

A comparison of preconditioners for incompressible flows

C. Vuik, M. ur Rehman, and A. Segal

Delft University of Technology

Delft Institute of Applied Mathematics

Delft, The Netherlands.

EPFL, Lausanne, Laboratory of Computational Engineering

January 8, 2009

Outline

- Introduction
- Solution technique
- IDR(s) method
- Preconditioning
- Numerical experiments
- Conclusions

The incompressible Navier Stokes equation

$$-\nu \nabla^2 \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = f \quad \text{in } \Omega$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega.$$

\mathbf{u} is the fluid velocity vector

p is the pressure field

$\nu > 0$ is the kinematic viscosity coefficient ($1/Re$).

$\Omega \subset \mathbf{R}^2$ or 3 is a bounded domain with the boundary condition:

$$\mathbf{u} = \mathbf{w} \quad \text{on } \partial\Omega_D, \quad \nu \frac{\partial \mathbf{u}}{\partial \mathbf{n}} - \mathbf{n}p = 0 \quad \text{on } \partial\Omega_N.$$

Linear system

The finite element discretization give rise to a non-linear system.

Matrix form after linearization:

$$\begin{bmatrix} F & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

where $F \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{m \times n}$, $f \in \mathbb{R}^n$ and $m \leq n$

- $F = A$ in **Stokes problem**, A is vector Laplacian matrix
- $F = \nu A + N$ in **Picard linearization**, N is vector-convection matrix
- $F = \nu A + N + W$ in **Newton linearization**, W is the Newton derivative matrix
- B is the divergence matrix

Sparse linear system, Symmetric indefinite (Stokes problem), nonsymmetric otherwise.

Saddle point problem having large number of zeros on the main diagonal

Iterative Solution Techniques

- **Classical Iterative Schemes:**

Methods based on matrix splitting, generates sequence of iterations

$$x_{k+1} = M^{-1}(Nx_k + b) = Qx_k + s, \text{ where } \mathcal{A} = M - N$$

Jacobi, Gauss Seidel, SOR, SSOR

- **Krylov Subspace Methods:**

$$x_{k+1} = x_k + \alpha_k p_k$$

Some well known methods are

CGNR[1975], QMR[1991], CGS[1989], Bi-CGSTAB[1992], GMRES[1986],
GMRESR[1994], GCR[1986], IDR(s)[2007]

IDR and IDR(s) (Induced Dimension Reduction)

- Sonneveld developed IDR the 1970's. IDR is a finite termination Krylov method for solving nonsymmetric linear systems.
- Analysis showed that IDR can be viewed as Bi-CG combined with linear minimal residual steps.
- This discovery led to the development of first CGS, and later of Bi-CGSTAB (by van der Vorst).

IDR and IDR(s) (continued)

- As a result of these developments the basic IDR-idea was abandoned for the Bi-CG-approach.
- Recently, Sonneveld and van Gijzen discovered that the IDR-approach was abandoned too soon and proposed a generalization of IDR: IDR(s).
- P. SONNEVELD AND M.B. VAN GIJZEN IDR(s): a family of simple and fast algorithms for solving large nonsymmetric systems of linear equations
SIAM J. Sci. Comput., 31, pp. 1035-1062, 2008

The IDR approach for solving $Ax = b$

Generate residuals $r_n = b - Ax_n$ that are in subspaces \mathcal{G}_j of decreasing dimension.

These nested subspaces are related by

$$\mathcal{G}_j = (\mathbf{I} - \omega_j \mathbf{A})(\mathcal{G}_{j-1} \cap \mathcal{S})$$

where

- \mathcal{S} is a fixed proper subspace of \mathbb{C}^N . \mathcal{S} can be taken to be the orthogonal complement of s randomly chosen vectors $p_i, i = 1 \cdots s$.
- The parameters $\omega_j \in \mathbb{C}$ are non-zero scalars.

It can be proved that ultimately $r_n \in \{\mathbf{0}\}$ (IDR theorem).

IDR versus Bi-CG

The $\text{IDR}(s)$ forces the residual to be in an increasingly small subspace, while Bi-CG constructs a residual in an increasingly large subspace. Yet, $\text{IDR}(s)$ is closely related to:

- Bi-CGSTAB: $\text{IDR}(1)$ and Bi-CGSTAB are mathematically equivalent.
- ML(k)BiCGSTAB (Yeung and Chan, 1999): This method generalizes Bi-CGSTAB using multiple 'shadow residuals'. Mathematically $\text{IDR}(s)$ and ML(k)BiCGSTAB differ in the selection of the parameters ω_j .

