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ANSWERS OF THE TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU)
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1. a The local truncation error is defined by

$$\tau_{n+1}(h) := \frac{y_{n+1} - z_{n+1}}{h}, \tag{1}$$

where $y_n := y(t_n)$ represents the exact solution and

$$z_{n+1} = y_n + hf(t_{n+1}, z_{n+1}), \tag{2}$$

represents the approximation of the numerical solution at t_{n+1} upon using y_n for the previous time step. Since, we use the test equation $y' = \lambda y$, we express y_{n+1} in terms of y_n as follows

$$y_{n+1} = y_n e^{\lambda h} = y_n(1 + h\lambda + \frac{1}{2}h^2\lambda^2 + O(h^3)). \tag{3}$$

From (2), we use the test equation and the geometric series

$$z_{n+1} = \frac{y_n}{1 - h\lambda} = y_n(1 + h\lambda + h^2\lambda^2 + O(h^3)). \tag{4}$$

Substitution of equations (3) and (4) into the definition of the local truncation error, gives

$$\tau_{n+1}(h) = \frac{y_n}{h}\left(-\frac{h^2\lambda^2}{2} + O(h^3)\right) = O(h). \tag{5}$$

b Using the test equation, we get

$$w_{n+1} = w_n + h\lambda w_{n+1}, \tag{6}$$

where w_n denotes the numerical approximation of y_n. The above equation implies

$$w_{n+1} = \frac{w_n}{1 - h\lambda} =: Q(h\lambda)w_n. \tag{7}$$

Here $Q(h\lambda)$ represents the amplification factor. For numerical stability, we require the modulus of the amplification factor to satisfy

$$|Q(h\lambda)| \leq 1, \text{ hence } |\frac{1}{1 - h\lambda}| = \frac{1}{|1 - h\lambda|} \leq 1. \tag{8}$$

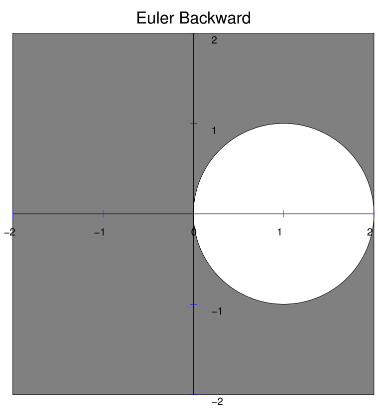

Figure 1: The region of stability of the backward Euler method (grey area).

From the above equation, it is clear that

$$|1 - h\lambda| \geq 1, \tag{9}$$

and with $\lambda = \mu + i\nu$, we get

$$(1 - h\mu)^2 + (h\nu)^2 \geq 1. \tag{10}$$

This area is the whole complex plane except the unit circle with center $(1, 0)$, see Figure 1.

c Consider the equations that we have to solve

$$\begin{aligned} y_1' &= y_1(1 - (y_1 + 2y_2)) =: f_1(y_1, y_2), \\ y_2' &= y_2(1 - (y_1 + y_2)) =: f_2(y_1, y_2), \end{aligned} \tag{11}$$

Here, we introduced the functions $f_1(y_1, y_2)$ and $f_2(y_1, y_2)$. Then, the Jacobi matrix is given by

$$J(y_1, y_2) := \begin{pmatrix} \frac{\partial f_1}{\partial y_1}(y_1, y_2) & \frac{\partial f_1}{\partial y_2}(y_1, y_2) \\ \\ \frac{\partial f_2}{\partial y_1}(y_1, y_2) & \frac{\partial f_2}{\partial y_2}(y_1, y_2) \end{pmatrix} = \begin{pmatrix} 1 - 2(y_1 + y_2) & -2y_1 \\ \\ -y_2 & 1 - (y_1 + 2y_2) \end{pmatrix}. \tag{12}$$

For the equilibrium $(0, 1)$, we have

$$J(y_1, y_2) := \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}. \tag{13}$$

Hence both eigenvalues are given by $\lambda_1 = -1$ and $\lambda_2 = -1$.

d - We have $\lambda_1 = -1$ and $\lambda_2 = -1$, hence with $h > 0$, this implies that $h\lambda < 0$ (thus real-valued), then from Figure 1, it is clear that the backward Euler is stable for any $h > 0$.

- Since the eigenvalues are real-valued and negative, we use

$$h \leq \frac{2}{|\lambda|}, \tag{14}$$

as stability bound for the forward Euler method. With $\lambda_1 = \lambda_2 = -1$, we get $h \leq 2$ as the maximum allowable time step to warrant numerical stability, based on linear stability analysis around $(0, 1)$.

e Applying the forward Euler time integration method to system (11), gives

$$\begin{aligned} u_{n+1} &= u_n + hu_n(1 - (u_n + 2v_n)), \\ v_{n+1} &= v_n + hv_n(1 - (u_n + v_n)). \end{aligned} \tag{15}$$

where $\underline{w}_n = (u_n, v_n)^T$ denotes the numerical solution with components u_n and v_n. Using $h = 1$ and $u_0 = 0.25$ and $v_0 = 0.5$, gives

$$\begin{aligned} u_1 &= u_0 + hu_0(1 - (u_0 + 2v_0)) = \tfrac{1}{4} + \tfrac{1}{4}(1 - (\tfrac{1}{4} + 1)), \\ v_1 &= v_0 + hv_0(1 - (u_0 + v_0)) = \tfrac{1}{2} + \tfrac{1}{2}(1 - (\tfrac{1}{4} + \tfrac{1}{2})). \end{aligned} \tag{16}$$

Hence $u_1 = \frac{3}{16} = 0.1875$ and $v_1 = \frac{5}{8} = 0.625$.

