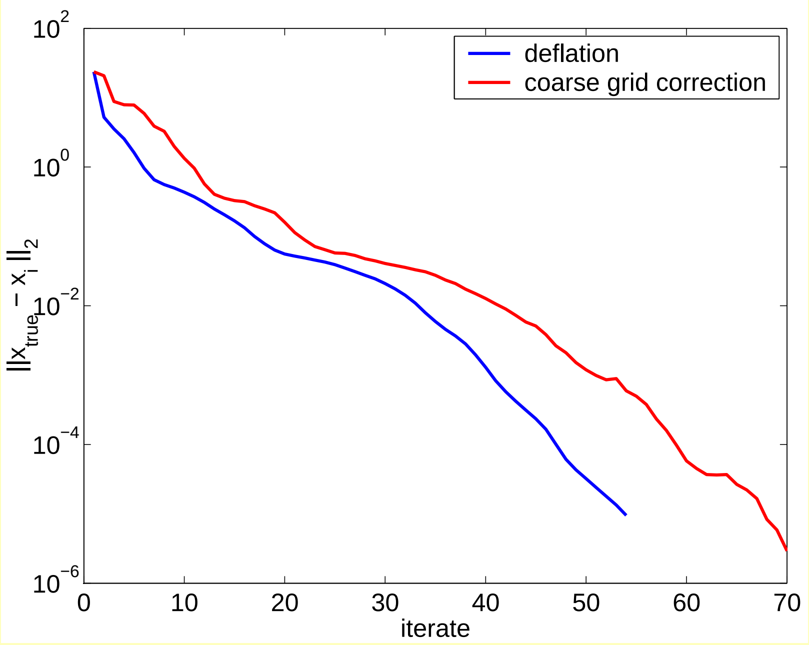


Error with Block IC



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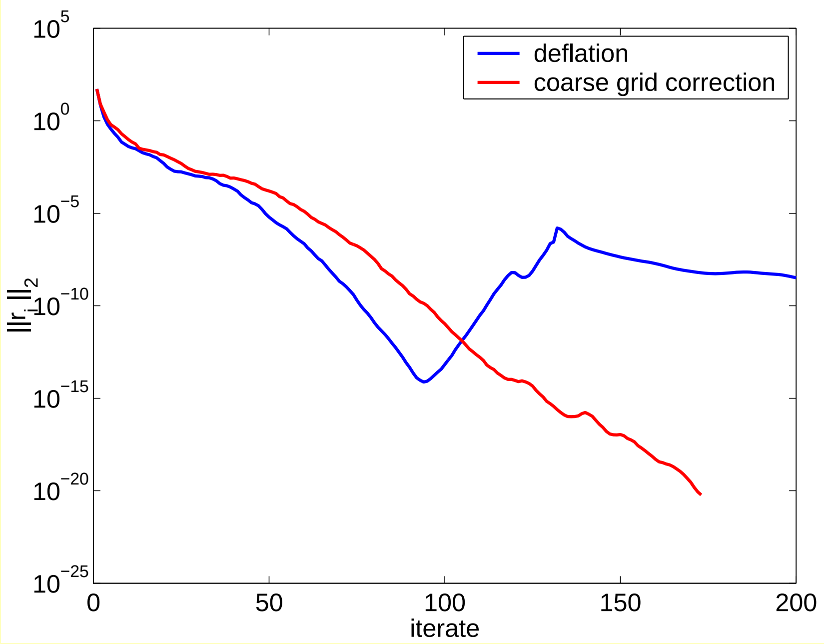
Too severe termination criterion

Residual with Block IC



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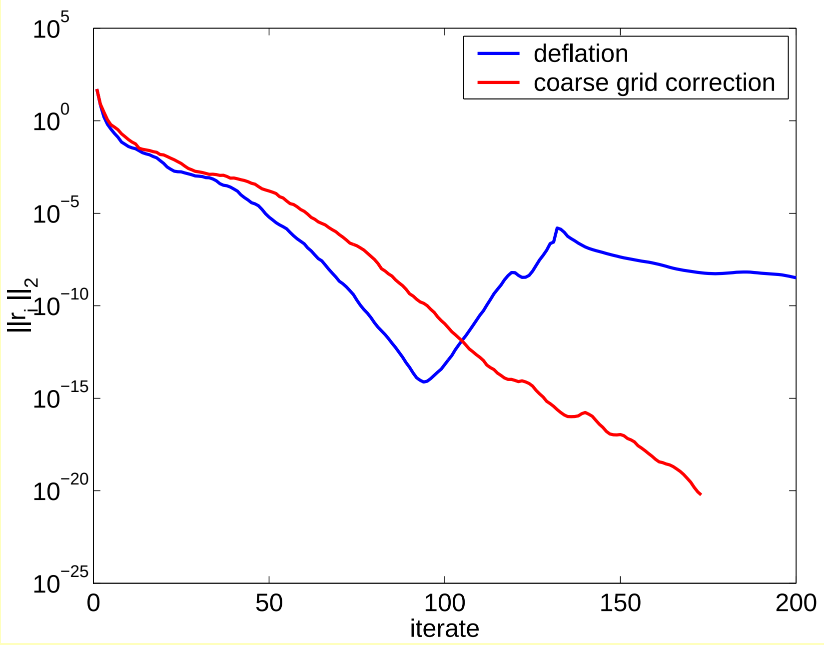
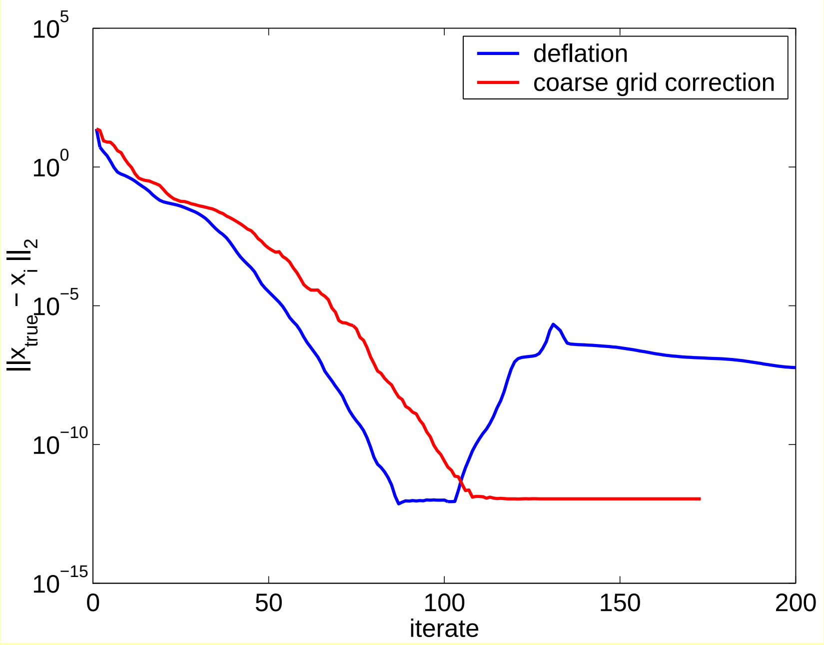