$\text{IDR}(s)$ uses simpler recurrences, less vector operations and memory than ML(k)BiCGSTAB, and is more flexible (e.g. to avoid break down).

Prototype IDR(s) algorithm.

```
while  $\|\mathbf{r}_n\| > TOL$  or  $n < MAXIT$  do  
  for  $k = 0$  to  $s$  do  
    Solve  $\mathbf{c}$  from  $\mathbf{P}^H d\mathbf{R}_n \mathbf{c} = \mathbf{P}^H \mathbf{r}_n$   
     $\mathbf{v} = \mathbf{r}_n - d\mathbf{R}_n \mathbf{c}; \mathbf{t} = \mathbf{A}\mathbf{v};$   
    if  $k = 0$  then  
       $\omega = (\mathbf{t}^H \mathbf{v}) / (\mathbf{t}^H \mathbf{t});$   
    end if  
     $d\mathbf{r}_n = -d\mathbf{R}_n \mathbf{c} - \omega \mathbf{t}; d\mathbf{x}_n = -d\mathbf{X}_n \mathbf{c} + \omega \mathbf{v};$   
     $\mathbf{r}_{n+1} = \mathbf{r}_n + d\mathbf{r}_n; \mathbf{x}_{n+1} = \mathbf{x}_n + d\mathbf{x}_n;$   
     $n = n + 1;$   
     $d\mathbf{R}_n = (d\mathbf{r}_{n-1} \cdots d\mathbf{r}_{n-s}); d\mathbf{X}_n = (d\mathbf{x}_{n-1} \cdots d\mathbf{x}_{n-s});$   
  end for  
end while
```

More information

More information: <http://ta.twi.tudelft.nl/nw/users/gijzen/IDR.html>

- P. SONNEVELD AND M.B. VAN GIJZEN IDR(s): a family of simple and fast algorithms for solving large nonsymmetric systems of linear equations
SIAM J. Sci. Comput., 31, pp. 1035-1062, 2008
- The relation of IDR(s) with Bi-CGSTAB, and how to derive generalizations of Bi-CGSTAB using IDR-ideas can be found in: Bi-CGSTAB as an induced dimension reduction method (with Sleijpen).
- A high quality IDR(s) implementation is described in: An elegant IDR(s) variant that efficiently exploits bi-orthogonality properties.

Preconditioning

A linear system $Ax = b$ is transformed into $P^{-1}Ax = P^{-1}b$ such that

- $P \approx A$
- Eigenvalues of $P^{-1}A$ are more clustered than A
- $Pz = r$ cheap to compute

Several approaches, we will discuss here

- Block triangular preconditioners
(LSC, Least Squares Commutator)
- SIMPLE-type block preconditioners
- Preconditioners comparison (with SILU[Rehman2008])
- Preconditioned IDR(s) and Bi-CGSTAB comparison

Block preconditioners

Block triangular preconditioner

$$\begin{bmatrix} F & B^T \\ B & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ BF^{-1} & I \end{bmatrix} \underbrace{\begin{bmatrix} F & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} I & F^{-1}B^T \\ 0 & I \end{bmatrix}}_{P_t}$$
$$P_t = \begin{bmatrix} F & B^T \\ 0 & S \end{bmatrix}, \quad S = -BF^{-1}B^T \text{ (Schur complement matrix)}$$

Subsystems: solve z_2 from $Sz_2 = r_2$, and z_1 from $Fz_1 = r_1 - B^T z_2$

Block preconditioners

Generalized eigenvalue problem

$$\begin{bmatrix} F & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \lambda \begin{bmatrix} F & B^T \\ 0 & S \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix},$$

This eigenvalue problem has the eigenvalues $\lambda = 1$ of multiplicity n and the remaining eigenvalues depend on the Schur complement

$$BF^{-1}B^T p = \mu_i S p,$$

$\mu_i = 1$ if $S = BF^{-1}B^T$, however

- In practice F^{-1} and S^{-1} are expensive.
- F^{-1} is obtained by an approximate solve
- S is first approximated and then solved inexactly

Block preconditioners

Least squares commutator (LSC) preconditioner

[Elman, Howle, Shadid, Silvester and Tuminaro, 2002]

$$S \approx -(BQ^{-1}B^T)(BQ^{-1}FQ^{-1}B^T)^{-1}(BQ^{-1}B^T)$$

Q is the diagonal of the velocity mass matrix.

