On complex shifted Laplace preconditioners for the vector Helmholtz equation

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Wim Mulder, René Edouard Plessix, Paul Urbach, Alex Kononov, Dwi Riyanti and Pim Hooghiemstra

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1. Introduction

The Helmholtz problem is defined as follows

\[-\partial_{xx} u - \partial_{yy} u - z_1 k^2(x, y) u = f, \quad \text{in} \quad \Omega,\]

Boundary conditions on \( \Gamma = \partial \Omega, \)

where:

- \( z_1 = \alpha_1 + i\beta_1 \) and \( k(x, y) \) is the wavenumber
- for "solid" boundaries: Dirichlet/Neumann
- for "fictitious" boundaries: Sommerfeld \( \frac{du}{dn} - ik u = 0 \)
- Perfectly Matched Layer (PML)
- Absorbing Boundary Layer (ABL)
Discretization

In general: Finite Difference/Finite Element Methods.

Particular to the present case: 5-point Finite Difference stencil, $O(h^2)$.

Linear system

$$Ax = b, \quad A \in \mathbb{C}^{N \times N}, \quad b, x \in \mathbb{C}^N,$$
Discretization

In general: Finite Difference/Finite Element Methods.

Particular to the present case: 5-point Finite Difference stencil, $\mathcal{O}(h^2)$.

Linear system

$$Ax = b, \quad A \in \mathbb{C}^{N \times N}, \quad b, x \in \mathbb{C}^N,$$

$A$ is a sparse, highly indefinite matrix for practical values of $k$.

Special property $A = A^T$.

For high resolution a very fine grid is required: $30 - 60$ grid-points per wavelength (or $\approx 5 - 10 \times k$) $\rightarrow A$ is extremely large!
Characteristic properties of the problem

- $A \in \mathbb{C}^{N \times N}$ is sparse
- wavenumber $k$ and grid size $N$ are very large
- wavenumber $k$ varies discontinuously
- real parts of the eigenvalues of $A$ are positive and negative
Application: geophysical survey

Marmousi model (hard)
Application: geophysical survey

Marmousi model (hard)
2. Spectrum of shifted Laplacian preconditioners

Operator based preconditioner $P$ is based on a discrete version of

$$-\partial_{xx}u - \partial_{yy}u - (\alpha_2 + i\beta_2)k^2(x, y)u = f, \quad \text{in } \Omega.$$ 

appropriate boundary conditions

Matrix $P^{-1}$ is approximated by an inner iteration process.

<table>
<thead>
<tr>
<th>$\alpha_2$</th>
<th>$\beta_2$</th>
<th>Type</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>Laplacian</td>
<td>Bayliss and Turkel, 1983</td>
</tr>
<tr>
<td>-1</td>
<td>0</td>
<td>Definite Helmholtz</td>
<td>Laird, 2000</td>
</tr>
<tr>
<td>0</td>
<td>-1</td>
<td>Complex</td>
<td>Erlangga, Vuik and Oosterlee, 2004, 2006</td>
</tr>
<tr>
<td>1</td>
<td>-0.5</td>
<td>'Optimal'</td>
<td></td>
</tr>
</tbody>
</table>
After discretization we obtain the (un)damped Helmholtz operator

\[ L - z_1 M, \]

where \( L \) and \( M \) are SPD matrices and \( z_1 = \alpha_1 + i\beta_1 \).

The preconditioner is then given by

\[ L - z_2 M, \]

where \( z_2 = \alpha_2 + i\beta_2 \) is chosen such that

- systems with the preconditioner are easy to solve,
- the outer Krylov process is accelerated significantly.
Spectrum of shifted Laplacian preconditioners

References: Manteuffel, Parter, 1990; Yserentant, 1988

Since $L$ and $M$ are SPD we have the following eigenpairs

$$Lv_j = \lambda_j M v_j, \text{ where, } \lambda_j \in \mathbb{R}^+$$

The eigenvalues $\sigma_j$ of the preconditioned matrix satisfy

$$(L - z_1 M)v_j = \sigma_j (L - z_2 M)v_j.$$ 

**Theorem 1**

Provided that $z_2 \neq \lambda_j$, the relation

$$\sigma_j = \frac{\lambda_j - z_1}{\lambda_j - z_2}$$

holds.
Theorem 2
If $\beta_2 = 0$, the eigenvalues $\sigma_r + i\sigma_i$ are located on the straight line in the complex plane given by

$$\beta_1 \sigma_r - (\alpha_1 - \alpha_2) \sigma_i = \beta_1.$$
Spectrum of shifted Laplacian preconditioners

