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ANSWERS OF THE TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU and CTB2400)
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1.

a The local truncation error is defined as

$$\tau_{n+1}(h) = \frac{y_{n+1} - z_{n+1}}{h}, \tag{1}$$

where z_{n+1} is given by

$$z_{n+1} = y_n + h\left(a_1 f(t_n, y_n) + a_2 f(t_n + h, y_n + hf(t_n, y_n)\right). \tag{2}$$

A Taylor expansion of f around (t_n, y_n) yields

$$f(t_n+h, y_n+hf(t_n, y_n)) = f(t_n, y_n)+h\frac{\partial f}{\partial t}(t_n, y_n)+hf(t_n, y_n)\frac{\partial f}{\partial y}(t_n, y_n)+O(h^2). \tag{3}$$

This is substituted into equation (2) to obtain

$$z_{n+1} = y_n+h\left(a_1 f(t_n, y_n) + a_2\left[f(t_n, y_n) + h\frac{\partial f}{\partial t}(t_n, y_n) + hf(t_n, y_n)\frac{\partial f}{\partial y}(t_n, y_n)\right]\right)+O(h^3). \tag{4}$$

A Taylor series for $y(t)$ around t_n gives for y_{n+1}

$$y_{n+1} = y(t_n + h) = y_n + hy'(t_n) + \frac{h^2}{2}y''(t_n) + O(h^3). \tag{5}$$

From the differential equation we know that:

$$y'(t_n) = f(t_n, y_n) \tag{6}$$

From the Chain Rule of Differentiation, we derive

$$y''(t_n) = \frac{df(t_n, y_n)}{dt} = \frac{\partial f(t_n, y_n)}{\partial t} + \frac{\partial f(t_n, y_n)}{\partial y}y'(t_n) \tag{7}$$

after substitution of the differential equation one obtains:

$$y''(t_n) = \frac{\partial f(t_n, y_n)}{\partial t} + \frac{\partial f(t_n, y_n)}{\partial y}f(t_n, y_n) \tag{8}$$

Equations (5) and (4) are substituted into relation (1) to obtain

$$\tau_{n+1}(h) = f(t_n, y_n)(1 - (a_1 + a_2)) + h\left(\frac{\partial f}{\partial t} + f\frac{\partial f}{\partial y}\right)\left(\frac{1}{2} - a_2\right) + O(h^2) \tag{9}$$

Hence

(a) $a_1 + a_2 = 1$ implies $\tau_{n+1}(h) = O(h)$;

(b) $a_1 + a_2 = 1$ and $a_2 = 1/2$, that is, $a_1 = a_2 = 1/2$, gives $\tau_{n+1}(h) = O(h^2)$.

b The test equation is given by

$$y' = \lambda y. \tag{10}$$

Application of the predictor step to the test equation gives

$$w^*_{n+1} = w_n + h\lambda w_n = (1 + h\lambda)w_n. \tag{11}$$

The corrector step yields

$$w_{n+1} = w_n + h\left(a_1\lambda w_n + a_2\lambda(1 + h\lambda)w_n\right) = (1 + (a_1 + a_2)h\lambda + a_2h^2\lambda^2)w_n. \tag{12}$$

Hence the amplification factor is given by

$$Q(h\lambda) = 1 + (a_1 + a_2)h\lambda + a_2h^2\lambda^2. \tag{13}$$

c Let $\lambda < 0$ (so λ is real), then, for stability, the amplification factor must satisfy

$$-1 \leq Q(h\lambda) \leq 1, \tag{14}$$

from the previous assignment, we have

$$-1 \leq 1 + (a_1 + a_2)h\lambda + a_2(h\lambda)^2 \leq 1 \Leftrightarrow -2 \leq (a_1 + a_2)h\lambda + a_2(h\lambda)^2 \leq 0. \tag{15}$$