- Two Poisson solves
- One velocity solve

SIMPLE(R) preconditioner

$$\begin{pmatrix} u^* \\ p^* \end{pmatrix} = \begin{pmatrix} u^k \\ p^k \end{pmatrix} + M_L^{-1} B_L \left(\begin{pmatrix} r_u \\ r_p \end{pmatrix} - A \begin{pmatrix} u^k \\ p^k \end{pmatrix} \right),$$
$$\begin{pmatrix} u^{k+1} \\ p^{k+1} \end{pmatrix} = \begin{pmatrix} u^* \\ p^* \end{pmatrix} + B_R M_R^{-1} \left(\begin{pmatrix} r_u \\ r_p \end{pmatrix} - A \begin{pmatrix} u^* \\ p^* \end{pmatrix} \right).$$

Where

$$B_R = \begin{pmatrix} I & -D^{-1}B^T \\ 0 & I \end{pmatrix}, \quad M_R = \begin{pmatrix} F & 0 \\ B & \hat{S} \end{pmatrix} \text{ and}$$
$$B_L = \begin{pmatrix} I & 0 \\ -BD^{-1} & I \end{pmatrix}, \quad M_L = \begin{pmatrix} F & B^T \\ 0 & \hat{S} \end{pmatrix}.$$

SIMPLE-type preconditioner

Assuming u^* and p^* equal zero, the steps in SIMPLE reduce to:

SIMPLE preconditioner[Vuik 2000]:

1. Solve $Fu^* = r_u$.
 2. Solve $\hat{S}\delta p = r_p - Bu^*$.
 3. update $u = u^* - D^{-1}B^T\delta p$.
 4. update $p = \delta p$.
- One Poisson solve
 - One velocity solve

SIMPLE-type preconditioner

Assuming u^k and p^k equal zero, the steps in SIMPLER reduce to:

SIMPLER preconditioner:

1. Solve $\hat{S}p^* = r_p - BD^{-1}r_u$
2. Solve $Fu^* = r_u - B^T p^*$.
3. Solve $\hat{S}\delta p = r_p - Bu^*$.
4. update $u = u^* - D^{-1}B^T\delta p$.
5. update $p = p^* + \delta p$.

Lemma: In the SIMPLER preconditioner/algorithm, both variants (one or two velocity solves) are identical.

- Two Poisson solve
- One velocity solve
- Gives faster convergence than SIMPLE

Improvements in SIMPLE-type preconditioners

- Relaxation parameter
- hSIMPLER
- MSIMPLER

Improvements in SIMPLE(R) preconditioners

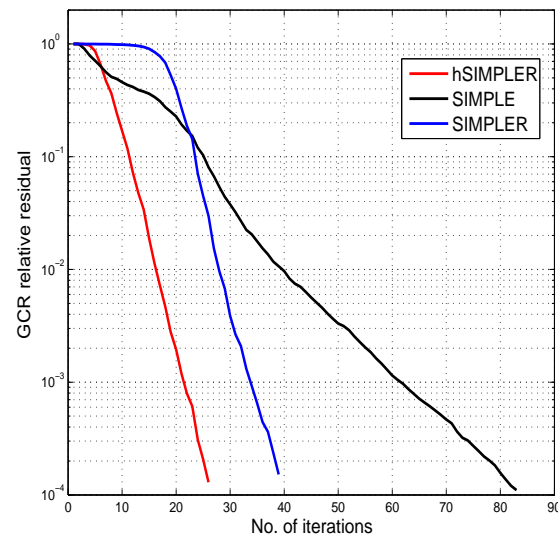
Relaxation parameter:

- Under-relaxation is well-known in SIMPLE-type methods.
- In SIMPLE preconditioner, velocity relaxation has no effect on the convergence, therefore only pressure is under-relaxed by a factor ω .
 $p = p^* + \omega\delta p$, where ω is chosen between 0 and 1.
- ω has no effect on convergence with SIMPLER due to extra pressure correction step.
- Faster convergence is achieved in some cases.
- Choice of ω is currently based on trial an error.