Error with Block IC



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4. Convergence of DICCG

Theorem

Let A be SPD, and A^*, $A - A^*$ be SPSD, and span$Z$ = null(A^*)
Furthermore preconditioner K is SPD and $K = LL^T$ then

$$K_{eff}(L^{-1}PAL^{-T}) \leq \frac{\lambda_n(L^{-1}AL^{-T})}{\lambda_{m+1}(L^{-1}A^*L^{-T})}.$$

1. Removing the smallest eigenvalues from the spectrum leads to the greatest improvement for PDE problems.
2. A good preconditioner for A^* may be attractive (Kaasschieter).
3. A preconditioner for A^* may increase the largest eigenvalue of $L^{-1}AL^{-T}$.

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Symmetric M-matrices (Stieltjes)

Block system:

$$\begin{bmatrix} A_{11} & \dots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{m1} & \dots & A_{mm} \end{bmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

Subdomain block Jacobi matrix $K(A) \in \mathbb{R}^{n \times n}$

$$K(A) = \begin{bmatrix} A_{11} & & \\ & \ddots & \\ & & A_{mm} \end{bmatrix}$$

Block matrices $K_{jj} = A_{jj}$ are Stieltjes matrices

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Symmetric M-matrices (Stieltjes)

Definition

Define the matrix A^* by

$$A^* = K - diag(r_1, ..., r_n),$$

where $r_i = \sum_{j=1}^{n} k_{ij}$ (i^{th} rowsum)

Block matrices A^*_{jj} have zero rowsums $\Rightarrow Z$ is a basis for null(A^*).

Theorem

If A is an irreducibly diagonally dominant Stieltjes matrix and A^* has only irreducible blocks, then the hypotheses of the previous theorem are met.