First, we consider the left inequality:

$$a_2(h\lambda)^2 + (a_1 + a_2)h\lambda + 2 \geq 0 \tag{16}$$

For $h\lambda = 0$, the above inequality is satisfied, further the discriminant is given by $(a_1 + a_2)^2 - 8a_2 < 0$. Here the last inequality follows from the given hypothesis. Hence the left inequality in relation (15) is always satisfied. Next we consider the right hand inequality of relation (15)

$$a_2(h\lambda)^2 + (a_1 + a_2)h\lambda \leq 0. \tag{17}$$

This relation is rearranged into

$$a_2(h\lambda)^2 \leq -(a_1 + a_2)h\lambda, \tag{18}$$

hence

$$a_2|h\lambda|^2 \leq (a_1 + a_2)|h\lambda| \Leftrightarrow |h\lambda| \leq \frac{a_1 + a_2}{a_2}, \qquad a_2 \neq 0. \tag{19}$$

This results into the following condition for stability

$$h \leq \frac{a_1 + a_2}{a_2|\lambda|}, \qquad a_2 \neq 0. \tag{20}$$

d The Jacobian, J, is given by

$$J = \begin{pmatrix} \dfrac{\partial f_1}{\partial y_1} & \dfrac{\partial f_1}{\partial y_2} \\ \\ \dfrac{\partial f_2}{\partial y_1} & \dfrac{\partial f_2}{\partial y_2} \end{pmatrix}. \tag{21}$$

Since $f_1(y_1, y_2) = -2y_1 - y_1y_2$ and $f_2(y_1, y_2) = 2y_1y_2 - y_2^2$, we obtain

$$J = \begin{pmatrix} -2 - y_2 & -y_1 \\ 2y_2 & 2y_1 - 2y_2 \end{pmatrix}. \tag{22}$$

Substitution of the initial values $y_1(0) = 2$ and $y_2(0) = 2$, gives

$$J = \begin{pmatrix} -4 & -2 \\ 4 & 0 \end{pmatrix}. \tag{23}$$

e The eigenvalues of the Jacobian at $y_1(0) = y_2(0) = 2$ are given by $\lambda_{1,2} = -2 \pm 2i$. For our case, we have

$$Q(h\lambda) = 1 + h\lambda + \frac{1}{2}(h\lambda)^2. \tag{24}$$

Since our eigenvalues are not real valued, it is required for stability that

$$|Q(h\lambda)| \leq 1. \tag{25}$$

Since the eigenvalues are complex conjugates, we can proceed with one of the eigenvalues, say $\lambda = -2 + 2i$ with $\lambda^2 = -8i$ to obtain

$$Q(h\lambda) \quad = 1 + h(-2 + 2i) + \tfrac{1}{2}h^2(-8i) \tag{26}$$

Substitution of $h = \frac{1}{2}$ shows that $Q(h\lambda) = 0$. This implies that $|Q(h\lambda)| = 0 \leq 1$ so the method is stable.

2. a. Given $y(x) = e^x(2 - x)$, then $y''(x) = -e^x x$, and hence $-y'' + y = 2e^x$ follows by simple addition. Furthermore, $y(0) = 2$ and $y'(x) = -e^x(x - 1)$ and hence $y'(1) = 0$. Hence the differential equation, as well as the boundary conditions are satisfied.

   b. Let $y_j = y(x_j)$, and let $x_n = 1$, hence $h = 1/n$, then

$$y_{j-1} = y(x_j - h) = y_j - hy'(x_j) + h^2/2y''(x_j) - h^3/3!y'''(x_j) + h^4/4!y''''(x_j) + O(h^5);$$

$$y_{j+1} = y(x_j + h) = y_j + hy'(x_j) + h^2/2y''(x_j) + h^3/3!y'''(x_j) + h^4/4!y''''(x_j) + O(h^5). \tag{27}$$

From the above expressions, it can be seen that

$$y''(x_j) = \frac{y_{j-1} - 2y_j + y_{j+1}}{h^2} + \frac{h^2}{12} y''''(x_j) + O(h^3), \tag{28}$$

and hence the error is $O(h^2)$. This gives the following discretisation

$$\frac{-w_{j-1} + 2w_j - w_{j+1}}{h^2} + w_j = 2e^{x_j}, \qquad \text{for } j = 1 \dots n, \tag{29}$$

where $x_j = jh$ and $w_j \approx y_j$ is the numerical (finite difference) solution neglecting the error.