Improvements in SIMPLE(R) preconditioners

hSIMPLER preconditioner:

In hSIMPLER (hybrid SIMPLER), first iteration of Krylov method preconditioned with SIMPLER is done with SIMPLE and SIMPLER is employed afterwards.



- Faster convergence than SIMPLER
- Effective in the Stokes problem

Improvements in SIMPLE(R) preconditioners

MSIMPLER preconditioner:

Making the following changes in SIMPLER leads to the MSIMPLER preconditioner.

$$\text{LSC: } \hat{S} \approx -(B\hat{Q}_u^{-1}B^T)(B\hat{Q}_u^{-1}\underbrace{F\hat{Q}_u^{-1}}_{\approx I}B^T)^{-1}(B\hat{Q}_u^{-1}B^T)$$

assuming $F\hat{Q}_u^{-1} \approx I$ (time dependent problems with a small time step)

$$\hat{S} = -B\hat{Q}_u^{-1}B^T$$

MSIMPLER uses this approximation for the Schur complement and updates scaled with \hat{Q}_u^{-1} .

- Convergence better than other variants of SIMPLE
- Cheaper than SIMPLER (in construction) and LSC (per iteration)

Numerical Experiments

- Driven Cavity flow (2D)
- Backward facing step flow (2D and 3D)
- Q2-Q1 finite element discretization [Taylor, Hood - 1973]
- Q2-P1 finite element discretization [Crouzeix, Raviart - 1973]
- GCR(20), Bi-CGSTAB, GMRES, IDR(s)
- The iteration is stopped if the linear systems satisfy $\frac{\|r^k\|_2}{\|b\|_2} \leq tol$,
- Experiments done with IFISS (Matlab program) and SEPRAN (industrial FEM code)

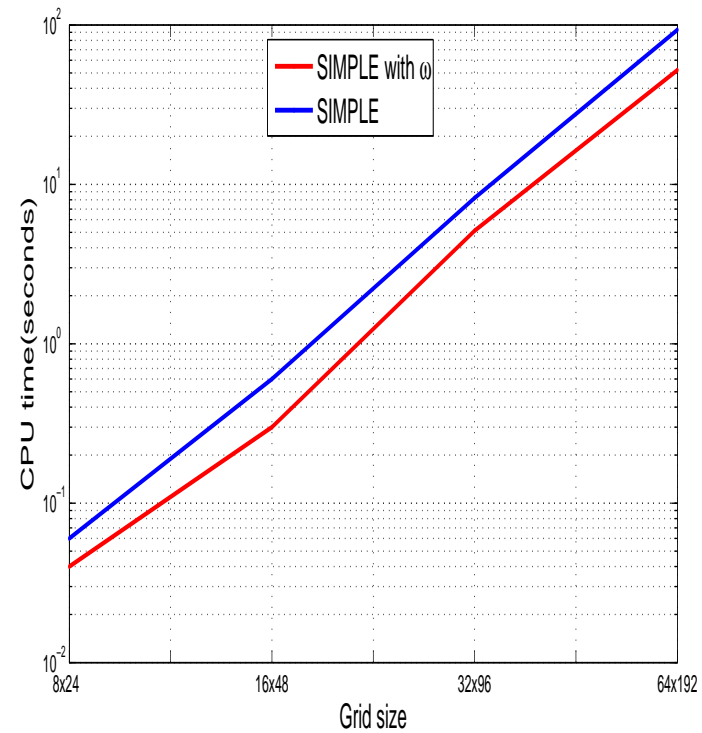
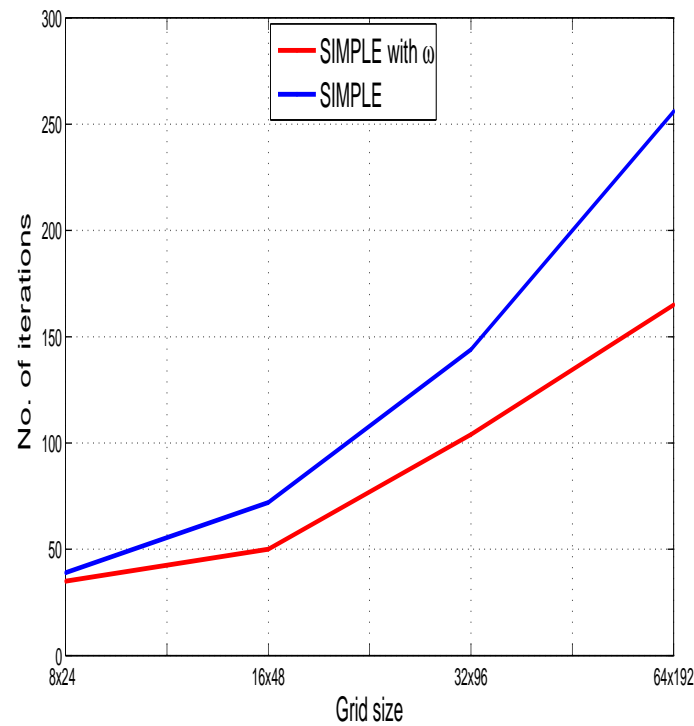
Numerical Experiments (SIMPLE type preconditioners)