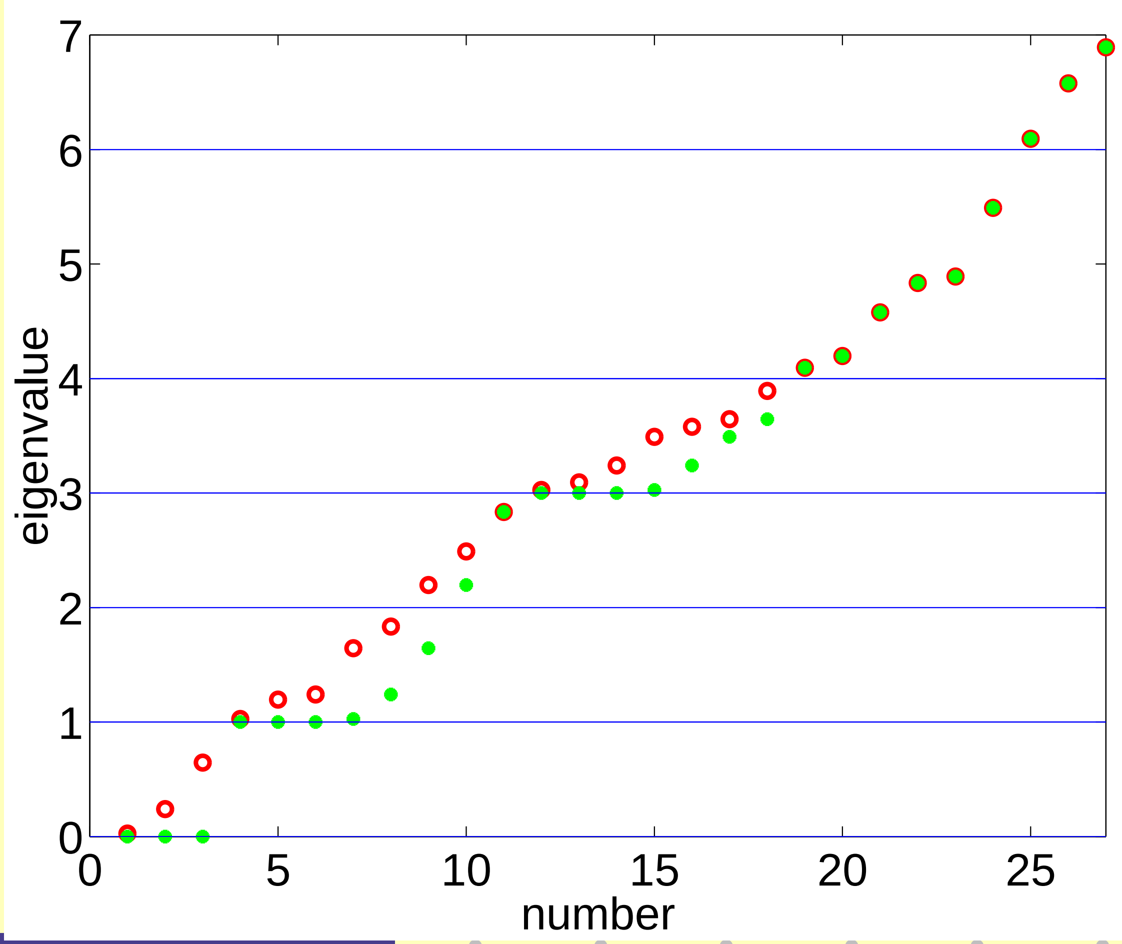
Spectral properties

$$A = \left(\begin{array}{ccc|ccc} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ \hline 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{array}\right)$$

$$A^* = \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{array}\right)$$

Spectral properties

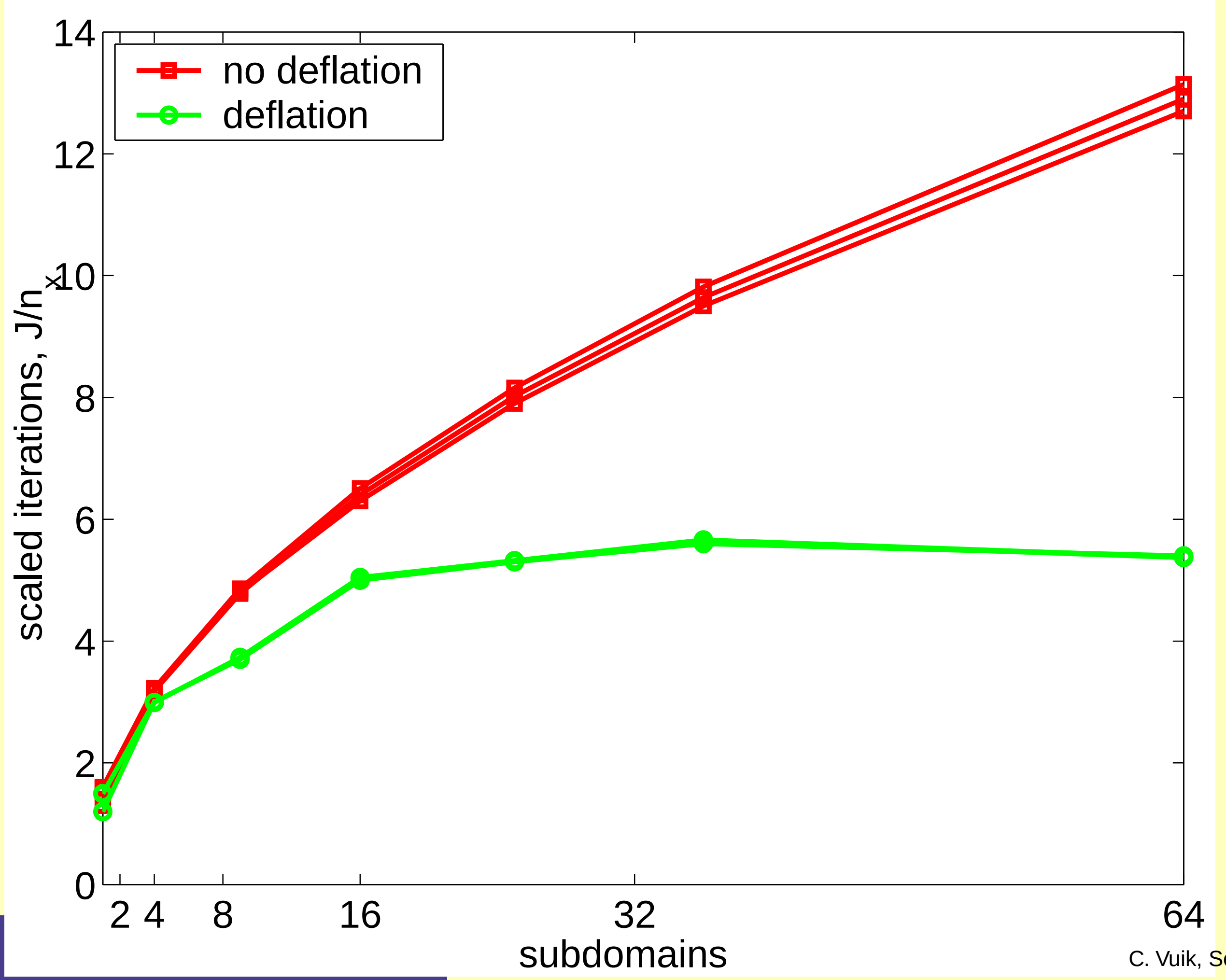
2D problem, 3×9 points, 3 blocks A A^* PA



Results (near grid independence)

Poisson equation $-\Delta u(x,y) = f$.