Theorem 2
If $\beta_2 = 0$, the eigenvalues $\sigma_r + i\sigma_i$ are located on the straight line in the complex plane given by

$$\beta_1 \sigma_r - (\alpha_1 - \alpha_2)\sigma_i = \beta_1.$$

Theorem 3
If $\beta_2 \neq 0$, the eigenvalues $\sigma_r + i\sigma_i$ are on the circle in the complex plane with center $c$ and radius $R$:

$$c = \frac{z_1 - \bar{z}_2}{z_2 - \bar{z}_2}, \quad R = \left| \frac{z_2 - z_1}{z_2 - \bar{z}_2} \right|.$$

Note that if $\beta_1 \beta_2 > 0$ the origin is not enclosed in the circle.
Using Sommerfeld boundary conditions, it impossible to write the matrix as $L - z_1 M$ where, $L$ and $M$ are SPD.

Generalized matrix

$$L + iC - z_1 M,$$

where $L$, $M$, and $C$ are SPD. Matrix $C$ contains Sommerfeld boundary conditions (or other conditions: PML, ABL).

Use as preconditioner

$$L + iC - z_2 M.$$
Suppose

\[(L + iC)v = \lambda_C Mv\]

then

\[(L + iC - z_1 M)v = \sigma_C (L + iC - z_2 M)v.\]

**Theorem 4**

Let \( \beta_2 \neq 0 \) then the eigenvalues \( \sigma_C \) are in or on the circle with center

\[c = \frac{z_1 - \bar{z}_2}{z_2 - \bar{z}_2}\]

and radius

\[R = \left| \frac{\bar{z}_2 - z_1}{z_2 - \bar{z}_2} \right|.\]
3. Shift with an SPD real part

Motivation: the preconditioned system is easy to solve.
### Optimal choices for $z_2$?

<table>
<thead>
<tr>
<th>Damping</th>
<th>Optimal $\beta_2$</th>
<th>&quot;optimal&quot; iterations</th>
<th>Minimum iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_1 = 0$</td>
<td>-1</td>
<td>56</td>
<td>54</td>
</tr>
<tr>
<td>$\beta_1 = -0.1$</td>
<td>-1.005</td>
<td>42</td>
<td>41</td>
</tr>
<tr>
<td>$\beta_1 = -0.5$</td>
<td>-1.118</td>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>$\beta_1 = -1$</td>
<td>-1.4142</td>
<td>13</td>
<td>13</td>
</tr>
</tbody>
</table>
**Optimal choices for \( z_2 \)?**

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<th>&quot;optimal&quot; iterations</th>
<th>Minimum iterations</th>
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</thead>
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<td>13</td>
<td>13</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Number of iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h ) ( f ) ( \beta_1 )</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>8</td>
</tr>
<tr>
<td>16</td>
</tr>
<tr>
<td>32</td>
</tr>
</tbody>
</table>
Superlinear convergence of GMRES

![Graph showing superlinear convergence of GMRES](image)

- **f = 8, beta1 = 0, no radiation condition**

- **Y-axis:** Scaled residual norm

- **X-axis:** Number of iterations

C. Vuik, June, 2008
Superlinear convergence of GMRES

![Graph](image-url)
Superlinear convergence of GMRES
4. General Shifted Laplacian preconditioner

No restriction on $\alpha_2$

For the outer loop $\alpha_2 = 1$ and $\beta_2 = 0$ is optimal. Convergence in 1 iteration. But, the inner loop does not converge with multi-grid (original problem).

However, it appears that multi-grid works well for $\alpha_2 = 1$ and $\beta_2 = -1$ and the convergence of the outer loop is much faster than for the choice $\alpha_2 = 0$ and $\beta_2 = -1$. 
Eigenvalues for Complex preconditioner $k = 100$ and $\alpha_2 = 1$

Spectrum is independent of the grid size

$\beta_2 = -1$

$\beta_2 = -0.5$
Eigenvalues for $\beta_1 = -0.025$ (damping) and $\alpha_2 = -1$, $\beta_2 = -0.5$

Spectrum is independent of the grid size and the choice of $k$.