Stokes backward facing step solved with preconditioned GCR(20) with *accuracy* of 10^{-6} , PCG used as an inner solver (SEPRAN), Green: **Low inner accuracy**, Yellow: **High inner accuracy**

Grid	SIMPLE	SIMPLER	hSIMPLER	MSIMPLER
	iter. (ts)	iter. (ts)	iter. (ts)	iter. (ts)
8 × 24	39(0.06)	26(0.05)	19(0.03)	11(0.02)
	37(0.14)	19(0.07)	17(0.06)	12(0.05)
16 × 46	72(0.6)	42(0.5)	31(0.34)	12(0.1)
	68(1.94)	30(0.86)	24(0.68)	15(0.44)
32 × 96	144(8.2)	NC	44(5.97)	16(0.9)
	117(34)	114(32)	37(10.6)	20(5.75)
64 × 192	256(93)	NC	89(141)	23(8.5)
	230(547)	NC	68(161)	25(60)

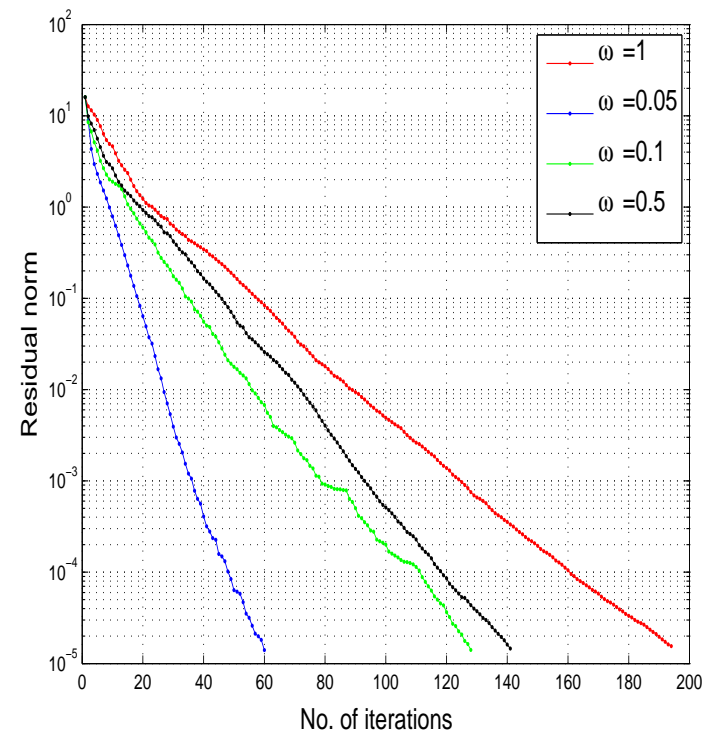
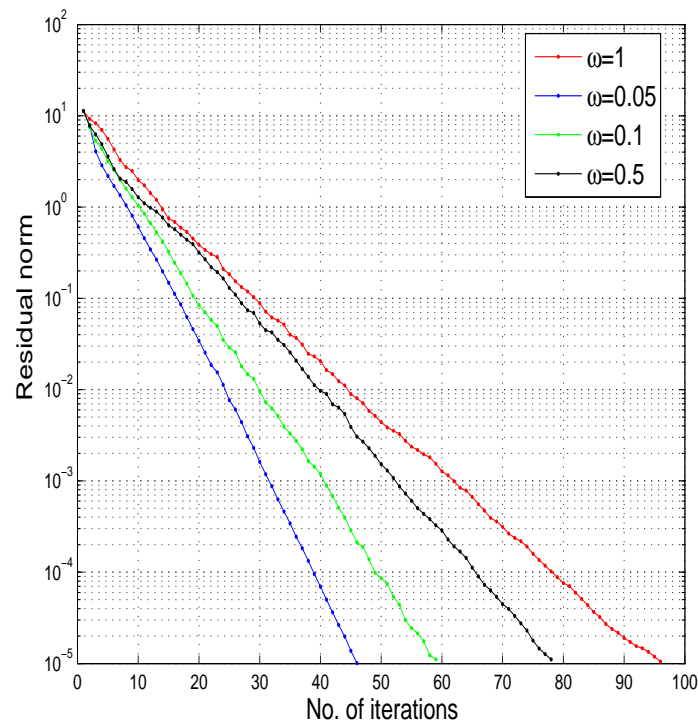
Numerical Experiments (SIMPLE type preconditioners)