2. (a) The Taylor polynomials around 0 are given by:

$$\begin{aligned} f(0) &= f(0) , \\ f(-h) &= f(0) - hf'(0) + \frac{h^2}{2} f''(0) - \frac{h^3}{6} f'''(\xi_1) , \\ f(-2h) &= f(0) - 2hf'(0) + 2h^2 f''(0) - \frac{(2h)^3}{6} f'''(\xi_2) . \end{aligned}$$

Here $\xi_1 \in (-h, 0)$, $\xi_2 \in (-2h, 0)$. We know that $Q(h) = \frac{\alpha_0}{h^2} f(0) + \frac{\alpha_{-1}}{h^2} f(-h) + \frac{\alpha_{-2}}{h^2} f(-2h)$, which should be equal to $f''(0) + O(h)$. This leads to the following conditions:

$$\begin{array}{rcccccl} f(0): & \frac{\alpha_0}{h^2} & + & \frac{\alpha_{-1}}{h^2} & + & \frac{\alpha_{-2}}{h^2} & = 0 , \\ f'(0): & & & -\frac{h\alpha_{-1}}{h^2} & - & \frac{2h\alpha_{-2}}{h^2} & = 0 , \\ f''(0): & & & \frac{h^2}{2h^2}\alpha_{-1} & + & \frac{2h^2\alpha_{-2}}{h^2} & = 1 . \end{array}$$

This can also be written as

$$\begin{array}{rcccccl} f(0): & \alpha_0 & + & \alpha_{-1} & + & \alpha_{-2} & = 0 , \\ f'(0): & & & -\alpha_{-1} & - & 2\alpha_{-2} & = 0 , \\ f''(0): & & & \frac{\alpha_{-1}}{2} & + & 2\alpha_{-2} & = 1 . \end{array}$$

(b) The truncation error follows from the Taylor polynomials:

$$f''(0) - Q(h) = f''(0) - \frac{f(0) - 2f(-h) + f(-2h)}{h^2} = -\left(\frac{\frac{2h^3}{6} f'''(\xi_1) - \frac{8h^3}{6} f'''(\xi_2)}{h^2} \right)$$

$$= hf'''(\xi).$$

(c) Note that

$$f''(0) - Q(h) = Kh \tag{17}$$

$$f''(0) - Q(\frac{h}{2}) = K(\frac{h}{2}) \tag{18}$$

Subtraction gives:

$$Q(\frac{h}{2}) - Q(h) = Kh - K\frac{h}{2} = K(\frac{h}{2}). \tag{19}$$

We choose $h = \frac{1}{2}$. Then $Q(h) = Q(\frac{1}{2}) = \frac{0-2\times 0.1250+1}{0.25} = 3$ and $Q(\frac{h}{2}) = Q(\frac{1}{4}) = \frac{0-2\times 0.0156+0.1250}{(\frac{1}{4})^2} = 1.5008$. Combining (18) and (19) shows that

$$f''(0) - Q(\frac{1}{4}) = Q(\frac{1}{4}) - Q(\frac{1}{2}) = -1.4992$$

(d) To estimate the rounding error we note that

$$|Q(h) - \hat{Q}(h)| = |\frac{(f(0) - \hat{f}(0)) - 2(f(-h) - \hat{f}(-h)) + (f(-2h) - \hat{f}(-2h))}{h^2}|$$

$$\leq \frac{|f(0) - \hat{f}(0)| + 2|f(-h) - \hat{f}(-h)| + |f(-2h) - \hat{f}(-2h)|}{h^2} \leq \frac{4\epsilon}{h^2},$$

so $C_1 = 4$. Since only 4 digits are given the rounding error is: $\epsilon = 0.00005$.

(e) The total error is bounded by

$$|f''(0) - \hat{Q}(h)| = |f''(0) - Q(h) + Q(h) - \hat{Q}(h)|$$

$$\leq |f''(0) - Q(h)| + |Q(h) - \hat{Q}(h)|$$

$$\leq 6h + \frac{4\epsilon}{h^2} = g(h)$$

This is minimal for h_{opt}, for which $g'(h_{opt}) = 0$. Note that $g'(h) = 6 - \frac{8\epsilon}{h^3}$. This implies that $h_{opt}^3 = \frac{4\epsilon}{3}$, so $h_{opt} = (\frac{4\epsilon}{3})^{\frac{1}{3}} \approx 0.0405$.