Iterations divided by the subdomain resolution $n_x \equiv n_y \in \{10,\ 50,\ 200\}$



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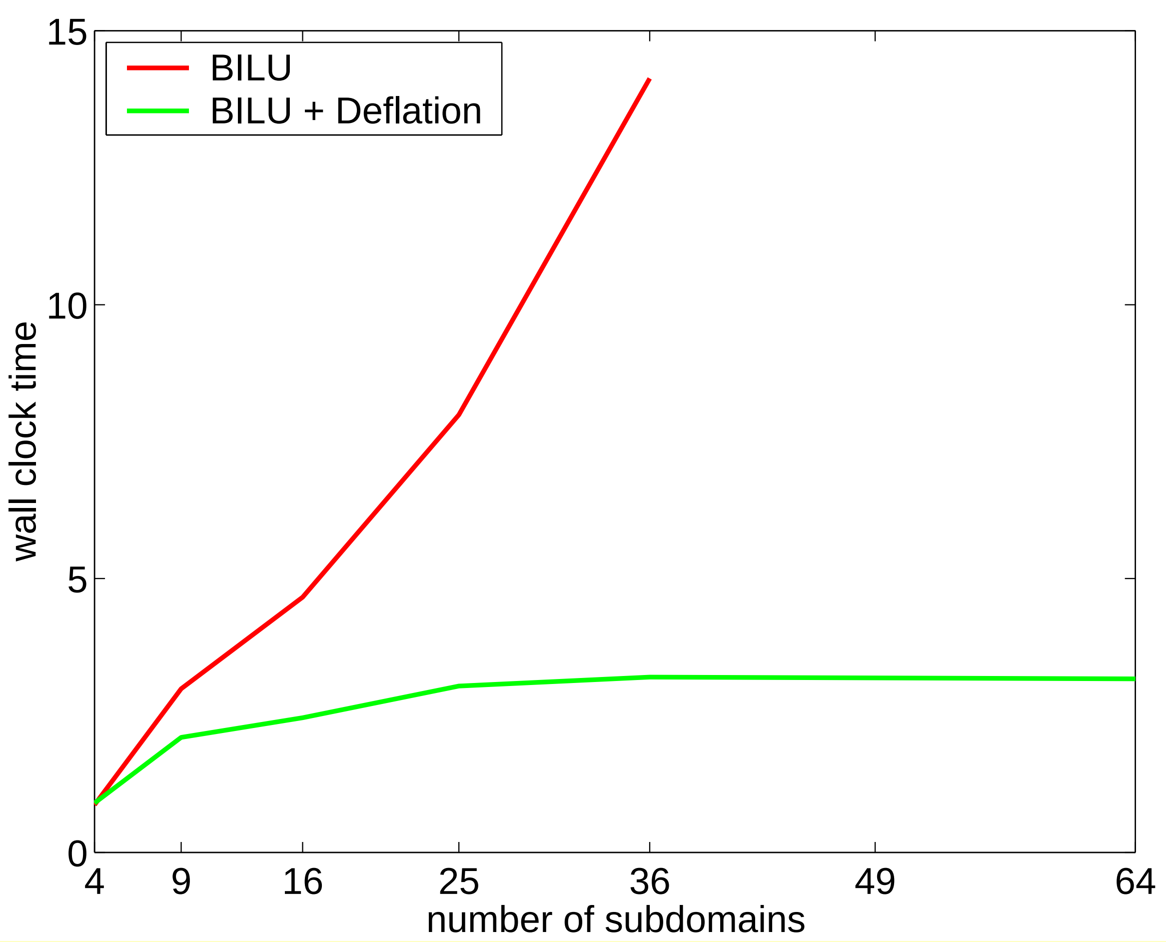
Results (parallel scalability)

subdomain grid size 50×50, wall clock time, Cray T3E



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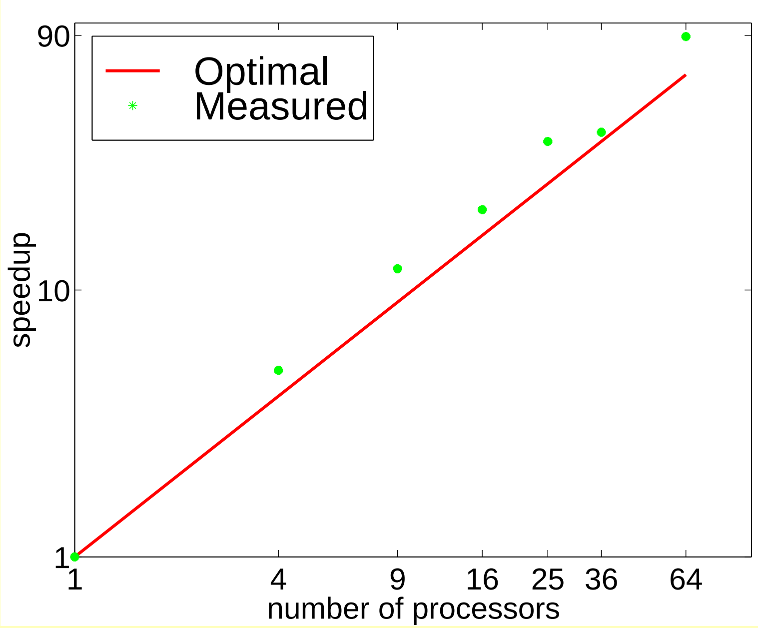
Results (parallel speedup)