SIMPLE with relaxation parameter



Numerical Experiments (SIMPLE type preconditioners)

Effect of relaxation parameter: The Stokes problem solved in Q2-Q1 discretized driven cavity problem with varying ω : 32×32 grid (Left), 64×64 grid (Right).



Numerical Experiments (overall comparison)

3D Backward facing step: Preconditioners used in the Stokes problem with preconditioned GCR(20) with *accuracy* of 10^{-6} (SEPRAN) using Q2-Q1 hexahedrons

Grid	SIMPLE	LSC	MSIMPLER
iter. (t_s) $\frac{\text{in-it-}u}{\text{in-it-}p}$			
$8 \times 8 \times 16$	44(4) $\frac{97}{342}$	16(1.9) $\frac{41}{216}$	14(1.4) $\frac{28}{168}$
$16 \times 16 \times 32$	84(107) $\frac{315}{1982}$	29(51) $\frac{161}{1263}$	17(21) $\frac{52}{766}$
$24 \times 24 \times 48$	99(447) $\frac{339}{3392}$	26(233) $\frac{193}{2297}$	17(77) $\frac{46}{1116}$
$32 \times 32 \times 40$	132(972) $\frac{574}{5559}$	37(379) $\frac{233}{2887}$	20(143) $\frac{66}{1604}$

Numerical Experiments (overall comparison)

3D Backward facing step: Preconditioners used in solving the Navier-Stokes problem with preconditioned GCR(20) with *accuracy* of 10^{-2} (SEPRAN) using Q2-Q1 hexahedrons

Re	LSC	MSIMPLER	SILU
	GCR iter. (t_s)	GCR iter. (t_s)	Bi-CGSTAB iter. (t_s)
$16 \times 16 \times 32$			
100	173(462)	96(162)	321(114)
200	256(565)	145(223)	461(173)
400	399(745)	235(312)	768(267)
$32 \times 32 \times 40$			
100	240(5490)	130(1637)	1039(1516)
200	NC	193(2251)	1378(2000)
400	675(11000)	295(2800)	1680(2450)

