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TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3 097 TU)
Wednesday 21 January 2004, 14:00-17:00

1. A heavy object of unknown origin comes down vertically. By chance a passer-by has recorded the last 15 meters of its fall on video. We would like to determine the velocity of the object when it strikes the earth. From the video we are able reconstruct the heights of the object at a number of points in time prior to the strike. The data are shown in the table below; in this table the moment of hitting the earth is set equal to 1. The heights are measured with a maximal error of 0.2 meters.

t (sec)	$y(t)$ (m)
0.0	14.9
0.1	14.0
0.2	12.9
0.3	11.5
0.4	10.2
0.5	8.8
0.6	7.3
0.7	5.5
0.8	3.8
0.9	2.0
1.0	0

   (a) We start by using a simple backward difference formula for the striking velocity $y'(1)$:
$$y'(1) \approx \frac{y(1) - y(1-h)}{h}. \tag{1}$$
   Derive an expression for the truncation error of this formula. (Note: applying (1), the whole set of table values may be used; so the value of h in (1) can be different from the stepsize 0.1 of the table.)

   (b) We conjecture, in view of the coarseness of the observations, that the results will be affected seriously by measuring errors. This conjecture is confirmed when we minimize the total error (= truncation error + measurement error). Determine h for which the total error is as small as possible. In this analysis you are allowed to assume that gravity accelerates the object uniformly, and you may put the acceleration at 10 m/sec/sec. Furthermore you may set the height at the moment of striking (t=1) exactly equal to 0 (no error involved).

[0]**please turn over**

(c) Give the approximation of $y'(1)$ using the analysis of part (b) and determine an upper bound for the total error.

(d) Suppose we are not satisfied with the accuracy obtained in (c) and look for better approximations. Give an advantage of using a 3- or more point formula for $y'(1)$ in this particular problem.

(e) A suitable 3-point formula is given by

$$y'(1) \approx \frac{3y(1) - 4y(1-h) + y(1-2h)}{2h}. \tag{2}$$

Give the truncation error of this formula.

(f) Approximate $y'(1)$ by (2) and provide an upper bound for the total error while choosing the optimal stepsize.

(g) Do you expect further accuracy improvements from 4- or more point formulas? Motivate your answer.

2. The following system of differential equations is given:

$$\frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u} + \boldsymbol{g}(t), \text{ with } \boldsymbol{u}(t_0) = \boldsymbol{u}_0,$$

where A is a 2×2 matrix: $A = \begin{pmatrix} -a & -1 \\ 1 & -a \end{pmatrix}, \quad a > 0.$

(a) Show that the eigenvalues of A are given by $\lambda_{1,2} = -a \pm i$.

(b) Show that the Euler Forward method applied to this system is only stable if $h \leq \frac{2a}{a^2+1}$.

(c) The Trapezoidal rule for $\frac{du}{dt} = f(t,u)$ is given by:

$$w_{j+1} = w_j + \frac{h}{2}[f(t_j, w_j) + f(t_{j+1}, w_{j+1})].$$

Show that the amplification factor $Q(h\lambda)$ is equal to $\frac{1+\frac{1}{2}h\lambda}{1-\frac{1}{2}h\lambda}$.

(d) Use this to show that the Trapezoidal applied to the system is always stable.

Choose $a = 2$, $\boldsymbol{g}(t) = \begin{pmatrix} t \\ t+1 \end{pmatrix}$ and $\boldsymbol{u}(0) = \boldsymbol{u}(t_0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $h = 0.5$.

(e) Do one step with the Euler Forward method.

(f) Do one step with the Trapezoidal rule.

For the answers of this test we refer to:
http://ta.twi.tudelft.nl/nw/users/vuik/wi3097/tentamen.html