480×480 grid, Cray T3E



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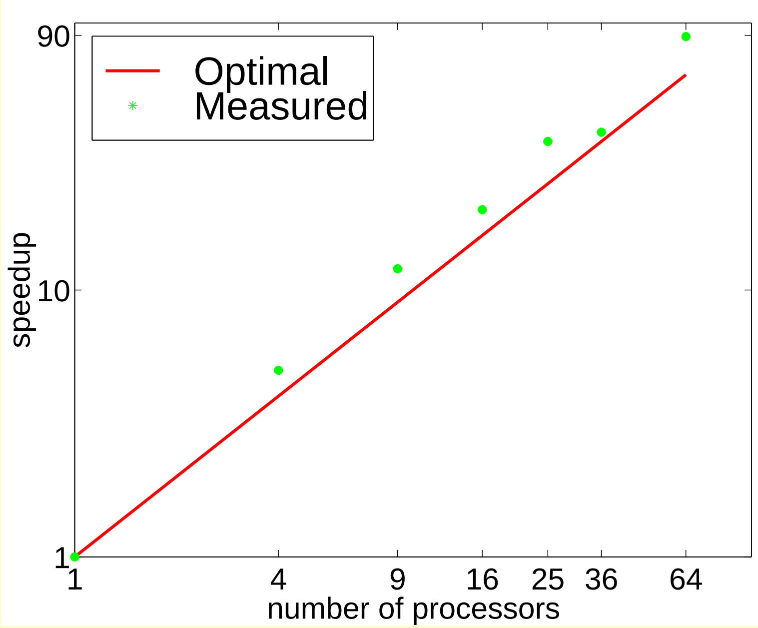
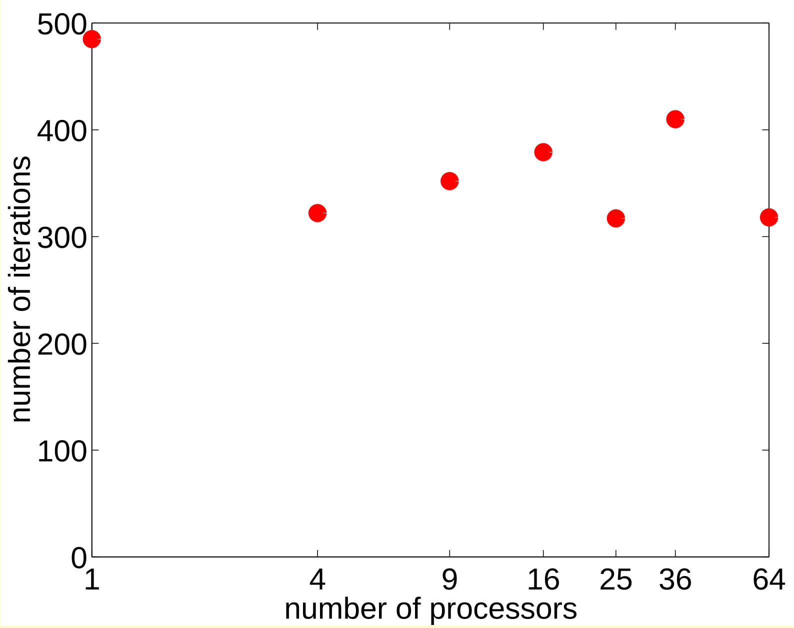
Results (parallel speedup)

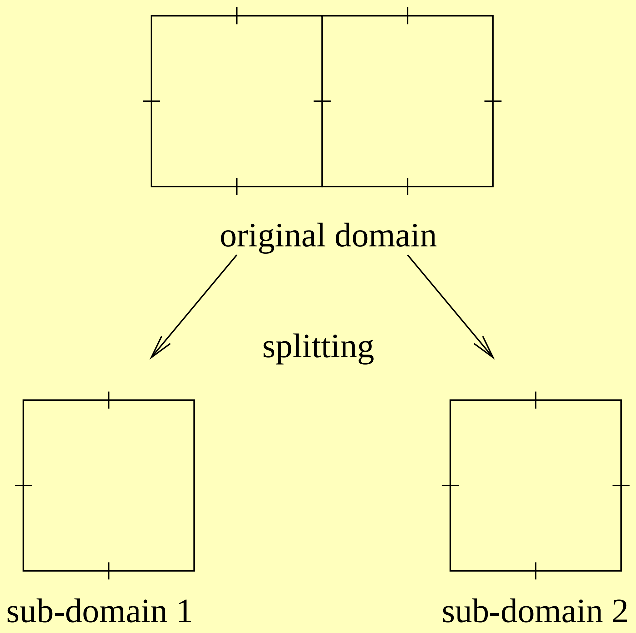
480×480 grid, Cray T3E

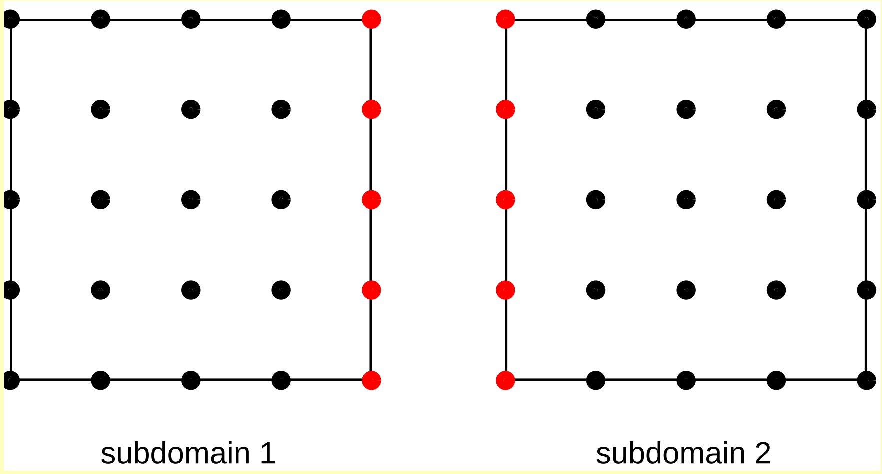




5. Vertex centered approach

Data distribution





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Variants for values at interfaces