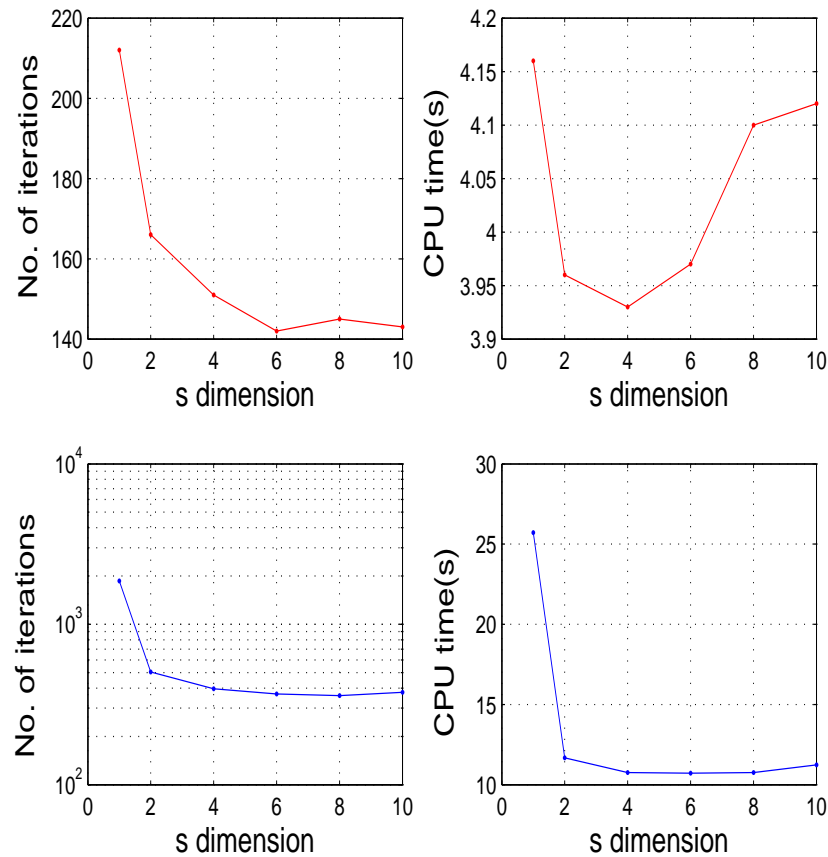
Numerical Experiments (overall comparison)

3D Lid driven cavity problem (tetrahedrons):The Navier-Stokes problem is solved with accuracy 10^{-4} , a linear system at each Picard step is solved with accuracy 10^{-2} using preconditioned Krylov subspace methods. Bi-CGSTAB is used as inner solver in block preconditioners(SEPRAN)

Re	LSC	MSIMPLER	SILU
	GCR iter. (t_s)	GCR iter. (t_s)	Bi-CGSTAB iter. (t_s)
$16 \times 16 \times 16$			
20	30(20)	20(16)	144(22)
50	57(37)	37(24)	234(35)
100	120(81)	68(44)	427(62)
$32 \times 32 \times 32$			
20	38(234)	29(144)	463(353)
50	87(544)	53(300)	764(585)
100	210(1440)	104(654)	1449(1116)

Numerical Experiments (IDR(s))

IDR(s): Top: 32×32 , Bottom: 64×64 driven cavity Stokes flow problem



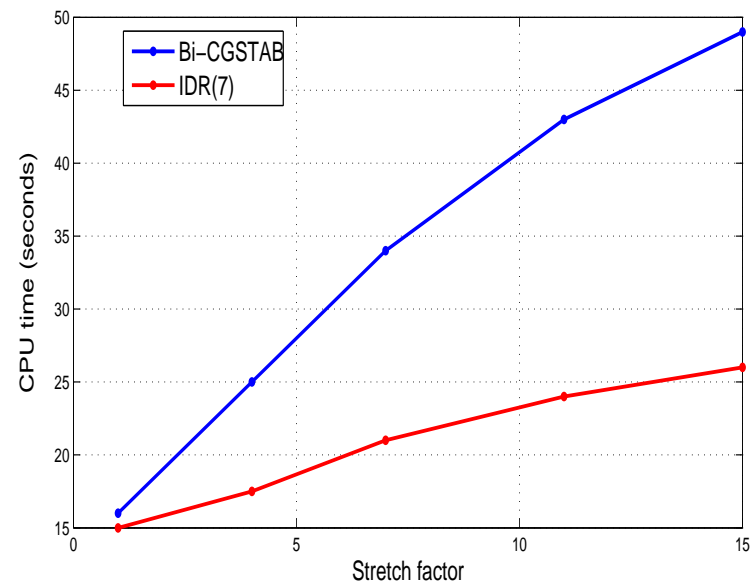
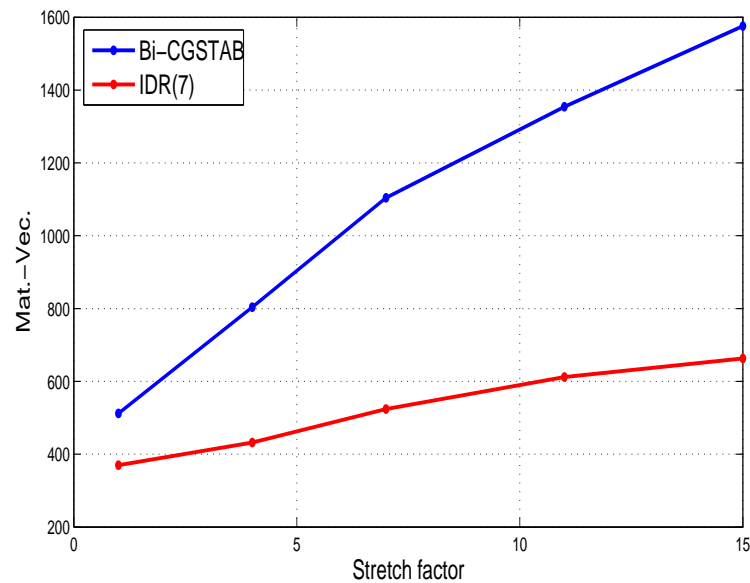
Numerical Experiments (IDR(s) vs Bi-CGSTAB)