$k = 100$

$k = 400$
5. Numerical experiments

Spectrum with inner iteration

(a) Spectrum (b) Spectrum
Sigsbee model

\[ dx = dz = 22.86 \text{ m}; \quad D = 24369 \times 9144 \text{ m}^2; \quad \text{grid points} \ 1067 \times 401. \]

<table>
<thead>
<tr>
<th>Bi-CGSTAB</th>
<th>5 Hz</th>
<th>10 Hz</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>CPU (sec)</td>
<td>Iter</td>
</tr>
<tr>
<td>NO preco</td>
<td>3128</td>
<td>16549</td>
</tr>
<tr>
<td>With preco</td>
<td>86</td>
<td>48</td>
</tr>
</tbody>
</table>

Note:  ► Without preconditioner, number of iterations \( > 10^4 \),
► With shifted Laplacian preconditioner, only 58 iterations.
6. Fighter radar signature

![Diagram of a fighter jet with various radar signature phenomena labeled](image-url)
Jet engine air intake scattering

- Jet engine air intake closed by jet engine compressor fan forms a large and deep open-ended cavity with varying cross section.
- Typical dimensions: $D \approx 30\lambda$ and $L \approx 200\lambda$ for X-band excitation (10 GHz).
Simulation of large cavity scattering

Different approaches for modelling deep cavity scattering:

- Asymptotic techniques based on ray-tracing and Geometrical/Physical Optics (become inaccurate for $L/D > 3$)
- Modal expansion methods (only for (piecewise) cylindrical cavities)
- Full wave methods based on discretisation of Maxwell equations (no restriction on cavity geometry but expensive for electrically large cavities)

Only full wave method offers flexibility to accurately model the actual geometry: FE for vector wave equation
Iterative solution of the linear system

Use preconditioned Krylov method to solve

\[-\nabla \times \nabla \times \mathbf{E} - k_0^2 \mathbf{E} = 0\]

Using as a preconditioner a single multigrid cycle of

\[-\nabla \times \nabla \times \mathbf{E} - \tilde{k}_0^2 \mathbf{E} = 0, \quad \tilde{k}_0^2 = (\alpha_2 + i\beta_2)k_0^2\]

- For exploratory study we used a Krylov method for the preconditioner solve, with fixed residual reduction factor or fixed number of iterations.
- Analyse different strategies for handling the discretised BIE
- Find optimal values of $\alpha_2$ and $\beta_2$. 
Iterative solution of the linear system

Use preconditioned GCR to solve

\[
\begin{pmatrix}
A_{11}(k_0) & A_{12}(k_0) \\
A_{21}(k_0) & A_{22}(k_0)
\end{pmatrix}
\begin{pmatrix}
E_i \\
E_a
\end{pmatrix}
=
\begin{pmatrix}
0 \\
b(k_0)
\end{pmatrix}
\]

Using the following block preconditioner:

\[
\begin{pmatrix}
A_{11}(\tilde{k}_0) & A_{12}(k_0) \\
0 & A_{22}(k_0)
\end{pmatrix}
\begin{pmatrix}
s_i \\
s_a
\end{pmatrix}
=
\begin{pmatrix}
r_i \\
r_a
\end{pmatrix}
\]

- Matrix-free formulation of $A_{11}(k_0)$ for large problems
- Solve $A_{11}(\tilde{k}_0)s_i = r_i - A_{12}(k_0)s_a$ with GCR for fixed $\epsilon$
- Precompute $(LU)_{A_{22}}$ and solve $(LU)_{A_{22}}s_a = r_a$
Iterative solution of the linear system
Iterative solution of the linear system
Iterative solution of the linear system
Iterative solution of the linear system

Future research: Multigrid solve of preconditioner essential for larger problems:

- CPU time in preconditioner solve
- Possibility to use short recurrence Krylov method.
7. Conclusions

- The shifted Laplacian operator leads to robust preconditioners for the Helmholtz equations with various boundary conditions.

- For real shifts the eigenvalues of the preconditioned operator are on a straight line.

- For complex shifts the eigenvalues of the preconditioned operator are on a circle.

- The proposed preconditioner (shifted Laplacian + multi-grid) is independent of the grid size and linearly dependent of $k$.

- With physical damping the proposed preconditioner is also independent of $k$.

- For the FEM vector Helmholtz problem a good multigrid method is needed for the shifted preconditioner.
Further information/research

- [http://ta.twi.tudelft.nl/nw/users/vuik/pub_it_helmholtz.html](http://ta.twi.tudelft.nl/nw/users/vuik/pub_it_helmholtz.html)
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  On a class of preconditioners for solving the Helmholtz equation
- Y.A. Erlangga, C.W. Oosterlee and C. Vuik
  A Novel Multigrid Based Preconditioner For Heterogeneous Helmholtz Problems
- M.B. van Gijzen, Y.A. Erlangga and C. Vuik
  Spectral analysis of the discrete Helmholtz operator preconditioned with a shifted Laplacian