$z_i = 1$ on Ω_i and $z_i = 0$ on $\Omega \setminus \bar{\Omega}_i$

1. **no overlap**

   $z_i = 1$ at one subdomain
   $z_i = 0$ at other subdomains

2. **complete overlap**

   $z_i = 1$ at all subdomains

3. **average overlap**

   $z_i = \frac{1}{n_{neighbors}}$ at all subdomains

4. **weighted overlap** $(-\text{div}(\sigma \nabla u) = f)$

   $z_i = \frac{\sigma(i)}{\sum \sigma(neighbors)}$

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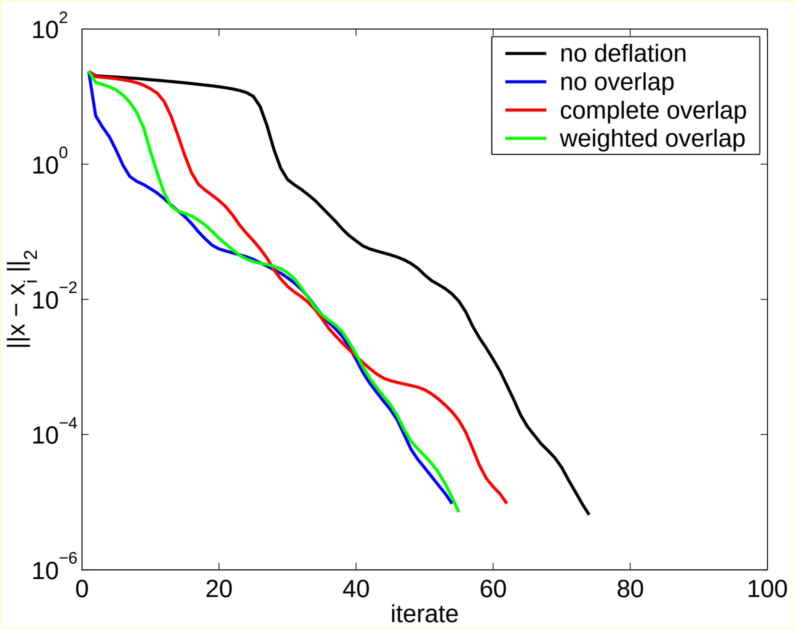
Results for constant coefficients

Error for Block IC and Deflation



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Results for constant coefficients

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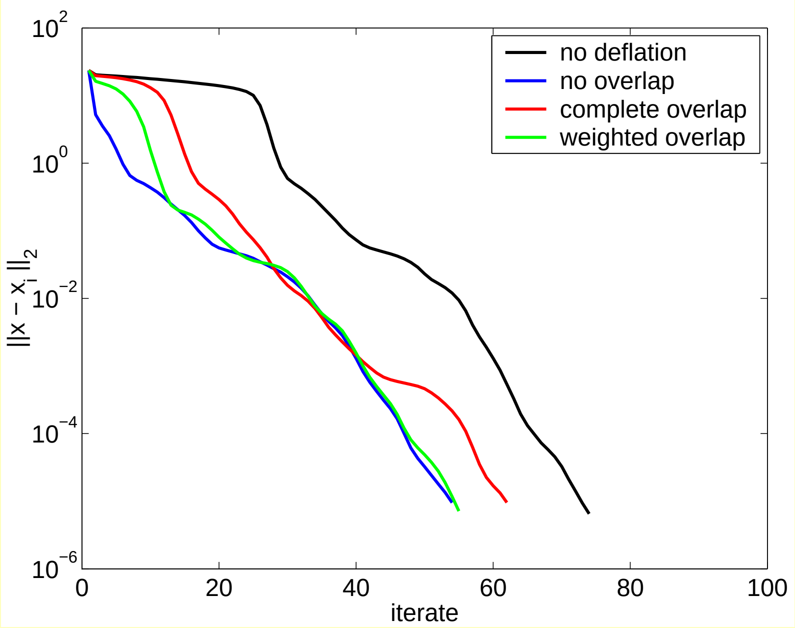
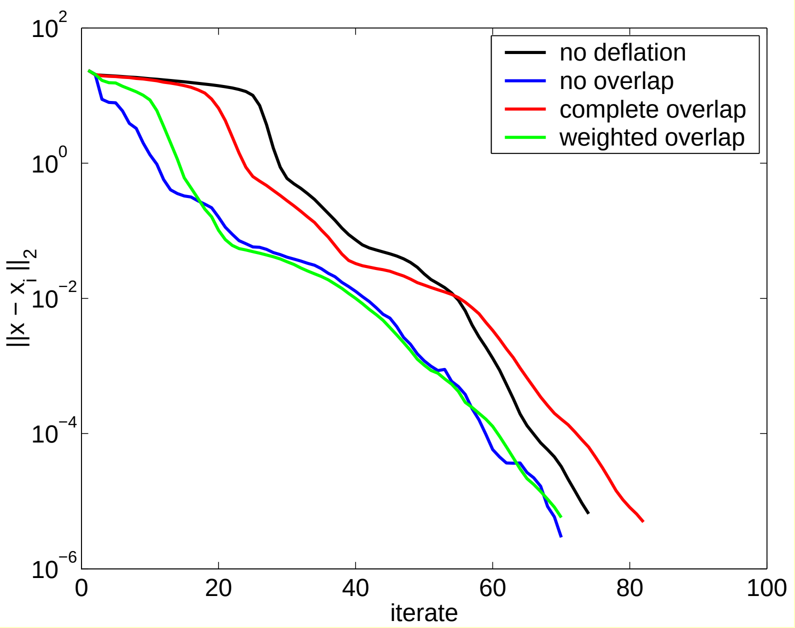