SILU preconditioner: Comparison of iterative methods for increasing grid size for the driven cavity Stokes flow problem.

Grid	Bi-CGSTAB	IDR(4)
	Mat.-Vec. (ts)	Mat.-Vec. (ts)
16×16	38(0.01)	33(0.01)
32×32	90(0.14)	75(0.14)
64×64	214(1.6)	159(1.4)
128×128	512(16)	404(15)
256×256	1386(183)	1032(156)

Numerical Experiments (IDR(s) vs Bi-CGSTAB)

SILU preconditioned: Comparison of iterative methods for increasing stretch factor for the driven cavity Stokes problem.



Numerical Experiments (IDR(s) vs Bi-CGSTAB)

SILU preconditioned: Comparison of iterative methods for the backward facing step Stokes problem.

Grid	Bi-CGSTAB	IDR(s)	
	Mat.-Vec.(ts)	Mat.-Vec.(ts)	s
32×96	214(1.3)	168(1.26)	4
64×96	NC	597(7.7)	4
96×96	NC	933(18)	4
128×96	NC	1105(31)	8

Numerical Experiments (IDR(s) vs Bi-CGSTAB(l))

SILU preconditioned: Comparison of iterative methods

Driven Cavity Stokes problem, stretch factor 10

Grid	Bi-CGSTAB(l)		IDR(s)	
	Mat.-Vec.(ts)	l	Mat.-Vec.(ts)	s
128×128	1104(36.5)	4	638(24.7)	6
256×256	5904(810)	6	1749(307)	8

Channel flow Stokes problem, length 100

Grid	Bi-CGSTAB(l)		IDR(s)	
	Mat.-Vec.(ts)	l	Mat.-Vec.(ts)	s
64×64	1520(12)	4	938(8.7)	8
128×128	NC	6	8224(335)	8

Conclusions

- Relaxation parameter improves performance of the SIMPLE preconditioner.
- hSIMPLER shows faster convergence than SIMPLER.
- MSIMPLER is at present the fastest of all SIMPLE-type preconditioners.
- In contrast with SIMPLER and hSIMPLER , SIMPLE and MSIMPLER are not sensitive to the accuracies that are used for the inner solvers.
- In all our experiments MSIMPLER proved to be cheaper than LSC. This concerns both the number of outer iterations, inner iterations and CPU time.
- In our experiments, MSIMPLER proved to be cheaper than SILU, especially when the problem is solved with high accuracy.
- $IDR(s)$ is faster and more robust than Bi-CGSTAB.
- $IDR(s)$ is faster and more robust than Bi-CGSTAB(l).

References

- ★ C. Vuik and A. Saghir and G.P. Boerstoel, "The Krylov accelerated SIMPLE(R) method for flow problems in industrial furnaces," *International Journal for Numerical methods in fluids*, 33 pp. 1027-1040, 2000.
- ★ M. ur Rehman and C. Vuik and G. Segal, "A comparison of preconditioners for incompressible Navier-Stokes solvers," *International Journal for Numerical methods in fluids*, 57, pp. 1731-1751, 2008.
- ★ M. ur Rehman and C. Vuik and G. Segal, "SIMPLE-type preconditioners for the Oseen problem," *International Journal for Numerical methods in fluids*, to appear.
- ★ Peter Sonneveld and Martin B. van Gijzen, "IDR(s): a family of simple and fast algorithms for solving large nonsymmetric linear systems," *SIAM J. Sci. Comput.*, 31, pp. 1035-1062, 2008