Furthermore, we use a virtual gridnode near $x = 1$, $x_{n+1} = 1 + h$, with

$$0 = y'(1) = \frac{y_{n+1} - y_{n-1}}{2h} - \frac{h^2}{6} y'''(1) + O(h^3), \tag{30}$$

hence the error is $O(h^2)$. Neglecting the error, and substitution into the discretisation equation $j = n$, yields

$$\frac{-2w_{n-1} + 2w_n}{h^2} + w_n = 2e. \tag{31}$$

Division by 2 to make the discretisation symmetric yields

$$\frac{-w_{n-1} + w_n}{h^2} + \frac{1}{2} w_n = e. \tag{32}$$

The boundary condition $y(0) = 2$ at $x = 0$ yields

$$\frac{2w_1 - w_2}{h^2} + w_1 = \frac{2}{h^2} + 2e^h. \tag{33}$$

c. For $j = 1$, we get, using $h = 1/3$,

$$18w_1 - 9w_2 + w_1 = \frac{2 + 1/9}{1/9} = 18 + 2e^{\frac{1}{3}} \tag{34}$$

For $j = 2$, we obtain

$$-9w_1 + 18w_2 - 9w_3 + w_2 = 2e^{2/3}. \tag{35}$$

For $j = 3 = n$, we obtain

$$-9w_2 + 9w_3 + \frac{1}{2} w_3 = 2e. \tag{36}$$

Hence, the system of equations reads

$$\begin{cases} 19w_1 - 9w_2 = 18 + 2e^{\frac{1}{3}}, \\ \\ -9w_1 + 19w_2 - 9w_3 = 2e^{2/3}, \\ \\ -9w_2 + 19/2 w_3 = 2e. \end{cases} \tag{37}$$

d. The exact solution is given by $y(x) = e^x(2-x)$ and its derivative of order k reads $y^{(k)}(x) = (2-x-k)e^x$. The error for the finite difference formula under consideration is determined by the derivatives of third order and larger. Since none of the derivatives $y^{(3+l)}(x)$, $l \geq 0$ vanishes for all values of x the error cannot be zero. We use the same argument to show that there is no finite difference formula which yields a nodally exact solution.

e. The linear Lagrangian interpolation polynomial, with nodes a and b, is given by

$$p_1(x) = \frac{x-b}{a-b}f(a) + \frac{x-a}{b-a}f(b). \tag{38}$$

This is evident from application of the given formula. We integrate function $f(x)$ by approximating $f(x)$ by $p_1(x)$, then it follows:

$$\begin{aligned} &\int_a^b f(x)\,\mathrm{d}x \approx \int_a^b p_1(x)\,\mathrm{d}x = \int_a^b \left\{ f(a)\frac{x-b}{a-b} + f(b)\frac{x-a}{b-a} \right\} \mathrm{d}x = \\ &= \left[\frac{1}{2}\frac{(x-a)^2}{b-a} f(b) \right]_a^b + \left[\frac{1}{2}\frac{(x-b)^2}{a-b} f(a) \right]_a^b = \frac{1}{2}(b-a)(f(a)+f(b)). \end{aligned} \tag{39}$$

This is the Trapezoidal rule.

f. The magnitude of the error of the numerical integration over interval $[a, b]$ is given by

$$\begin{aligned} &|\int_a^b f(x)\,\mathrm{d}x - \int_a^b p_1(x)\,\mathrm{d}x| = |\int_a^b \left(f(x) - p_1(x)\right)\,\mathrm{d}x| = \\ &|\int_a^b \frac{1}{2}(x-a)(x-b)f''(\chi(x))\,\mathrm{d}x| \leq \frac{1}{2}\max_{x\in[a,b]}|f''(x)| \int_a^b |(x-a)(x-b)|dx = \\ &\frac{1}{12}(b-a)^3 \max_{x\in[a,b]}|f''(x)|. \end{aligned} \tag{40}$$