Error for Block IC and CGC



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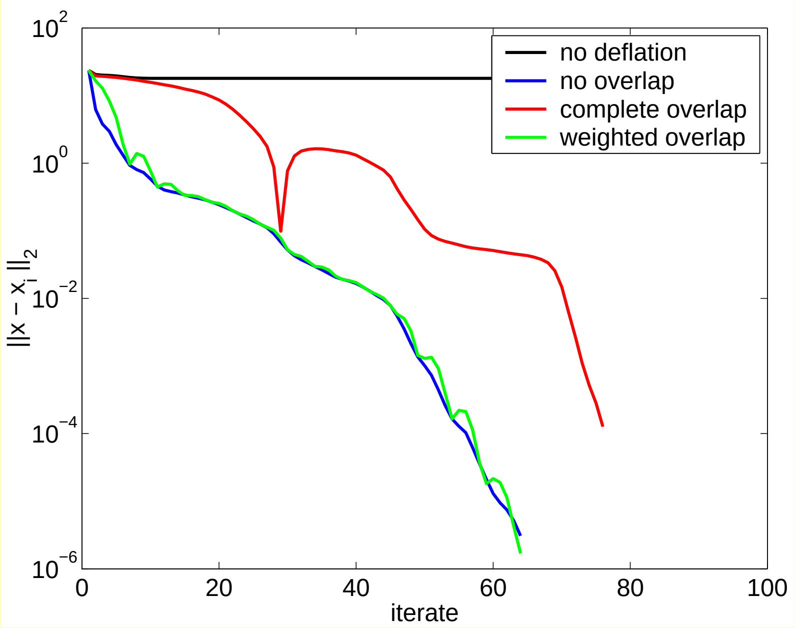
Results for disontinuous coefficients

Error for Block IC and Deflation



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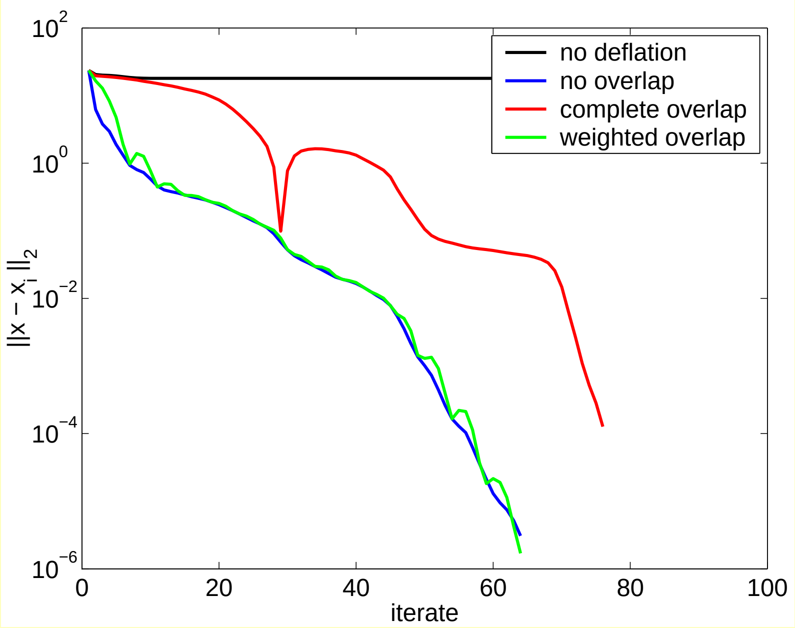
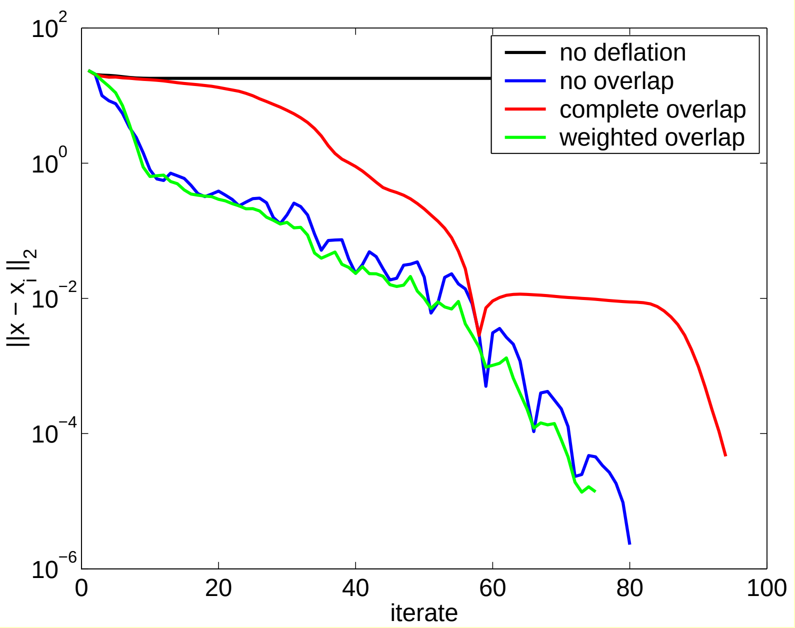
Results for disontinuous coefficients

Error for Block IC and Deflation



Error for Block IC and CGC



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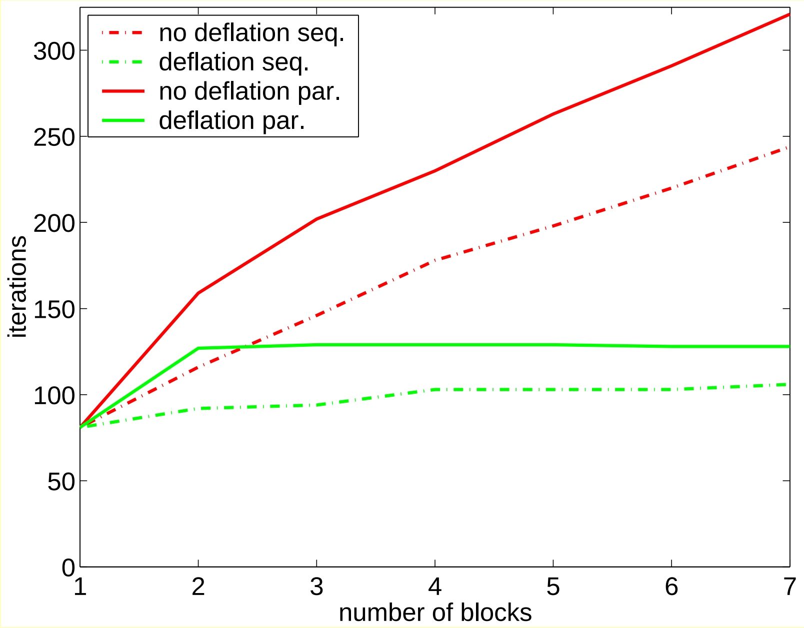
Results for Finite Element Method

Iterations



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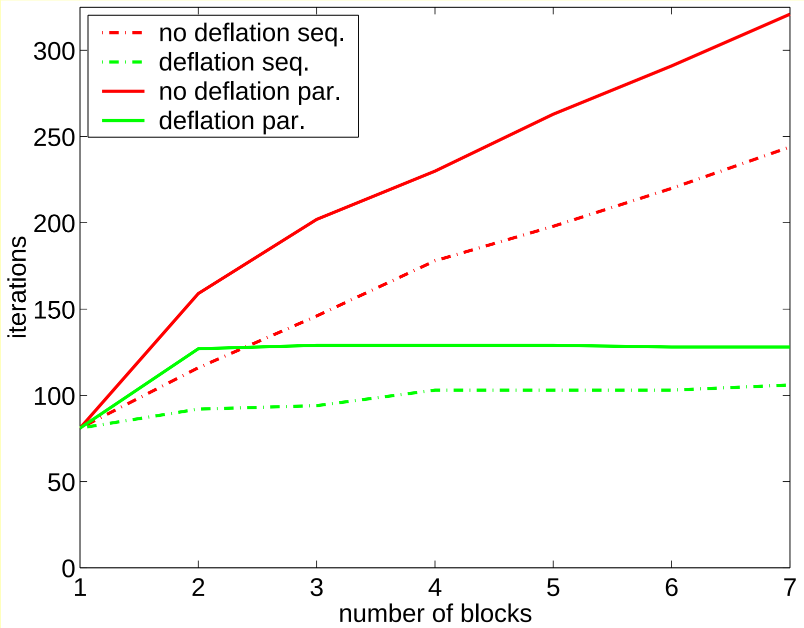
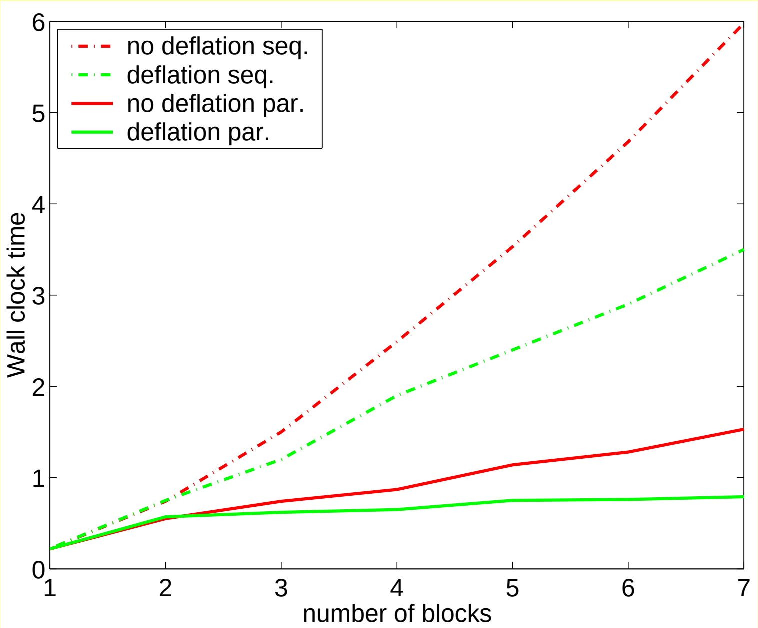
Results for Finite Element Method

Iterations



Wall clock time



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6. Conclusions

- DICCG is more efficient than CICCG.
- Block preconditioned Krylov methods combined with Deflation or CGC are well parallelizable (scalable, good speed up).
- The theoretical results give insight into an optimal subdomain distribution and a suitable preconditioner.
- For the vertex centered case, the weighted overlap strategy is optimal
- Choices for the deflation vectors lead to comparable results in DICCG and CICCG.
- DICCG is a robust and efficient method to solve diffusion problems with discontinuous coefficients.

Further information

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Overview

Krylov	Ar
Preconditioned Krylov	$L^{-T}L^{-1}Ar$
Block Preconditioned Krylov	$\sum_{i=1}^{m}(L_i^{-T}L_i^{-1})Ar$
Block Preconditioned Deflated Krylov	$\sum_{i=1}^{m}(L_i^{-T}L_i^{-1})PAr$

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